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Magnetic Quantum Otto Engine for the Single-Particle Landau Problem

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Abstract: We study the effect of the degeneracy factor in the energy levels of the well-known Landau problem for a magnetic quantum Otto engine. The scheme of the cycle is composed of two quantum adiabatic processes and two quantum isomagnetic processes driven by a quasi-static modulation of external magnetic field intensity. We derive the analytical expression of the relation between the magnetic field and temperature along the adiabatic process and, in particular, reproduce the expression for the efficiency as a function of the compression ratio.

Keywords: quantum thermodynamics; degeneracy effects ; magnetic quantum engine.

1. Introduction

Quantum thermodynamics is one of the most interesting topics in physics today. The possibility to create an alternative and efficient nanoscale device, like its macroscopic counterpart, introduces the concept of the quantum engine, which was proposed by Scovil and Schultz-Dubois in the 1950's [1]. The key point here is the quantum nature of the working substance and of course the quantum versions of the laws of thermodynamics [2–18]. The combination of these two simple facts leads to very interesting studies of well-known macroscopic engines of thermodynamics, such as Carnot, Stirling and Otto, among others [2–4]. The maximum efficiency of these engines is governed by the Carnot efficiency, and this cannot be surpassed unless the reservoirs are also of a quantum nature, for example, modeled as quantum coherent states or squeezed thermal states [4–6].

The classical Otto engine consists of two isochoric processes and two adiabatic processes. If the working substance is a classical ideal gas, the first approximation for efficiency depends on the quotient of the temperatures in the first adiabatic compression [19,20]. This expression is reduced with the specific condition along the adiabatic trajectory for this kind of gas, given by $TV^{\gamma-1} = cnt.$, where $\gamma = C_P/C_V$ obtaining the expression $\eta = 1 - \frac{1}{r^{\gamma-1}}$, where r is defined as a "compression ratio" that is defined as V_1/V_2 (with $V_1 > V_2$) [19]. On the other hand, the quantum Otto engine consists of two quantum adiabatic processes, which keeps invariant the probability occupation for the level of energy, and two quantum isochoric processes, in order to keep constant some parameters in the Hamiltonian. In this context, the quantum harmonic Otto cycle is a hot research topic, fully addressed by Kosloff and Rezek [21], and with a recent experimental realization employing a single ion in harmonic trap [22]. In the magnetic scenario, it is useful to think that the "isochoric process" is replaced by "isomagnetic " ones [12,13]. Therefore, the constant value in the process corresponds to the value of cyclotron frequency (or effective frequency depending on the case), which is proportional to the intensity of the magnetic field. This kind of approach is developed in the Ref. [13], for the case of a graphene under strain and the presence of the external magnetic field, exhibiting that the Carnot efficiency is achieved more quickly with the combination of these two effects as opposed to only applying strain to the sample.

The Landau levels of energy in condensed matter physics constitutes a very well-known case and a typical academic problem. The thermodynamics is fully addressed in the works of Kumar et al. [23] and one important point is the degeneracy factor present in the partition function, and consequently also present in the entropy. So, if we consider a thermodynamic cycle, where it is proposed to control the magnetic field along the adiabatic trajectories, it leads to very interesting new results and can be contrasted with the harmonic case. In this same framework, we highlight the work of Mehta and Ramandeep [24], who worked on a quantum Otto engine in the presence of level degeneracy, finding an enhancement of work and efficiency for two-level particles with a degeneracy in the excited state.

This work proposes to study the magnetic Otto cycle for the Landau problem and to understand the role of the degeneracy factor along the cycle. In particular, we found an analytical dependence between the magnetic field and temperature along the adiabatic process, and we use these results to calculate the efficiency of this cycle. We compare this efficiency with that corresponding to the harmonic trap with the same parametrization to see how strong the effect of this factor is on the results.

