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Article

Grey relational analysis for hesitant fuzzy sets and interval-valued hesitant fuzzy sets with applications to MADM problems

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Abstract: Quantitative and qualitative fuzzy measures have been proposed to hesitant fuzzy sets (HFSs) from different points. However, few of the existing HFSs fuzzy measures refer to the grey relational analysis (GRA) theory. Actually, the GRA theory is very useful in the fuzzy measure domain, which has been developed for such the intuitionistic fuzzy sets. Therefore, in this paper, we apply the GRA theory to the HFSs and interval-valued hesitant fuzzy sets (IVHFS) domain. We propose the HFSs grey relational degree, HFSs slope grey relational degree, HFSs synthetic grey relational degree and IVHFSs grey relational degree, which describe the closeness, the variation tendency and both the closeness and variation tendency of HFSs and closeness of IVHFSs, respectively, greatly enriching the fuzzy measures of HFSs. Furthermore, with the help of the TOPSIS method, we develop the grey relational based MADM methodology to solve the HFSs and IVHFSs MADM problems. Finally, combined with two practical MADM examples about energy policy selection with HFSs information and emergency management evaluation with IVHFSs information, we obtain the most desirable decision results, and compared with the previous methods, the validity, effectiveness and accuracy of the proposed grey relational degree for HFSs and IVHFSs are demonstrated in detail.

Keywords: Grey relational analysis (GRA); Hesitant Fuzzy Sets (HFSs); Interval-valued hesitant fuzzy sets (IVHFS); grey relational degree; grey relational based MADM methodology

1. Introduction

The introduction should briefly place the study in a broad context and highlight why it is important. It should define the purpose of the work and its significance. The current state of the research field should be reviewed carefully and key publications cited. Please highlight controversial and diverging hypotheses when necessary. Finally, briefly mention the main aim of the work and highlight the principal conclusions. As far as possible, please keep the introduction comprehensible to scientists outside your particular field of research. References should be numbered in order of appearance and indicated by a numeral or numerals in square brackets, e.g., [1] or [2,3], or [4–6]. See the end of the document for further details on references.

In 2009, Torra [1, 2] originally introduced the hesitant fuzzy sets (HFSs) and several basic operations for it. HFSs is one of the most efficient decision making techniques to deal with imprecise and vague information and a growing number of studies focus on not only the properties but also the applications of it.

Xia and Xu [3, 4], Liao and Xu [5, 6] first introduced some basic operations, aggregation operators, score function and variance for HFSs. Afterwards, Xu and Xia [7-9] proposed a variety of distance, similarity, entropy and correlation measures for hesitant fuzzy sets. Farhadinia [10] also investigated the relationship between the distance, similarity and entropy measure for HFSs and developed division and subtraction formulas for HFSs [11]. Li, Zeng and Zhao [12] introduced the concept of hesitance degree for HFE and presented some new distance measures between HFSs

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which contained the hesitance degrees, later they applied these new distances to solve pattern recognition problems [13]. Zhao, Xu and Liu [14] proposed a new axiomatic framework of entropy measures for HFEs by taking fully into account two faces of uncertainty associated with an HFE. Chen, Xu and Xia [15], Liao, Xu and Zeng [16] proposed some novel correlation coefficients of hesitant fuzzy sets for clustering analysis and medical diagnosis.

Besides a variety of operations, properties and fuzzy measures on HFSs, the hesitant fuzzy sets has shown its advantages in such the real fields as decision-making [3, 4, 6, 9 16-21], feature selection [22], pattern recognition [13], cluster analysis [15, 16] and linguistic computing [23-25]. He et al [17] first introduced the expected value and the geometric average value of hesitant multiplicative element (HME) to group decision making problems. Xu and Zhang [18] developed a novel approach based on TOPSIS and the maximizing deviation method for solving MADM problems with hesitant fuzzy information. Qin, Liu and Pedrycz [19] investigated multiple attribute decision making (MADM) problems with hesitant fuzzy attribute based on Frank triangular norms. Zhang et al [20] proposed an interval programming method for solving MAGDM problems with hesitant fuzzy alternatives based on LINMAP. Ashtiani and Azgomi [21] proposed a hesitant fuzzy multi-criteria decision making based computational trust model capable of taking into account the fundamental building blocks corresponding to the concept of trust. Ebrahimpour and Eftekhari [22] proposed an innovative method to deal with feature subset selection with HFSs based on Maximum Relevancy and Minimum Redundancy approach. Rodríguez, Martínez and Herrera et al [23] introduced the concept of a hesitant fuzzy linguistic term set (HLFSs) to provide a linguistic and computational basis. Liao et al [24] developed a method to solve the MCDM problem within the context of HLFSs. Wang et al [25] developed a likelihood-based TODIM approach for the selection and evaluation with multi-hesitant fuzzy linguistic information.

Chen, Xu and Xia [26] extended the HFSs and first introduced the interval-valued hesitant fuzzy set (IVHFS) to describe uncertain evaluation information in group decision making (GDM) processes, presented some operational laws and a score function for IVHFS and also proposed correlation coefficients for it [15]. Subsequently they [27] derived the properties and relationships of fundamental operations on IVHFSs for Algebraic t-norm and t-conorm and presented the operations based on Archimedean t-norm and t-conorm and investigated their properties. Farhadinia [28, 11] introduced the division and subtraction formulas for IVHFSs and discussed the distance, similarity and entropy measure of IVHFSs and applied into clustering analysis. Fernández, Alonso and Bustince et al [29, 30] introduced finite interval-valued hesitant fuzzy sets, defined a new order, entropy between them considering the fuzziness, lack of knowledge and hesitance and applied in the business selection. Verma [31] proposed four new operations on IVHFS and study their properties and relations in details. Gitinavard, Mousavi and Vahdani [32] introduced a novel multi-criteria weighting and ranking model with interval-valued hesitant fuzzy setting and applied to location and supplier selection problems. Zhang [33] developed two interval-valued hesitant fuzzy QUALIFLEX outranking methods to handle MCDM problems concerning the selection of green suppliers. Jin, Ni and Chen et al [34] developed two interval-valued hesitant fuzzy prioritized aggregation operators with the help of Einstein operations to investigate interval-valued hesitant fuzzy multi-attribute group decision-making problems. Zhang, Li and Mu et al [35] proposed a new rough set model that combines interval-valued hesitant fuzzy sets with multigranulation rough sets over two universes and applied to steam turbine fault diagnosis.

Despite of the qualitative and quantitative studies of HFSs and IVHFSs, the present work for them mainly focuses on such fuzzy measures as distance, similarity, entropy and correlation coefficients measure, few studies referred to the critical fuzzy measure over HFSs and IVHFSs: grey relational analysis. Meanwhile the correlation coefficients measure can only calculate the linear fashion of two HFSs and IVHFSs, they can not measure the closeness. Therefore, the applications of the HFSs and IVHFSs correlation coefficients are controversial, they are only one aspect of the real fuzzy measures. For the above reasons, it is essential to apply the grey relational analysis for HFSs and IVHFSs to measure the closeness of the sets.

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Actually, the traditional grey relational analysis of the fuzzy set takes an important occupation in the fuzzy measure field. It can measure the closeness of two fuzzy sets just like the distance and similarity measure. Many researchers have focused on the grey relational analysis of fuzzy set and proposed several approaches to solve decision making problems. Wei [36-39] established a series of grey relational analysis (GRA) method to investigate the multiple attribute decision-making problems with intuitionistic fuzzy information, 2-tuple linguistic information and the dynamic hybrid multiple attribute decision information. Zhang, Liu and Zhai [40], Zhang, Jin and Liu [41], Guo [42] also developed the grey relational analysis method for solving MCDM problems with interval-valued triangular fuzzy numbers, intuitionistic trapezoidal fuzzy number and hybrid multiple attribute information respectively. Kong, Wang and Wu [43] presented a new algorithm based on grey relational analysis to discuss fuzzy soft set decision problems. Kuo and Liang [44] combined the concepts of VIKOR and grey relational analysis to present an effective fuzzy MCDM method. Tang [45], Li, Wen and Xie [46] proposed a novel fuzzy soft set approach in decision making based on grey relational analysis and Dempster-Shafer theory of evidence respectively. Unfortunately, only a few of these grey relational analyses referred to HFSs and IVHFSs, so grey relational analysis for HFSs and IVHFSs is necessary and urgent.Wei and Li [47] establish an optimization model based on GRA to get the weight vector of the HFSs criteria, Sun, Guan, Yi and Zhou [48] defined the difference and slope of the HFSs to form a grey relational degree, which is inspiring to be used in our this paper.

Consequently, the motivation of this paper is to extend the concept of grey relational analysis to HFSs and IVHFSs and develop a methodology to solve MADM problems with HFSs and IVHFSs information. The novelties of this paper concentrate on the five aspects: (1) Investigate the grey relational analysis to HFSs and propose the grey relational coefficient and grey relational degree of the HFSs for the first time. (2) Propose the slope grey relational coefficient and slope grey relational degree of the HFSs. (3) Propose the synthetic grey relational coefficient and synthetic grey relational degree of the HFSs. (4) Extend the grey relational analysis to IVHFSs and proposed the grey relational coefficient and grey relational degree of the IVHFSs. (5) Develop a MADM methodology with HFSs and IVHFSs information.

The rest of the paper is as follows: Section 2 briefly reviews the concepts of HFSs, IVHFSs and grey relational analysis theory. In Section 3, we define the grey relational coefficient and grey relational degree for HFSs for the first time and propose some extended HFSs grey relational expression as the slope and synthetic grey relational degree. Furthermore, we extend the grey relational analysis to IVHFSs. In Section 4, we develop a hesitant fuzzy MADM methodology based on the grey relational analysis between HFSs and IVHFSs. In Section 5, we apply the proposed grey relational hesitant fuzzy MADM methodology to the practical MADM problems. Finally, the paper ends with some concluding remarks and future challenges in Section 6.

2. Preliminaries

In this section, we recall the HFSs, IVHFSs and grey relational analysis theory.

2.1. Hesitant fuzzy sets and interval-valued hesitant fuzzy sets

When an expert makes a decision, he may hesitate to choose the exact membership degree in [0, 1]. For such a circumstance where there are several membership degrees of one element to a set, Torra [1, 2] developed the hesitant fuzzy set (HFS), which is a kind of generalized fuzzy set where the membership degree of an element to a certain set can be illustrated as several different values between 0 and 1. HFSs is good at dealing with the situations that people have disagreements or hesitancy when deciding something.

Definition 1. [1, 2]. Suppose that $X = \{x_1, x_2, \cdots, x_n\}$ is a reference set, a hesitant fuzzy set (HFS) A on X is defined in terms of a function $h_A(x)$ when applied to X returns a subset of [0, 1], i.e.

$$A = \{\langle x, h_A(x)\rangle | x \in X\} \tag{1}$$

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Where $h_A(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to the set A. For convenience, Xia and Xu [5] call $h_A(x)$ a hesitant fuzzy element (HFE), which is a basic unit of HFS.

In many real problems, due to insufficiency in available information, it may be difficult to exactly quantify the attribute with a crisp number, but can be represented by an interval number within [0, 1]. Thus, Chen, Xu and Xia [26] introduced the concept of interval-valued hesitant fuzzy sets (IVHFSs), which permit the membership degrees of an element to a given set to have a few different interval values.

Definition 2. [26]. Suppose that $X = \{x_1, x_2, \cdots, x_n\}$ is a reference set, $D[0,1]$ is the set of all closed subintervals of [0, 1]. An interval-valued hesitant fuzzy sets (IVHFSs) $\tilde{A}$ on X is defined as

$$\tilde{A} = \{\langle x, \tilde{h}_{\tilde{A}}(x)\rangle | x \in X\} \tag{2}$$

Where $\tilde{h}_{\tilde{A}}(x)$ denotes all possible interval-valued membership degrees of the element, is a set of some different values in [0, 1], represents the possible membership degrees of the element $x \in X$ to the set $\tilde{A}$. For convenience, they call $\tilde{h}_{\tilde{A}}(x)$ an interval-valued hesitant fuzzy element (IVHFE), which is a basic unit of IVHFS.

$$\tilde{h}_{\tilde{A}}(x) = \{\tilde{\gamma} \,|\, \tilde{\gamma} \in \tilde{h}_{\tilde{A}}(x)\} \tag{3}$$

Where $\tilde{\gamma}$ is an interval number, $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$, $\tilde{\gamma}^L$ and $\tilde{\gamma}^U$ represent the lower and upper limits of $\tilde{\gamma}$, respectively.

2.2. Grey relational analysis Theory

Grey relational theory was originally introduced by Deng [49]. It has been widely applied to decision making, pattern recognition and some other problems under uncertainty, particularly under the discrete data and fuzzy information.

Definition 3. [47]. For reference set $X_0 = (x_0(j), j = 1,2,\cdots,k)$ and $X_i = (x_i(j), j = 1,2,\cdots,k)$, the grey relational coefficient is defined by

$$r(x_0(j), x_i(j)) = \frac{\min_i \min_j |x_0(j) - x_i(j)| + \rho \cdot \max_i \max_j |x_0(j) - x_i(j)|}{|x_0(j) - x_i(j)| + \rho \cdot \max_i \max_j |x_0(j) - x_i(j)|} \tag{4}$$

Where ρ is the distinguished coefficient, $\rho \in [0,1]$.

The grey relational degree is defined as:

$$\gamma(X_0, X_i) = \frac{1}{k} \cdot \sum_{j=1}^{k} r(x_0(j), x_i(j)) \tag{5}$$

Take the weight into consideration, let the weight vector of X_i is $\boldsymbol{w} = \{w_1, w_2, \cdots, w_k\}$, $\sum_{j=1}^{k} w_j = 1$, $j = 1, 2, \cdots, k$, the grey relational degree is extended to the weighted grey relational degree:

$$\gamma(X_0, X_i) = \sum_{j=1}^{k} w_j \cdot r(x_0(j), x_i(j)) \tag{6}$$

The traditional grey relational theory describes the closeness of two variables, which is necessary in the decision making and pattern recognition fields. In this paper, we will extend it to the HFSs and IVHFSs domain.

3. Grey relational analysis for HFSs and IVHFSs

In this section, we apply grey relational theory to the HFSs and IVHFSs domain and define some novel HFSs expressions based on grey relational theory.

3.1. Grey relational definition of HFEs and HFSs

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Definition 4. For two hesitant fuzzy sets on the fixed set $X=\{x_1,x_2,\cdots,x_n\}$, $A=\{\langle x_i,h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\}$ and $B_j=\{\langle x_i,h_{B_j}(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n, j=1,2,\cdots,m\}$ with $h_A(x_i)=\{\gamma_{Ai1},\gamma_{Ai2},\cdots,\gamma_{Ail_{Ai}}\}$, $h_{B_j}(x_i)=\{\gamma_{B_ji1},\gamma_{B_ji2},\cdots,\gamma_{B_jil_{B_ji}}\}$, $i=1,2,\cdots,n$, $j=1,2,\cdots,m$, then we define the grey relational coefficient between HFEs $h_A(x_i)$ and $h_{B_j}(x_i)$ as:

$$r(h_A(x_i),h_{B_j}(x_i))=\frac{\min_j\min_i\{d(h_A(x_i),h_{B_j}(x_i))\}+\rho\cdot\max_j\max_i\{d(h_A(x_i),h_{B_j}(x_i))\}}{d(h_A(x_i),h_{B_j}(x_i))+\rho\cdot\max_j\max_i\{d(h_A(x_i),h_{B_j}(x_i))\}} \tag{7}$$

Where $d(h_A(x_i),h_{B_j}(x_i))$ is the distance between HFEs $h_A(x_i)$ and $h_{B_j}(x_i)$, which can be calculated according to following equations:

$$d_{hne}(h_A(x_i),h_{B_j}(x_i))=\left[(\frac{1}{l_{Ai}}\sum_{k=1}^{l_{Ai}}\left|\gamma_{Aik}-\gamma_{B_jik}\right|^2)\right]^{1/2} \tag{8}$$

Actually, there have been proposed a variety of distance measures for HFEs, for more detail, please refer to reference [7, 8, 11-13].