2. Partition Function for the Single-Particle Landau Problem

We consider the case for an electron with an effective mass m^* and charge e placed in a magnetic field, where the Hamiltonian of this problem working in the symmetric gauge leads to the known expression

$$\hat{H} = \frac{1}{2m^*}\left[\left(p_x - \frac{eBy}{2}\right)^2 + \left(p_y + \frac{exB}{2}\right)\right], \tag{1}$$

and the corresponding Landau levels display the energy spectrum

$$\mathcal{E}_n = \hbar\omega_B\left(n + \frac{1}{2}\right),. \tag{2}$$

Here, $n = 0, 1, 2, \ldots$is the quantum number, and

$$\omega_B = \frac{eB}{m^*} \tag{3}$$

is the standard definition for the cyclotron frequency [12,13,23]. With the definition of the parameter ω_B, we can define the Landau radius that captures the effect of the intensity of the magnetic field, given by $l_B = \sqrt{\hbar/(m^*\omega_B)}$. The energy spectrum for each level is degenerate with a degeneracy $g(B)$ given by [23]

$$g(B) = \frac{eB}{2\pi\hbar}\mathcal{A}, \tag{4}$$

with $\mathcal{A}$ being the area of the box perpendicular to the magnetic field B. So, with this approach it is straightforward to calculate the partition function to Landau problem, and it turns out to be

$$Z = \frac{m^*\omega_B\mathcal{A}}{4\pi\hbar}\text{csch}\left(\frac{\beta\hbar\omega_B}{2}\right), \tag{5}$$

which corresponds to standard partition function for a harmonic oscillator in the canonical ensemble, with a degeneracy for level equal to $g(B)$.

3. Quantum Thermodynamics and Magnetic Quantum Otto Engine

3.1. The First Law of Quantum Thermodynamics

The first law of quantum thermodynamics is fully addressed in many works [2–18] and gives us the possibility to explore different quantum cycles and compare them with the classical analogues. To

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derivate this law simply, consider a Hamiltonian with an explicit dependence of some parameter that we will call μ in a generic form [25]. So, you have a set of eigenvectors of $\hat{H}$ that satisfy the eigenvalue problem

$$\hat{H}|n;\mu\rangle = \mathcal{E}_n|n;\mu\rangle, \tag{6}$$

where n represents a set of indexes that label the spectrum of the Hamiltonian and $|n;\mu\rangle$ constitutes the set of eigenvectors of $\hat{H}$. On the other hand, the density matrix is diagonal in the energy eigenbasis. Therefore, the ensemble-average energy $E = \langle\hat{H}\rangle$ is reduced to

$$E = \sum_n P_n(\mu)\mathcal{E}_n(\mu), \tag{7}$$

for a given occupation distribution with probabilities $P_n(\mu_j)$ in the nth eigenstate.

The statistical ensemble just described can be submitted to an arbitrary quasi-static process, involving the modulation of the parameter μ, and hence the ensemble average energy changes accordingly,

$$\begin{aligned} dE &= \textstyle\sum_n \left(\mathcal{E}_n(\mu)dP_n(\mu) + P_n(\mu)d\mathcal{E}_n(\mu)\right) \\ &= \delta Q + \delta W. \end{aligned} \tag{8}$$

The last equation corresponds to the first law of quantum thermodynamics [2–18,21–26]. The first term in Eq.(8) is associated with the energy exchange, while the second term represents the work done. That is, the work performed corresponds to the change in the eigenenergies $\mathcal{E}_n(\mu)$. It is in agreement with the fact that work can only be carried out through a change in generalized coordinates of the system, which in turn gives rise to a change in the eigenenergies [9,10].

The usual expression for the entropy is given by the von Neumann form in the eigenenergy base as

$$S(\mu) = -k_B \sum_n P_n(\mu) \ln\left[P_n(\mu)\right], \tag{9}$$

where the coefficients $P_n(\mu)$ satisfy that $0 \leq P_n(\mu) \leq 1$ and the normalization condition

$$\sum_n P_n(\mu) = 1. \tag{10}$$

3.2. Magnetic Quantum Otto Engine

As mentioned before, the result of the efficiency of the conventional Otto cycle can be written in the form that the results only depend on the quotient of the temperatures in the first adiabatic compression. By using the properties of the ideal gas, the efficiency can be rewritten as follows:

$$\eta = 1 - \frac{1}{r^{\gamma-1}}, \tag{11}$$

where $\gamma = C_P/C_V$ is the quotient of the two specific heat (at constant pressure and at constant volume) and r is know as "compression ratio" which is defined as $\frac{V_1}{V_2}$ (with $V_1 > V_2$).