Based on grey relational coefficient between HFEs, the grey relational degree between HFSs A and B_j is defined as:

$$\gamma(A,B_j)=\frac{1}{n}\cdot\sum_{i=1}^{n}r(h_A(x_i),h_{B_j}(x_i)) \tag{9}$$

In practical applications, the elements $\{x_1,x_2,\cdots,x_n\}$ in the universe X have different weights. Take the weight into consideration, let the weight vector of X is $\boldsymbol{w}=\{w_1,w_2,\cdots,w_n\}$, $\sum_{i=1}^{n}w_i=1$, $i=1,2,\cdots,n$, we extend the HFSs grey relational degree to the weighted HFSs grey relational degree as:

$$\gamma_w(A,B_j)=\sum_{i=1}^{n}w_i\cdot r(h_A(x_i),h_{B_j}(x_i)) \tag{10}$$

3.2. Slope grey relational definition for HFEs and HFSs

The HFSs grey relational degree in section 3.1 mainly focus on the closeness of the two HFSs, here we extend a novel HFSs grey relational degree called HFSs slope grey relational degree to represent the linear fashion of HFSs. As a departure, we introduce two new concepts of HFEs and HFSs. Sun et al. [48] defined the difference and slope of the HFSs.

Definition 5. [48] For hesitant fuzzy sets $A=\{\langle x_i,h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\}$ with $h_A(x_i)=\{\gamma_{Ai1},\gamma_{Ai2},\cdots,\gamma_{Ail_{Ai}}\}$ on the fixed set $X=\{x_1,x_2,\cdots,x_n\}$, we define the difference of the HFSs as

$$\Delta A=\{\langle x_i,\Delta h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\} \tag{11}$$

Where $\Delta h_A(x_i)$ denotes the difference of the HFEs.

$$\Delta h_A(x_i)=\{\Delta\gamma_{Ai1},\Delta\gamma_{Ai2},\cdots,\Delta\gamma_{Aik},\cdots,\Delta\gamma_{Ail_{Ai}-1}\} \tag{12}$$

Where

$$\Delta\gamma_{Aik}=\gamma_{Aik+1}-\gamma_{Aik}, k=1,2,\cdots,l_{Ai}-1 \tag{13}$$

Definition 6. [48] For hesitant fuzzy sets $A=\{\langle x_i,h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\}$ with $h_A(x_i)=\{\gamma_{Ai1},\gamma_{Ai2},\cdots,\gamma_{Ail_{Ai}}\}$ on the fixed set $X=\{x_1,x_2,\cdots,x_n\}$, the difference of A is $\Delta A=\{\langle x_i,\Delta h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\}$ with $\Delta h_A(x_i)=\{\Delta\gamma_{Ai1},\Delta\gamma_{Ai2},\cdots,\Delta\gamma_{Aik},\cdots,\Delta\gamma_{Ail_{Ai}-1}\}$, we define the slope of the HFSs as

$$A'=\{\langle x_i,h_{A'}(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\} \tag{14}$$

Where $h_{A'}(x_i)$ denotes the slope of the HFEs.

$$h_{A'}(x_i)=\left\{\gamma_{A'i1},\gamma_{A'i2},\cdots,\gamma_{A'ik},\cdots,\gamma_{A'il_{Ai}-1}\right\} \quad (15)$$

Where

$$\gamma_{A'ik}=\frac{\Delta\gamma_{Aik}}{\bar{h}_A(x_i)}=\frac{\gamma_{Aik+1}-\gamma_{Aik}}{\bar{h}_A(x_i)}, k=1,2,\cdots,l_{Ai}-1 \quad (16)$$

Where $\bar{h}_A(x_i)$ is the mean of the HFE $h_A(x_i)$

$$\bar{h}_A(x_i)=\frac{1}{l_{Ai}}\sum_{k=1}^{l_{Ai}}\gamma_{Aik}\ , i=1,2,\cdots,n \quad (17)$$

The difference and slope of the HFSs can denote the linear fashion of the HFSs clearly, which is useful in practice. Based on the two concepts, we give the a novelHFSs slope grey relational coefficient, which extends Sun et al.'s definition [48].

Definition 7. For two hesitant fuzzy sets on the fixed set $X=\{x_1,x_2,\cdots,x_n\}$, $A=\{\langle x_i,h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\}$ and $B_j=\left\{\left\langle x_i,h_{B_j}(x_i)\right\rangle \mid x_i\in X, i=1,2,\cdots,n, j=1,2,\cdots,m\right\}$ with $h_A(x_i)=\{\gamma_{Ai1},\gamma_{Ai2},\cdots,\gamma_{Ail_{Ai}}\}$, $h_{B_j}(x_i)=\left\{\gamma_{B_ji1},\gamma_{B_ji2},\cdots,\gamma_{B_jil_{B_ji}}\right\}$, $i=1,2,\cdots,n$, $j=1,2,\cdots,m$, the difference of them are $\Delta A=\{\langle x_i,\Delta h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\}$, $\Delta h_A(x_i)=\left\{\Delta\gamma_{Ai1},\Delta\gamma_{Ai2},\cdots,\Delta\gamma_{Aik},\cdots,\Delta\gamma_{Ail_{Ai}-1}\right\}$, $\Delta\gamma_{Aik}=\gamma_{Aik+1}-\gamma_{Aik}$, $k=1,2,\cdots,l_{Ai}-1$ and $\Delta B_j=\left\{\left\langle x_i,\Delta h_{B_j}(x_i)\right\rangle \mid x_i\in X, i=1,2,\cdots,n, j=1,2,\cdots,m\right\}$, $\Delta h_{B_j}(x_i)=\left\{\Delta\gamma_{B_ji1},\Delta\gamma_{B_ji2},\cdots,\Delta\gamma_{B_jik},\cdots,\Delta\gamma_{B_jil_{B_ji}-1}\right\}$, $\Delta\gamma_{B_jik}=\gamma_{B_jik+1}-\gamma_{B_jik}$, $k=1,2,\cdots,l_{B_ji}-1$, the slope of them are $A'=\left\{\left\langle x_i,h_{A'}(x_i)\right\rangle \mid x_i\in X, i=1,2,\cdots,n\right\}$ with $h_{A'}(x_i)=\left\{\gamma_{A'i1},\gamma_{A'i2},\cdots,\gamma_{A'ik},\cdots,\gamma_{A'il_{Ai}-1}\right\}$ and $B'_j=\left\{\left\langle x_i,h_{B'_j}(x_i)\right\rangle \mid x_i\in X, i=1,2,\cdots,n\right\}$ with $h_{B'_j}(x_i)=\left\{\gamma_{B'_ji1},\gamma_{B'_ji2},\cdots,\gamma_{B'_jik},\cdots,\gamma_{B'_jil_{Ai}-1}\right\}$, we define the HFSs slope grey relational coefficient between the HFEs $h_A(x_i)$ and $h_{B_j}(x_i)$ as:

$$r_s(h_A(x_i),h_{B_j}(x_i))=\frac{1}{l_{Ai}-1}\sum_{k=1}^{l_{Ai}-1}\xi_s(\Delta h_A(x_i),\Delta h_{B_j}(x_i)) \quad (18)$$

Where

$$\xi_s(\Delta h_A(x_i),\Delta h_{B_j}(x_i))=\frac{1+\left|\gamma_{A'ik}\right|}{1+\left|\gamma_{A'ik}\right|+\left|\gamma_{A'ik}-\gamma_{B'_jik}\right|}=\frac{1+\left|\dfrac{\gamma_{Aik+1}-\gamma_{Aik}}{\bar{h}_A(x_i)}\right|}{1+\left|\dfrac{\gamma_{Aik+1}-\gamma_{Aik}}{\bar{h}_A(x_i)}\right|+\left|\dfrac{\gamma_{Aik+1}-\gamma_{Aik}}{\bar{h}_A(x_i)}-\dfrac{\gamma_{B'_jik+1}-\gamma_{B'_jik}}{\bar{h}_{B'_j}(x_i)}\right|} \quad (19)$$

Based on HFSs slope grey relational coefficient, the HFSs slope grey relational degree is defined as:

$$\gamma_s(A,B_j)=\frac{1}{n}\cdot\sum_{i=1}^{n}r_s(h_A(x_i),h_{B_j}(x_i)) \quad (20)$$

Take the weight into consideration, let the weight vector of X is $\boldsymbol{w}=\{w_1,w_2,\cdots,w_n\}$, $\sum_{i=1}^{n}w_i=1$, $i=1,2,\cdots,n$, we extend the HFSs slope grey relational degree to the weighted HFSs slope grey relational degree as:

$$\gamma_{sw}(A,B_j)=\sum_{i=1}^{n}w_i\cdot r_s(h_A(x_i),h_{B_j}(x_i)) \quad (21)$$

3.3. Synthetic grey relational definition for HFSs

Definition 8. For two hesitant fuzzy sets on the fixed set $X=\{x_1,x_2,\cdots,x_n\}$, $A=\{\langle x_i,h_A(x_i)\rangle \mid x_i\in X, i=1,2,\cdots,n\}$ and $B_j=\left\{\left\langle x_i,h_{B_j}(x_i)\right\rangle \mid x_i\in X, i=1,2,\cdots,n, j=1,2,\cdots,m\right\}$ with $h_A(x_i)=\{\gamma_{Ai1},\gamma_{Ai2},\cdots,\gamma_{Ail_{Ai}}\}$, $h_{B_j}(x_i)=\left\{\gamma_{B_ji1},\gamma_{B_ji2},\cdots,\gamma_{B_jil_{B_ji}}\right\}$, the slope of them are $A'=\left\{\left\langle x_i,h_{A'}(x_i)\right\rangle \mid x_i\in X, i=1,2,\cdots,n\right\}$ with $h_{A'}(x_i)=\left\{\gamma_{A'i1},\gamma_{A'i2},\cdots,\gamma_{A'ik},\cdots,\gamma_{A'il_{Ai}-1}\right\}$ and

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$B_j^{'} = \left\{ \left\langle x_i, h_{B_j^{'}}(x_i) \right\rangle | x_i \in X, i = 1,2,\cdots,n \right\}$ with $h_{B_j^{'}}(x_i) = \left\{ \gamma_{B_j^{'}i1}, \gamma_{B_j^{'}i2}, \cdots, \gamma_{B_j^{'}ik}, \cdots, \gamma_{B_j^{'}il_{Ai}-1} \right\}$, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,m$, then we define the HFEs synthetic grey relational coefficient between HFEs $h_A(x_i)$ and $h_{B_j}(x_i)$ as:

$$r_c(h_A(x_i), h_{B_j}(x_i)) = \frac{1 + \xi \cdot \max_j \max_i \{d(h_A(x_i), h_{B_j}(x_i))\} + \eta \cdot \max_j \max_i \{d(h_{A^{'}}(x_i), h_{B_j^{'}}(x_i))\}}{1 + \lambda_1 \cdot d(h_A(x_i), h_{B_j}(x_i)) + \lambda_2 \cdot d(h_{A^{'}}(x_i), h_{B_j^{'}}(x_i)) + \xi \cdot \max_j \max_i \{d(h_A(x_i), h_{B_j}(x_i))\} + \eta \cdot \max_j \max_i \{d(h_{A^{'}}(x_i), h_{B_j^{'}}(x_i))\}} \quad (22)$$

Where $\lambda_1, \lambda_1 > 0$, which indicate the importance of the closeness and linear fashion of HFSs, respectively, which satisfied $\lambda_1 + \lambda_2 = 1$, ξ and η denote the distinguished coefficient of the closeness and linear fashion, $d(h_A(x_i), h_{B_j}(x_i))$ and $d(h_{A^{'}}(x_i), h_{B_j^{'}}(x_i))$ are the distance between HFEs $h_A(x_i)$ and $h_{B_j}(x_i)$ and the distance between the slope of HFEs $h_A(x_i)$ and $h_{B_j}(x_i)$, respectively, $d(h_{A^{'}}(x_i), h_{B_j^{'}}(x_i))$ can be calculated by

$$d_{hne}(h_{A^{'}}(x_i), h_{B_j^{'}}(x_i)) = \left[\left(\frac{1}{l_{Ai} - 1} \sum_{k=1}^{l_{Ai}-1} \left| \gamma_{A^{'}ik} - \gamma_{B_j^{'}ik} \right|^2 \right) \right]^{1/2} \quad (23)$$

If the numbers of values in different HFEs of HFSs are different, we have to extend the shorter one until both of them have the same length when we compare them. We can extend them according to the optimistic or the pessimistic methods in [7,8].

Based on HFEs synthetic grey relational coefficient, the HFSs synthetic grey relational degree is defined as:

$$\gamma_c(A, B_j) = \frac{1}{n} \cdot \sum_{i=1}^{n} r_c(h_A(x_i), h_{B_j}(x_i)) \quad (24)$$

Take the weight into consideration, let the weight vector of X is $\boldsymbol{w} = \{w_1, w_2, \cdots, w_n\}$, $\sum_{i=1}^{n} w_i = 1$, $i = 1,2,\cdots,n$, we extend the HFSs synthetic grey relational degree to the weighted HFEs synthetic grey relational degree as:

$$\gamma_{cw}(A, B_j) = \sum_{i=1}^{n} w_i \cdot r_c(h_A(x_i), h_{B_j}(x_i)) \quad (25)$$

The HFSs synthetic grey relational degree takes the considerations of both the closeness and linear fashion of HFSs together, which can better represent the fuzzy measure between the HFSs.

3.4. Grey relational definition of IVHFEs and IVHFSs

An interval-valued hesitant fuzzy sets (IVHFSs) $\tilde{A}$ on X is defined as

$$\tilde{A} = \{\left\langle x, \tilde{h}_{\tilde{A}}(x) \right\rangle | x \in X\} \quad (26)$$

Where $\tilde{h}_{\tilde{A}}(x)$ denotes all possible interval-valued membership degrees of the element is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to the set $\tilde{A}$. For convenience, they call $\tilde{h}_{\tilde{A}}(x)$ an interval-valued hesitant fuzzy element (IVHFE), which is a basic unit of IVHFS.

$$\tilde{h}_{\tilde{A}}(x) = \{\tilde{\gamma} \,|\, \tilde{\gamma} \in \tilde{h}_{\tilde{A}}(x)\} \quad (27)$$

Here $\tilde{\gamma}$ is an interval number, $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$, $\tilde{\gamma}^L$ and $\tilde{\gamma}^U$ represent the lower and upper limits of $\tilde{\gamma}$, respectively.