On the other hand, the quantum "conventional" Otto engine is composed of two quantum isochoric processes and two quantum adiabatic processes. For the first process mentioned, the occupation probabilities $P_n(\mu)$ change and thus the entropy S changes, until the working substance finally reaches thermal equilibrium with the heat bath. For the case of the quantum adiabatic process, the population distributions remain unchanged, that is $dP_n(\mu) = 0$. Thus, no transition occurs between levels, and no heat is exchanged during this process. It is important to recall that a classical adiabatic process does not necessarily require the occupation probabilities to be kept invariant [26].

This case has been considered in several works for different quantum systems[3,9,13,24,27] where the key findings are an expression for the efficiency present in Eq.(11) and establishing the

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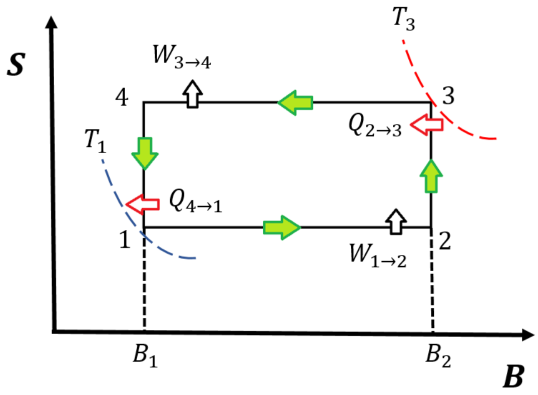

Figure 1. The magnetic quantum Otto engine represented as an entropy (S) versus a magnetic field (B) diagram.

value of γ for that case. For the magnetic case, the two quantum isochoric trajectories are replaced by two "isomagnetic" ones, where the principal finding is the possibility to keep constant the value of the magnetic field intensity along the process while heat is exchanged between the system and the reservoirs [13].

Let us consider a cycle by devising a sequence of quasi-static trajectories, as depicted in Fig. 1. First, the system, while submitted to an external magnetic field B_1, is brought into thermal equilibrium with macroscopic thermostats at temperature T_1. In equilibrium, the probabilities $P_n(\mu)$ take the Boltzmann form and can work with a partition function in the canonical ensemble, $Z(\mu, T)$. So, the Helmholtz free energy can be defined by $F(\mu, T) = -k_B T \ln Z(\mu, B)$ and the entropy given by Eq.(9) can be written as $S(\mu, T) = \frac{E(\mu,T)}{T} + k_B \ln Z(\mu, T)$, where the ensemble average energy is given by

$$E(\mu, T) = k_B T^2 \frac{\partial}{\partial T} \ln Z(\mu, T). \tag{12}$$

Here, the μ parameter is related to the intensity of the magnetic field, so $\mu \to B$ for the case under the study. Then, the system is submitted to a quantum isoentropic process from $1 \to 2$, increasing the magnitude of the magnetic field from B_1 to B_2. The systems perform work along the isoentropic trajectory according to

$$\begin{aligned} W_{1\to2} = \int_{B_1}^{B_2} dB \left(\frac{\partial E}{\partial B}\right)_{P_n(B)=cnt.} &= \int_{B_1}^{B_2} dB \left(\frac{\partial E}{\partial B}\right)_S \\ &= E\left(T_2, B_2\right) - E\left(T_1, B_1\right). \end{aligned} \tag{13}$$

For the case of the "isomagnetic" heating process with the intensity of magnetic field equal to B_2 from $2 \to 3$, no work is done, but heat is absorbed. The heat absorbed ($Q_{2\to3}$) is given by the expression

$$\begin{aligned} Q_{2\to3} &= \int_{T_2}^{T_3} dT \left(\frac{\partial E}{\partial T}\right)_{B_2} \\ &= E\left(T_3, B_2\right) - E\left(T_2, B_2\right). \end{aligned} \tag{14}$$

In the same way discussed before, the isoentropic trajectory from $3 \to 4$, the system performs work in the form

$$W_{3\to4} = E\left(T_4, B_1\right) - E\left(T_3, B_2\right). \tag{15}$$

A physical interpretation of the work performed by the engine is obtained by considering the statistical mechanical definition of the ensemble-average magnetization, $M = -\left(\frac{\partial E}{\partial B}\right)_S$. Hence, the works defined in Eqs. (13) and (15) can also be interpreted as $W = -\int M dB$ [12,13].