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Definition 9. For two interval-valued hesitant fuzzy sets on the fixed set $X=\{x_1,x_2,\cdots,x_n\}$, $\tilde{A}=\{\langle x,\tilde{h}_{\tilde{A}}(x_i)\rangle | x_i\in X, i=1,2,\cdots,n\}$ and $\tilde{B}_j=\left\{\langle x_i,\tilde{h}_{\tilde{B}_j}(x_i)\rangle | x_i\in X, i=1,2,\cdots,n, j=1,2,\cdots,m\right\}$ with $\tilde{h}_{\tilde{A}}(x_i)=\left\{\tilde{\gamma}_{\tilde{A}i1},\tilde{\gamma}_{\tilde{A}i2},\cdots,\tilde{\gamma}_{\tilde{A}il_{\tilde{A}i}}\right\}$, $\tilde{h}_{B_j}(x_i)=\left\{\tilde{\gamma}_{\tilde{B}_ji1},\tilde{\gamma}_{\tilde{B}_ji2},\cdots,\tilde{\gamma}_{\tilde{B}_jil_{\tilde{B}_ji}}\right\}$, $i=1,2,\cdots,n$, $j=1,2,\cdots,m$, then we define the grey relational coefficient between HFEs $\tilde{h}_{\tilde{A}}(x_i)$ and $\tilde{h}_{B_j}(x_i)$ as:

$$r(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i))=\frac{\min\limits_j\min\limits_i\{d(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i))\}+\rho\cdot\max\limits_j\max\limits_i\{d(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i))\}}{d(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i))+\rho\cdot\max\limits_j\max\limits_i\{d(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i))\}} \tag{28}$$

Where $d(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i))$ is the distance between IVHFEs $\tilde{h}_{\tilde{A}}(x_i)$ and $\tilde{h}_{B_j}(x_i)$, which can be calculated according by

$$d_{hne}(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i))=\left[(\frac{1}{2l_{Ai}}\sum_{k=1}^{l_{Ai}}\left(\left|\tilde{\gamma}^L_{\tilde{A}ik}-\tilde{\gamma}^L_{\tilde{B}_jik}\right|^2+\left|\tilde{\gamma}^U_{\tilde{A}ik}-\tilde{\gamma}^U_{\tilde{B}_jik}\right|^2\right))\right]^{1/2} \tag{29}$$

For more distance between IVHFEs, please refer to Ref. [11, 26-31].

Based on grey relational coefficient between IVHFEs, the grey relational degree between IVHFSs $\tilde{A}$ and $\tilde{B}_j$ is defined as:

$$\gamma(\tilde{A},\tilde{B}_j)=\frac{1}{n}\cdot\sum_{i=1}^{n}r(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i)) \tag{30}$$

Take the weight into consideration, let the weight vector of X is $\boldsymbol{w}=\{w_1,w_2,\cdots,w_n\}$, $\sum_{i=1}^{n}w_i=1$, $i=1,2,\cdots,n$, we extend the IVHFSs grey relational degree to the weighted IVHFSs grey relational degree as:

$$\gamma_w(\tilde{A},\tilde{B}_j)=\sum_{i=1}^{n}w_i\cdot r(\tilde{h}_{\tilde{A}}(x_i),\tilde{h}_{B_j}(x_i)) \tag{31}$$

4. The grey relational based MADM methodology with HFSs and IVHFSs information

In this section, we investigate the grey relational analysis to MADM problems with HFSs and IVHFSs information, and propose the grey relational based MADM methodology with the help of the TOPSIS [50] method.

Suppose that a hesitant fuzzy MADM problem has m alternatives $A_i(i=1,2,\cdots,m)$, each alternative have n hesitant fuzzy attribute $C_j(j=1,2,\cdots,n)$, denote $h_{A_i}(C_j)=\left\{\gamma_{A_i1},\gamma_{A_i2},\cdots,\gamma_{A_ik},\cdots,\gamma_{A_il_{ij}}\right\}$ represent the hesitant fuzzy information of the alternatives A_i on the attribute C_j, l_{ij} is the number of the membership values in $h_{A_i}(C_j)$, let $\boldsymbol{w}=\{w_1,w_2,\cdots,w_n\}$ be the relative weight vector of the attribute, satisfying the normalization conditions: $0\le w_j\le 1$ and $\sum_{j=1}^{n}w_j=1$. Then all the hesitant fuzzy information can be concisely expressed in matrix format as:

$$\boldsymbol{A}=\begin{bmatrix} h_{A_1}(C_1) & h_{A_1}(C_2) & \cdots & h_{A_1}(C_n) \\ h_{A_2}(C_1) & \ddots & \cdots & h_{A_2}(C_n) \\ \vdots & \vdots & h_{A_i}(C_j) & \vdots \\ h_{A_m}(C_1) & h_{A_m}(C_2) & \cdots & h_{A_m}(C_n) \end{bmatrix}_{m\times n} \tag{32}$$

Then according to the TOPSIS approach, we propose the grey relational based MADM methodology with hesitant fuzzy matrix as follows:

Step 1: Determine the positive ideal solution (PIS) and negative ideal solution (NIS) of each attribute in the normalized hesitant fuzzy decision matrix to form two new positive and negative HFSs:

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$$A^{+}=\left\{\left\langle C_j, h_A^{+}(C_j)\right\rangle \mid C_j\in C, j=1,2,\cdots,n\right\} \tag{33}$$

$$A^{-}=\left\{\left\langle C_j, h_A^{-}(C_j)\right\rangle \mid C_j\in C, j=1,2,\cdots,n\right\} \tag{34}$$

Where $h_A^{+}(C_j)$ and $h_A^{-}(C_j)$ are the new positive and negative HFEs:

$$h_A^{+}(C_j)=\{\gamma_1^{+},\gamma_2^{+},\cdots,\gamma_k^{+},\cdots,\gamma_{l_j^{+}}^{+}\},\ \gamma_k^{+}=\left(\max_{1\le i\le m}\{\gamma_{A_ik}\}\text{ if } \gamma_{A_ik}\in\Omega_b, \min_{1\le i\le m}\{\gamma_{A_ik}\}\text{ if } \gamma_{A_ik}\in\Omega_c\right) \tag{35}$$

$$h_A^{-}(C_j)=\{\gamma_1^{-},\gamma_2^{-},\cdots,\gamma_k^{-},\cdots,\gamma_{l_j^{-}}^{-}\},\ \gamma_k^{-}=\left(\min_{1\le i\le m}\{\gamma_{A_ik}\}\text{ if } \gamma_{A_ik}\in\Omega_b, \max_{1\le i\le m}\{\gamma_{A_ik}\}\text{ if } \gamma_{A_ik}\in\Omega_c\right) \tag{36}$$

Where Ω_b and Ω_c are related to benefit attribute and cost attribute, l_j^{+} and l_j^{-} are the number of the membership values in the new positive and negative HFEs respectively, and $l_j^{+}=l_j^{-}$.

Step 2: Calculate the HFSs positive and negative grey relational degree between each alternative and the PIS and NIS, respectively. Here, we can calculate them by the proposed three grey relational degrees, the HFSs grey relational degree, HFSs slope grey relational degree and HFSs synthetic grey relational degree.

$$\gamma_w^{+}(A_i,A^{+})=\sum_{j=1}^{n} w_j\cdot r(h_{A_i}(C_j),h_A^{+}(C_j)) \tag{37}$$

$$\gamma_w^{-}(A_i,A^{-})=\sum_{j=1}^{n} w_j\cdot r(h_{A_i}(C_j),h_A^{-}(C_j)) \tag{38}$$

$$\gamma_{sw}^{+}(A_i,A^{+})=\sum_{j=1}^{n} w_j\cdot r_s(h_{A_i}(C_j),h_A^{+}(C_j)) \tag{39}$$

$$\gamma_{sw}^{-}(A_i,A^{-})=\sum_{j=1}^{n} w_j\cdot r_s(h_{A_i}(C_j),h_A^{-}(C_j)) \tag{40}$$

$$\gamma_{cw}^{+}(A_i,A^{+})=\sum_{j=1}^{n} w_j\cdot r_c(h_{A_i}(C_j),h_A^{+}(C_j)) \tag{41}$$

$$\gamma_{cw}^{-}(A_i,A^{-})=\sum_{j=1}^{n} w_j\cdot r_c(h_{A_i}(C_j),h_A^{-}(C_j)) \tag{42}$$

Where the equations (37-42) can be obtained according to definition 4, 7, 8.