Similarly, we obtain the heat released to the low temperature sink in the quantum "isomagnetic" cooling process from $4 \rightarrow 1$

$$\begin{aligned} Q_{4\rightarrow 1} &= \int_{T_4}^{T_1} dT \left(\frac{\partial E}{\partial T}\right)_{B_1} \\ &= E\left(T_1, B_1\right) - E\left(T_4, B_1\right). \end{aligned} \tag{16}$$

The efficiency of the engine is then given by the expression

$$\eta = \left|\frac{W_{1\rightarrow 2} + W_{3\rightarrow 4}}{Q_H}\right| = 1 - \left|\frac{E(T_1, B_1) - E(T_4, B_1)}{E(T_3, B_2) - E(T_2, B_2)}\right|. \tag{17}$$

If we have the analytic function for the entropy, the intermediate temperatures T_2 and T_4 must be determined to reduce the expression for efficiency and can take on two different forms:

- Deducting the relation between the magnetic field and the temperature along an isoentropic trajectory solving the differential equation of first order given by

$$dS(B,T) = \left(\frac{\partial S}{\partial B}\right)_T dB + \left(\frac{\partial S}{\partial T}\right)_B dT = 0, \tag{18}$$

which can be written as

$$\frac{dB}{dT} = -\frac{C_B}{T\left(\frac{\partial S}{\partial B}\right)_T}, \tag{19}$$

where C_B is the specific heat at constant magnetic field.

- The other possibility is connecting the value for the entropy in two isoentropic trajectories in the form

$$\begin{aligned} S(B_1, T_1) &= S(B_2, T_2) \\ S(B_2, T_3) &= S(B_1, T_4), \end{aligned} \tag{20}$$

finding the function for magnetic field in terms of the temperature through numerical calculation. Finally, we parametrize this dependency in the efficiency by defining the ratio

$$r = \frac{l_{B_1}}{l_{B_2}}, \tag{21}$$

which represents the analogy of the compression ratio for the classical case. It is important to remember that the Landau radius is inversely proportional to the magnitude of the magnetic field. Therefore, for a major (minor) magnitude of the field, the Landau radius is smaller (bigger), and the r is well defined.

It is important to highlight the work of Zheng and Poletti [27], where they derived a general form for the efficiency of quantum Otto cycles with power law trapping potentials, corresponding to Eq. (11), and showed that γ must be equal to three. We remark that this result requires that two conditions be met. First, it is only valid for the non-degenerate cases, or more specifically, when the degeneracy is independent of the parameter that rules the cycle. The second condition is that the expansion process, when the system goes to $\omega' \rightarrow \omega''$, must follow the following relation

$$\kappa \equiv \frac{\mathcal{E}_n(\omega')}{\mathcal{E}_n(\omega'')} = \left(\frac{\omega'}{\omega''}\right)^{\alpha}, \tag{22}$$

where α depends on the power of the potentials under study. For example, for a conventional harmonic trap, the spectrum of energy is always $\mathcal{E}_n = \hbar\omega\left(n + \frac{1}{2}\right)$ and you quickly obtain the result previously discussed. Moreover, this value of γ is valid for a family of trapping potentials that fulfills the state equation $PV = 2\langle E\rangle$ [27].

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The case of the Landau problem is different. The energy spectrum that respects the condition of Eq.(22) has the structure of a harmonic trap; however, the degeneracy factor is a function of the magnetic field and the size of the system. Therefore, the results previously discussed do not hold, because in the first ($1 \to 2$) and third ($3 \to 4$) process, the change in the intensity of magnetic field leads to a change in the degeneracy factor, thus this problem must be analyzed carefully.

3.2.1. Magnetic Quantum Otto Engine for the Landau Problem

We show that the representation of the partition function for this case can be taken in the form of Eq.(5). The thermodynamic quantities are present in the work of Kumar *et al.* [23], given by

$$\mathcal{F} = -\frac{1}{\beta} \ln \left[\frac{g(B)}{2} \text{csch} \left(\frac{\beta \hbar \omega_B}{2} \right) \right], \tag{23}$$

$$E = \frac{\hbar \omega_B}{2} \coth \left(\frac{\beta \hbar \omega_B}{2} \right), \tag{24}$$

$$S_L = \frac{\hbar \omega_B}{2T} \coth \left(\frac{\beta \hbar \omega_B}{2} \right) + k_B \ln \left[\frac{g(B)}{2} \text{csch} \left(\frac{\beta \hbar \omega_B}{2} \right) \right], \tag{25}$$

and the specific heat

$$C_B = k_B \beta^2 \left(\frac{\hbar \omega_B}{2} \right)^2 \text{csch}^2 \left(\frac{\beta \hbar \omega_B}{2} \right). \tag{26}$$

First, we highlight that Eq.(23) leads to a natural consequence that the entropy contains the degeneracy terms, due to relation $S = \frac{1}{T}(E - F)$. It is in fact due to the structure of von Neumann entropy, because the probability coefficients contain the information of the degeneracy factor. For example, in thermal equilibrium, this coefficient takes the Boltzmann form, so

$$P_n(\mu) = [Z(\mu, T)]^{-1} g(\mu)\, e^{-\frac{\mathcal{E}_n(\mu)}{k_B T}}. \tag{27}$$

An opposite case occurs for the expected value of energy and the specific heat at constant field because these two physical quantities are obtained as the derivative in the temperature of the partition function.

To clarify the importance of the degeneracy, we analyze the following case. Instead of the term $\frac{g(B)}{2}$ appearing in Eqs.(23) and (25), we put a factor one, corresponding to treat a single oscillator, and we call the entropy for that case just $S(T, B)$. It easy to show that the dependence of the magnetic field on the temperature for the isoentropic trajectories in the non-degenerate scenario obeys the proportionality $B \propto T$. This trivial relation gives us the possibility to obtain the relations between the temperatures along the cycle given by $\frac{T_1}{T_2} = \frac{T_3}{T_4}$, and the efficiency is reduced to a very well-known expression

$$\eta = 1 - \frac{\omega(B_1)}{\omega(B_2)}, \tag{28}$$

which can be rewritten as

$$\eta = 1 - \frac{1}{\left(\frac{l_{B_1}}{l_{B_2}} \right)^2} \equiv 1 - \frac{1}{r^{3-1}}, \tag{29}$$

and we get the result $\gamma = 3$, as described in the work of Zheng and Poletti [27].

4. Results and Discussion

For the Landau case, it is useful to rewrite the term of the degeneracy factor in the entropy as $\frac{g}{2} = \frac{\Phi(B)}{2\Phi_0}$, where $\Phi(B)$ is the magnetic flux quanta and Φ_0 is the universal quantum of magnetic flux, given by $h/2e$. Moreover, we define this degeneracy term in the entropy as $\frac{g}{2} = \lambda B$, where $\lambda = \frac{A}{2\Phi_0}$. Thus, the entropy for this case given by Eq.(25) depends on three variables, $S_L \equiv S_L(T, B, \lambda)$.

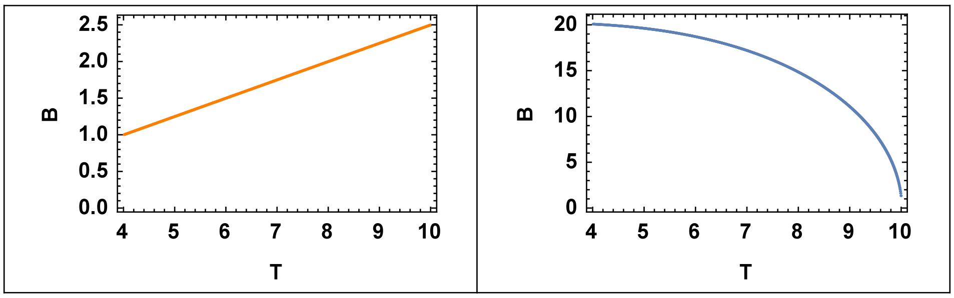

Figure 2. Comparison for two isoentropic trajectories between the case without the degeneracy factor (left frame) and the case with the degeneracy factor $\frac{\Phi(B)}{2\Phi_0}$ (rigth frame). We select the factor $\frac{A}{2\Phi_0} \propto 10^8\ T^{-1}$ for this example.