Step 3: Construct the relative closeness to the ideal solution based on the calculated positive and negative grey relational degree. The relative closeness of the alternative $A_i(i=1,2,\cdots,m)$ with respect to the ideal solution are defined as

$$\eta_i=\frac{\gamma_w^{+}(A_i,A^{+})}{\gamma_w^{+}(A_i,A^{+})+\gamma_w^{-}(A_i,A^{-})},\quad i=1,2,\cdots,m \tag{43}$$

$$\eta_{si}=\frac{\gamma_{sw}^{+}(A_i,A^{+})}{\gamma_{sw}^{+}(A_i,A^{+})+\gamma_{sw}^{-}(A_i,A^{-})},\quad i=1,2,\cdots,m \tag{44}$$

$$\eta_{ci}=\frac{\gamma_{cw}^{+}(A_i,A^{+})}{\gamma_{cw}^{+}(A_i,A^{+})+\gamma_{cw}^{-}(A_i,A^{-})},\quad i=1,2,\cdots,m \tag{45}$$

Step 4: Rank the alternatives according to the decreasing order of their relative closeness. That is, the best alternative is the one with the greatest relative closeness to the ideal solution.

Remark: The interval-valued hesitant fuzzy MCDM process is similar to the process of the hesitant fuzzy MCDM, for simplify, we do not repeat it again, it can be deduced easily.

The process of the grey relational based MADM methodology with hesitant fuzzy information and interval-valued hesitant fuzzy information is shown as the following figure 1.

5. MADM applications

In this section, we apply the proposed grey relational based MADM methodology to deal with MADM problems with HFSs and IVHFSs information respectively.

5.1. Apply the proposed grey relational based MADM methodology to energy policy selection

In this section, followed by the MADM example with hesitant fuzzy information concerning energy policy selection in Ref. [7, 10, 12, 18, 51-53], to validate the proposed grey relational based MADM methodology.

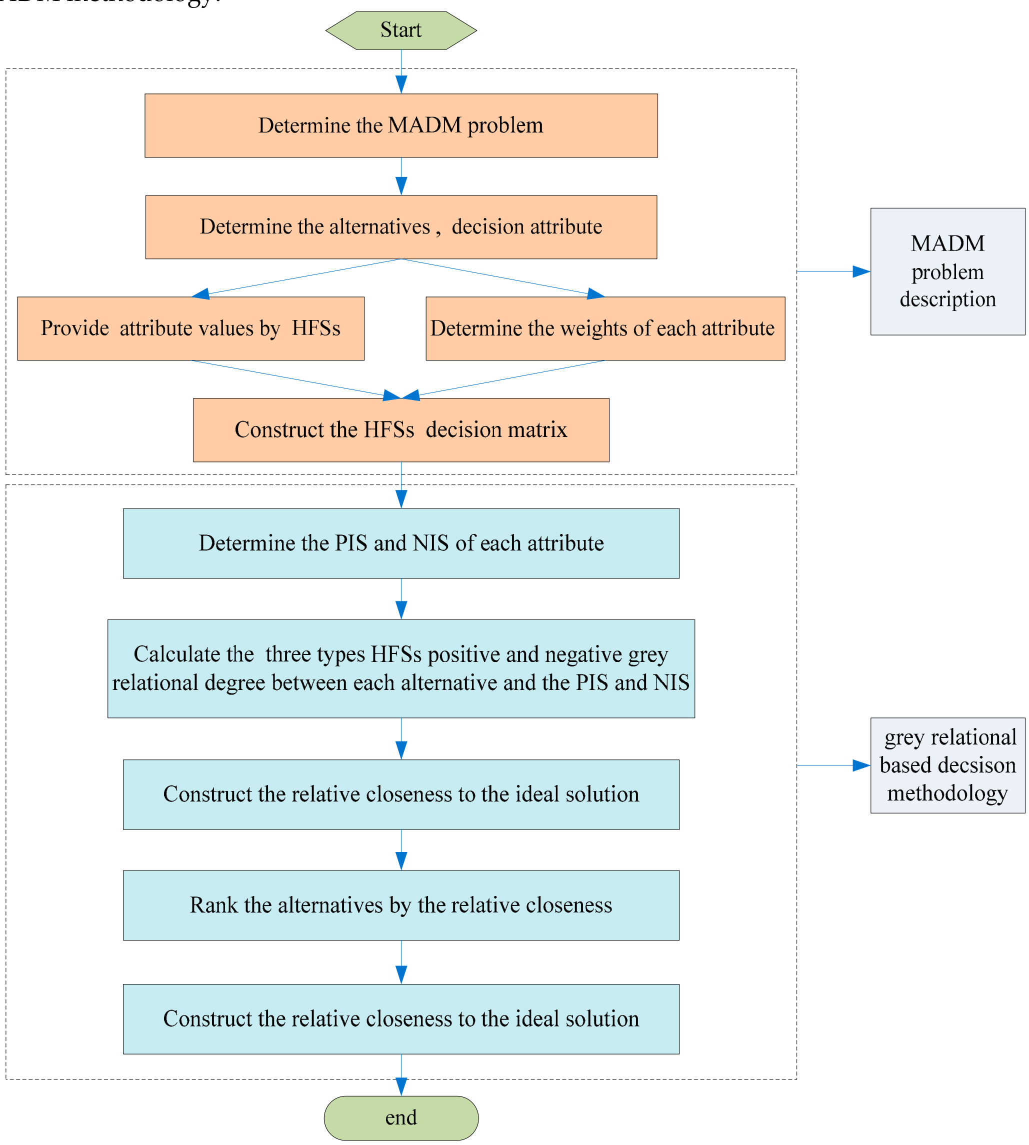

Figure 1. The process of the grey relational based MADM methodology with HFSs and IVHFSs information

In this section, followed by the MADM example with hesitant fuzzy information concerning energy policy selection in Ref. [7, 10, 12, 18, 51-53], to validate the proposed grey relational based MADM methodology.

Example 1 [7, 10, 12, 18, 50-52]. Suppose that there are five alternatives (energy projects) $A_i (i = 1, 2, \cdots, 5)$ to be selected, and four attributes to be considered: $P_j (j = 1, 2, 3, 4)$. The attribute

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weight vector is $w=(0.15,0.3,0.2,0.35)^T$. The attribute values are described in hesitant fuzzy sets, shown in Table 1. What we want to do is get the most desirable alternative from these hesitant fuzzy information.

Table 1. The hesitant fuzzy decision making information for the 4 criteria of 5 alternatives

Alternatives	Attributes			
	Technological	Environmental	Socio-political	Economic
A1	{0.5,0.4,0.3}	{0.9,0.8,0.7,0.1}	{0.5,0.4,0.2}	{0.9,0.6,0.5,0.3}
A2	{0.5,0.3}	{0.9,0.7,0.6,0.5,0.2}	{0.8,0.6,0.5,0.1}	{0.7,0.4,0.3}
A3	{0.7,0.6}	{0.9,0.6}	{0.7,0.5,0.3}	{0.6,0.4}
A4	{0.8,0.7,0.4,0.3}	{0.7,0.4,0.2}	{0.8,0.1}	{0.9,0.8,0.6}
A5	{0.9,0.7,0.6,0.3,0.1}	{0.8,0.7,0.6,0.4}	{0.9,0.8,0.7}	{0.9,0.7,0.6,0.3}

Because the numbers of values in different HFEs of HFSs are different, we can not directly process these hesitant fuzzy information. We have to extend the shorter one until both of them have the same length when we compare them. In this example, we extend these data in the pessimistic view as the reference [18] do. Actually, the extension of the HFSs plays an important role in the ultimate decision making, different extension methods may obtain the different results, so how to select the appropriate extension methods remains to be further research. The extended hesitant fuzzy decision making information are shown in Table 2.

Table 2. The extended hesitant fuzzy decision making information for the 4 criteria of 5 alternatives

Alternatives	Attributes			
	Attribute1	Attribute2	Attribute3	Attribute4
A1	{0.5,0.4,0.3,0.3,0.3}	{0.9,0.8,0.7,0.1,0.1}	{0.5,0.4,0.2,0.2}	{0.9,0.6,0.5,0.3}
A2	{0.5,0.3,0.3,0.3,0.3}	{0.9,0.7,0.6,0.5,0.2}	{0.8,0.6,0.5,0.1}	{0.7,0.4,0.3,0.3}
A3	{0.7,0.6,0.6,0.6,0.6}	{0.9,0.6,0.6,0.6,0.6}	{0.7,0.5,0.3,0.3}	{0.6,0.4,0.4,0.4}
A4	{0.8,0.7,0.4,0.3,0.3}	{0.7,0.4,0.2,0.2,0.2}	{0.8,0.1,0.1,0.1}	{0.9,0.8,0.6,0.6}
A5	{0.9,0.7,0.6,0.3,0.1}	{0.8,0.7,0.6,0.4,0.4}	{0.9,0.8,0.7,0.7}	{0.9,0.7,0.6,0.3}

Step 1: All the attribute are benefit type, we select each maximum HFE in the five alternatives HFSs on the four attribute to construct the hesitant fuzzy PIS A^+ and each minimum HFE to construct the hesitant fuzzy the NIS A^-:

$$A^+=[\{0.9,0.7,0.6,0.6,0.6\},\{0.9,0.8,0.7,0.6,0.6\},\{0.9,0.8,0.7,0.7\},\{0.9,0.8,0.6,0.6\}],$$

$$A^-=[\{0.5,0.3,0.3,0.3,0.1\},\{0.7,0.4,0.2,0.1,0.1\},\{0.5,0.1,0.1,0.1\},\{0.6,0.4,0.3,0.3\}].$$

Step 2: Calculate the HFSs positive and negative three types grey relational degrees between each alternative and the PIS and NIS, respectively. We assume the distinguished coefficient $\rho=0.5$, $\lambda_1=\lambda_2=0.5$, $\xi=\eta=0.5$. The results are shown as Table 3.

Table 3. Positive and negative grey relational degree from the PIS and the NIS

Methods	Relational degrees	Alternatives				
		A1	A2	A3	A4	A5
HFSs grey relational degree	γ_w^+	0.4934	0.4984	0.5922	0.6250	0.6949
	γ_w^-	0.6942	0.7630	0.6867	0.7325	0.5583
HFSs slope grey relational degree	γ_{sw}^+	0.8854	0.8882	0.8901	0.9021	0.9107
	γ_{sw}^-	0.8086	0.8288	0.8265	0.8929	0.8188
HFSs synthetic grey relational degree	γ_{cw}^+	0.8640	0.8676	0.9054	0.8658	0.9251
	γ_{cw}^-	0.8366	0.8605	0.8493	0.8883	0.8160

We also compare the three types positive and negative grey relation degree with each other as shown in Figure 2 and Figure 3.

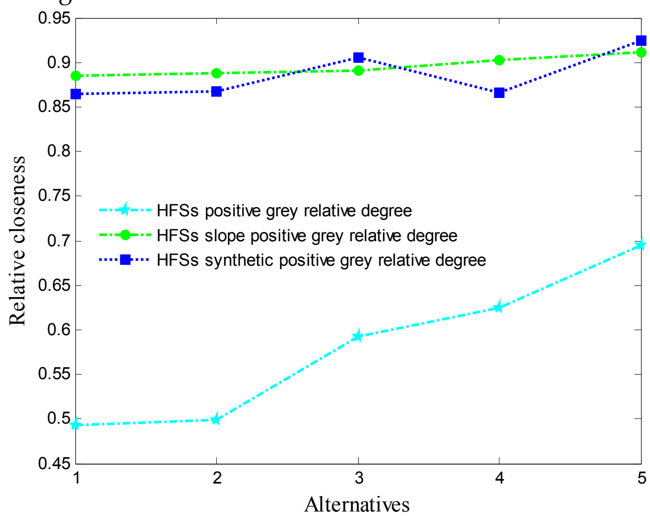

Figure 2. The three types positive grey relational degree

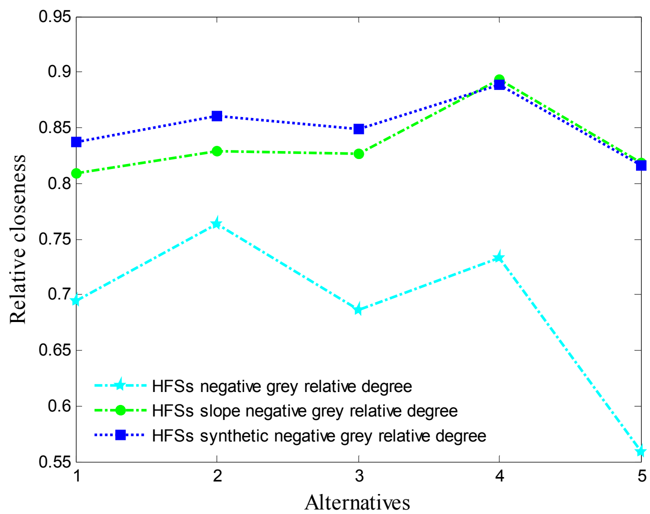

Figure 3. The three types negative grey relational degree

Step 3: Construct the relative closeness to the ideal solution based on the above calculated three types positive and negative grey relational degree. The relative closeness of the alternative

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$A_i (i = 1, 2, \cdots, m)$ with respect to the ideal solution are shown as Table 4 and the compared effect of the three methods is shown in Figure 4.

Table 4. The three types grey relative closeness of the 5 alternatives to the ideal solution

Relative closeness	Alternatives					Rankings
	A1	A2	A3	A4	A5	
HFSs grey relative closeness	0.4155	0.3951	0.4631	0.4604	0.5545	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
HFSs slope grey relative closeness	0.5227	0.5173	0.5185	0.5026	0.5266	$A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$
HFSs synthetic grey relative closeness	0.5081	0.5020	0.5160	0.4936	0.5313	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$

Step 4: Rank the alternatives according to the decreasing order of the three types relative closeness, shown as Table 4 too.

Consequently, we select the best one with the greatest relative closeness to the ideal solution and all the three types relative closeness indicate that the decision result is alternative A_5.

Compared the decision results, we can see that the decision result is consistent with the decision result from reference [7, 10, 12, 18, 52, 53], which derived from different distance and similarity measures. It proves that the three types grey relation methods are effective in the hesitant fuzzy decision problems, successfully apply the grey relation analysis theory into the hesitant fuzzy domain.