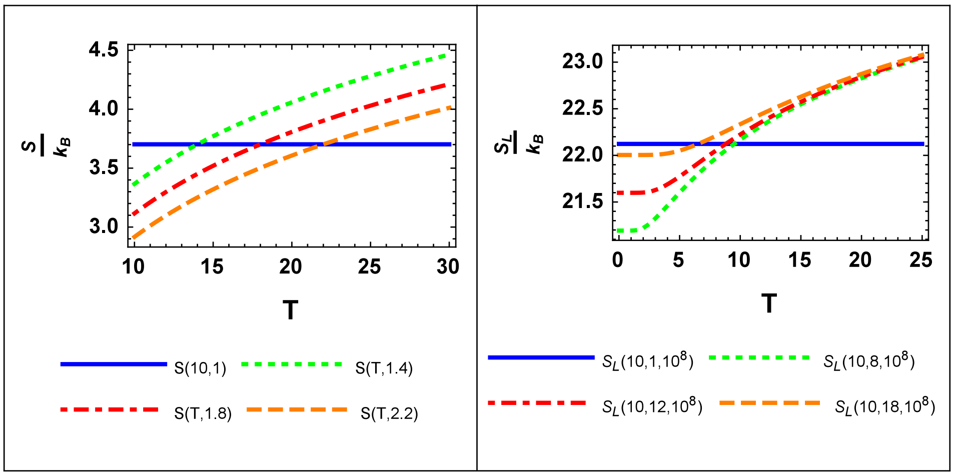

Figure 3. The isoentropic trajectories behavior for the two cases under discussion. In the left frame we plot the non-degenerate case $S(T,B) = S(10,1)$ and in the right frame the degenerate case $S_L(T,B,10^8) = S_L(10,1,10^8)$.

If the dependence of magnetic field and temperature in the adiabatic process for the Landau case is analyzed, we clearly see that the condition for entropy $S_L(T_0,B_0,\lambda) = S_L(T,B,\lambda)$ yields a relation between the magnetic field and temperature which will not depend on λ. This is because the degeneracy term $g(B)$ is associated with a logarithmic term, so the degeneracy effect in the cycle is only reflected in the magnetic field dependence of $g(B)$. To reinforce this idea, we calculate the structure of first order differential equation for the adiabatic processes through Eq.(19), which for this case has the form

$$\frac{dB}{dT} = -\frac{C_1^2 \frac{B^2}{T^3} \mathrm{csch}^2\left(C_1 \frac{B}{T}\right)}{\frac{1}{B} - C_1^2 \frac{B}{T^2} \mathrm{csch}^2\left(C_1 \frac{B}{T}\right)}, \tag{30}$$

where C_1 is a constant given by $C_1 = \frac{e\hbar}{2k_B m}$. This previous equation has the analytical solution (see Appendix A for details) given by

$$C_1 \frac{B}{T} \coth\left(C_1 \frac{B}{T}\right) + \ln\left(C_1 B\right) - \ln\left[\sinh\left(C_1 \frac{B}{T}\right)\right] = \mathcal{C}_2, \tag{31}$$

where $\mathcal{C}_2$ is an integration constant. Note that the additional term in the differential equation which provides $g(B)$, is the factor $(1/B)$ in the dominator of Eq.(30). If this term does not exist, the

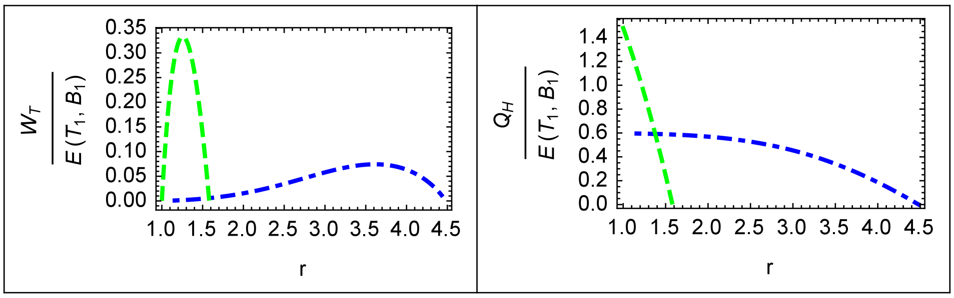

Figure 4. Total work (left frame) and input heat (right frame) versus the r parameter along the cycle for the case with degeneracy (dot dashed line) and without degeneracy (dashed line).

differential equation has a simple form $\frac{dB}{dT} = \frac{B}{T}$ and obtains the result previously discussed for the non-degenerate case.

In Fig. 2, we see the behavior of the magnetic field versus the temperature along an isoentropic trajectory, showing the linear dependence between the magnetic field and the temperature in the case of $g = 1$ (non-degenerate) and for the case of high degeneracy. In order to see the scale of entropy for $S_L(T, B, \lambda)$ for real values, we select $\lambda \propto 10^8\ T^{-1}$, which means an active area of $A \propto 10^{-7} m^2$, by using the fact that the universal flux quantum has an order of $\Phi_0 \propto 10^{-15}\ Wb$. In the left panel of Fig. 2, we plot the solution for the case $S(T, B) = S(10, 1)$, and in the right panel we plot the solution for the case $S(T, B, 10^8) = S(10, 1, 10^8)$. The contrast is evident, in the simple scenario for an increase in the magnetic field, and we obtain an increase in the temperature. However, for the case with degeneracy, the rise in the magnetic field leads to a decrease in the temperature. The explanation of this fact lies in the behavior of the entropy at low temperatures, because of $S(T, B, \lambda)_{T\to 0} \sim k_B \ln(g)$, where g is directly proportional to B. This is discussed in Fig. 3 where we show the entropy behaviors in these two different scenarios. In the non-degenerate case, when we increase the magnetic field, the function $S(T, B)$ intersects the starting value of the entropy always in a higher value than the initial one, reflected in the left frame of Fig. 3. It explains the linearity that we obtain in a plot B vs T for the left panel of Fig. 2. The opposite occurs for the degenerate case, the function $S_L(T, B, 10^8)$, which intersects the starting value of the entropy always in a lower value than the initial one, as we see in the right frame of Fig. 3. From this same figure, we can conclude that the entropy function for the degenerate case collapses to approximately the same value for higher temperature for different values of the magnetic field. Thus, if we consider changes of the external magnetic field as a result of the parameter that rules the operation of the engine, we have a region of the temperature and magnetic field where it is valid to raise this cycle.

As discussed in Appendix A, with an adequate analysis of asymptotic behaviours of Eq.(30), we found a critical temperature, given by

$$T_c = e^{(\mathcal{C}_2 - 1)}, \tag{32}$$

which corresponds to the value of the temperature when the magnetic field goes to zero and a critical value for the magnetic field when it starts to become constant, given by the expression

$$B_c = \frac{e^{\mathcal{C}_2}}{2C_1}. \tag{33}$$

Therefore, we have the two points for an initial value constant $\mathcal{C}_2$ where it makes sense to carry out the cycle. For the exponential form of the Eq.(33), the critical value of the constant magnetic field is always a large quantity. For a real example of a starting field and temperature, we can consider the example of Fig. 2, where the initial values of the intensity field and temperature are 1 T and a 10 K, respectively. The approximate value for the critical values are $T_c \approx 10.1\ K$ and $B_c \approx 20.1\ T$.

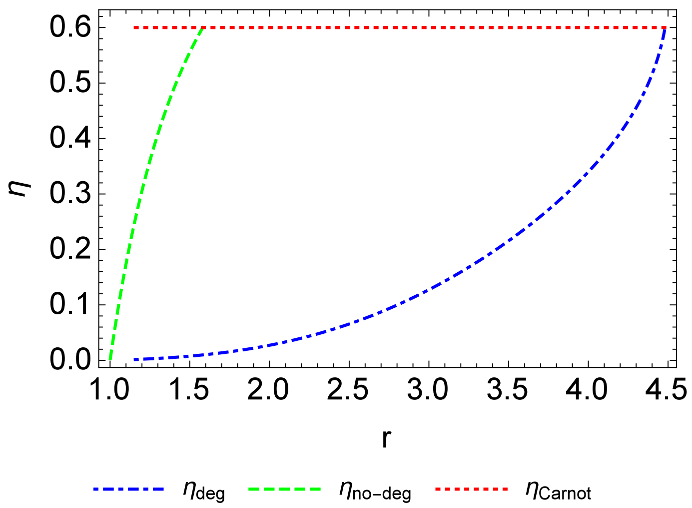

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Figure 5. Efficiency for different cases of interest. For this case, the dotted red line corresponds to the value of Carnot cycle for a machine operating between the two temperatures $T_1 = 4K$ and $T_3 = 10K$