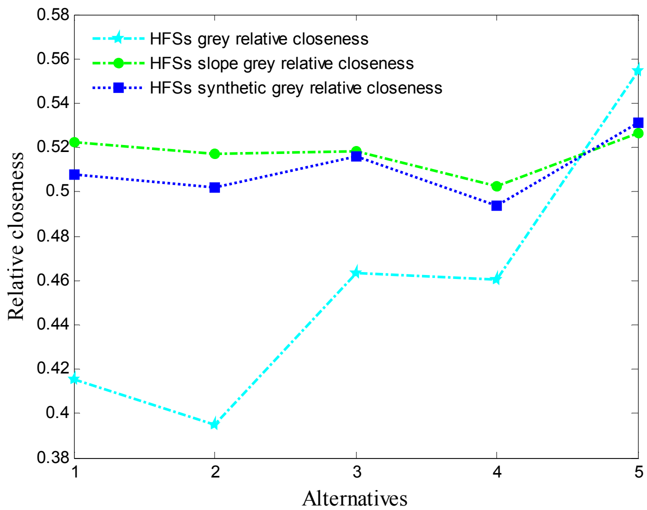

Figure 4. The three types relative closeness

5.2. Apply the proposed grey relational based MADM methodology to emergency management evaluation example

In this section, followed by the MADM example with interval-valued hesitant fuzzy information concerning emergency management evaluation problems, these data are extracted from Reference [34].

Suppose that there are four alternatives $A_i (i = 1, 2, 3, 4)$ to be evaluated by evaluators, each alternative has these six attributes $C_i (i = 1, 2, \cdots, 6)$, and assume the attribute weight is known, let the weight $w = (0.1074, 0.1205, 0.2101, 0.1428, 0.2474, 0.1718)^T$. The evaluated values are expressed by interval-valued hesitant fuzzy information in an interval-valued hesitant fuzzy decision matrix, shown in Table 5.

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Table 5. The Interval-valued hesitant fuzzy attributes information

Attribute	Alternatives			
	A1	**A2**	**A3**	**A4**
Attribute1	{[0.7,0.9],[0.7,0.8],[0.6,0.8]}	{[0.5,0.7],[0.5,0.6],[0.4,0.6]}	{[0.3,0.5],[0.2,0.4],[0.2,0.3]}	{[0.6,0.7],[0.5,0.7],[0.5,0.6]}
Attribute2	{[0.4,0.5],[0.2,0.3],[0.1,0.3]}	{[0.5,0.7],[0.5,0.5],[0.4,0.5]}	{[0.8,0.9],[0.7,0.8],[0.6,0.8]}	{[0.4,0.6],[0.3,0.6],[0.3,0.4]}
Attribute3	{[0.2,0.4],[0.2,0.3],[0.1,0.3]}	{[0.8,1.0],[0.7,0.9],[0.6,0.8]}	{[0.3,0.4],[0.2,0.4],[0.1,0.4]}	{[0.3,0.5],[0.2,0.4],[0.2,0.3]}
Attribute4	{[0.5,0.8],[0.4,0.7],[0.4,0.6]}	{[0.9,1.0],[0.7,0.9],[0.6,0.8]}	{[0.2,0.3],[0.1,0.3],[0.1,0.2]}	{[0.3,0.5],[0.3,0.4],[0.2,0.4]}
Attribute5	{[0.2,0.5],[0.2,0.4],[0.1,0.4]}	{[0.8,0.9],[0.7,0.9],[0.7,0.8]}	{[0.1,0.3],[0.0,0.2],[0.0,0.1]}	{[0.1,0.2],[0.0,0.2],[0.0,0.1]}
Attribute6	{[0.8,0.9],[0.7,0.8],[0.7,0.7]}	{[0.9,1.0],[0.8,1.0],[0.8,0.9]}	{[0.6,0.8],[0.6,0.7],[0.5,0.5]}	{[0.6,0.7],[0.4,0.6],[0.4,0.5]}

We utilize the proposed grey relational based MADM methodology to evaluate the alternatives with IVHFSs information in the following steps:

Step 1: All the attribute are benefit type, we select each maximum IVHFE in the five alternatives IVHFSs on the four attribute to construct the interval-valued hesitant fuzzy PIS A^+ and each minimum IVHFE to construct the interval-valued hesitant fuzzy the NIS A^-:

$$A^+=[\{[0.7,0.9],[0.7,0.8],[0.6,0.8]\},\{[0.8,0.9],[0.7,0.8],[0.6,0.8]\},\{[0.8,1.0],[0.7,0.9],[0.6,0.8]\},\{[0.9,1.0],[0.7,0.9],[0.6,0.8]\},\{[0.8,0.9],[0.7,0.9],[0.7,0.8]\},\{[0.9,1.0],[0.8,1.0],[0.8,0.9]\}],$$

$$A^-=[\{[0.3,0.5],[0.2,0.4],[0.2,0.3]\},\{[0.4,0.5],[0.2,0.3],[0.1,0.3]\},\{[0.2,0.4],[0.2,0.3],[0.1,0.3]\},\{[0.2,0.3],[0.1,0.3],[0.1,0.2]\},\{[0.1,0.2],[0.0,0.2],[0.0,0.1]\},\{[0.6,0.7],[0.4,0.6],[0.4,0.5]\}]$$

Step 2: Calculate the IVHFSs positive and negative grey relational degrees between each alternative and the PIS and NIS, respectively. We assume the distinguished coefficient $\rho=0.5$. The result is shown as Table 6.

Table 6. IVHFSs positive and negative grey relational degree from the PIS and the NIS

Methods	Relational degrees	Alternatives			
		A1	**A2**	**A3**	**A4**
IVHFSs grey relational degree	γ_w^+	0.5456	0.9102	0.4834	0.4498
	γ_w^-	0.7027	0.4360	0.8293	0.8292

Step 3: Construct the relative closeness to the ideal solution based on the calculated IVHFSs positive and negative grey relational degree. The IVHFSs relative closeness of the alternative $A_i (i=1,2,3,4)$ are shown as Table 7.

Table 7. The IVHFSs relative closeness of the 4 alternatives to the ideal solution

Relative closeness	Alternatives				Rankings
	A1	**A2**	**A3**	**A4**	
IVHFSs grey relative closeness	0.4371	0.6761	0.3683	0.3517	$A_2 \succ A_1 \succ A_3 \succ A_4$

Step 4: Rank the alternatives according to the decreasing order of The IVHFSs relative closeness also shown in Table 7.

It can be clearly seen from Table 7 that the decision result is the alternative A_2. It is consistent with the decision result from reference [34], which illustrates the validity and accuracy of the proposed IVHFSs grey relational MADM methodology.

Combined with two practical MADM examples about energy policy selection with HFSs information and emergency management evaluation with IVHFSs information, we can see that the proposed grey relational based MADM methodology can deal with the HFSs and IVHFSs MADM problems well. In addition, the decision results are the same with the previous methods, which

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demonstrates the grey relational based MADM methodology's effectiveness. Furthermore, it is the first time to apply the grey relational analysis theory to the HFSs and IVHFSs field, which greatly enrich the fuzzy measures of HFSs and is significant in the development of the HFSs.

6. Conclusions

In this paper, we apply the grey relational analysis theory to the HFSs and IVHFSs domain and propose three types grey relational degree: the HFSs grey relational degree, HFSs slope grey relational degree and HFSs synthetic grey relational degree, which describe the closeness, the variation tendency and both the closeness and variation tendency of the HFSs, respectively. We also propose the grey relational degree for the IVHFSs. We deduce these grey relational degrees for HFSs and IVHFSs in detail. Additionally, we develop the HFSs grey relational based MADM methodology based on the TOPSIS method to solve the HFSs and IVHFSs MADM problems. Finally, combined with two practical MADM examples about energy policy selection with HFSs information and emergency management evaluation with IVHFSs information, we obtain the appropriate decision results. Compared with the decision results with the previous methods, it illustrates the validity, effectiveness and accuracy of the proposed grey relational based MADM methodology.

In the future, we will apply the proposed grey relational analysis methodology for HFSs and IVHFSs to some other fields as pattern recognition and deep learning. Also, we will attempt to apply the grey analysis theory to the Dual hesitant fuzzy sets and hesitant fuzzy linguistic sets.

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