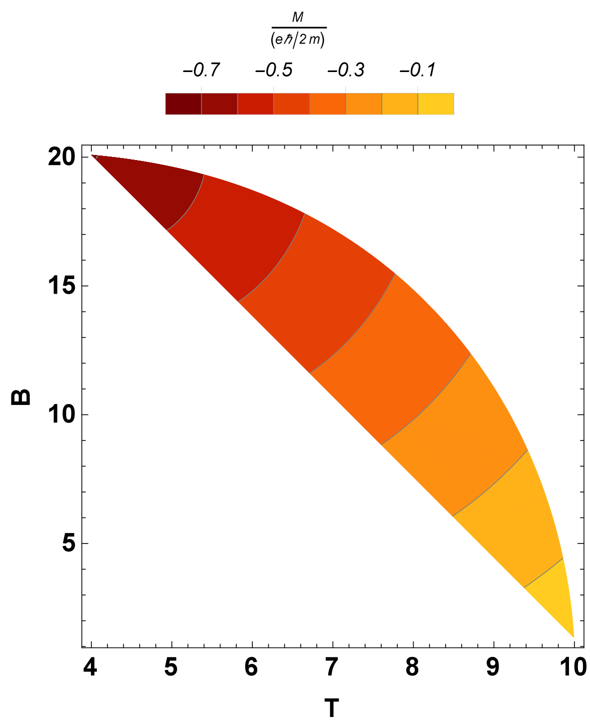

Figure 6. Magnetization as a function of B and T along the adiabatic trajectories.

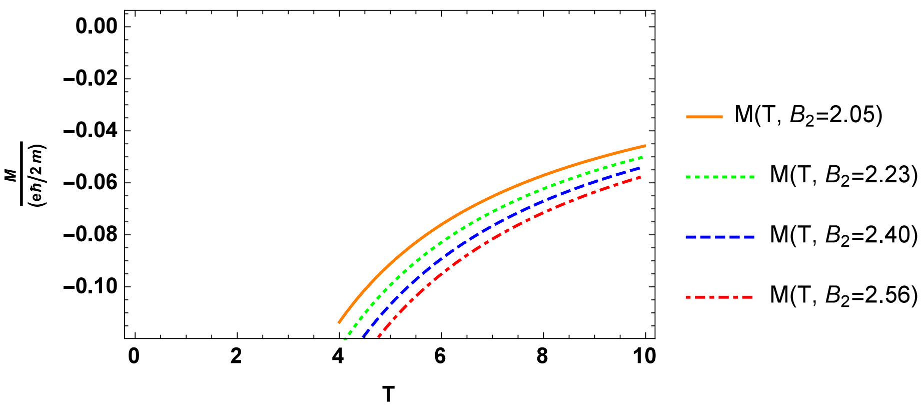

Figure 7. Magnetization along the first iso-magnetic trajectory as a function of T in the range of 4 K to 10 K. We select the different values for B_2 that we obtain from numerical calculations.

For the starting point previously indicated, we can consider a cycle for the degenerate case like that in Fig. 1 operating between the temperatures 4 K and 10 K. However, due to the behavior of temperature along the adiabatic trajectory described in Fig. 2, initially we brought the system into thermal equilibrium at $T_1 > T_3$. Thus, for that case, the heat defined by $Q_{4\rightarrow 1}$ corresponds to the heat absorbed, and for the heat released the correct definition is given by $Q_{2\rightarrow 3}$, contrary to the non-degenerate case. To reinforce this idea, we display in the right frame of Fig. 4 the behavior of heat along the cycle for the degenerate case and non-degenerate case. The convention of the sign (positive for heat absorbed) is satisfied along the entire operation of the engine. The positive work condition, which plays an essential role for a wheel defined thermal engine, is shown in the left frame of Fig. 4 for both cases. From the same figure, we can extract relevant information about the r parameter. For a machine operating between two reservoirs of 4 K and 10 K, we obtain

$$r^{max}_{no-deg} = 1.58 \quad \text{and} \quad r^{max}_{deg} = 4.47, \tag{34}$$

which represents the maximum value that can be taken for the compression ratio along the cycle and corresponds to the point where the Carnot efficiency is obtained. These results are natural only to see the Fig. (3) for this example. For the non-degenerate case, it is only necessary to increase the field by a factor of 2.5, but for the degenerate case it is necessary to increase the field by a factor of 20 to reach the Carnot efficiency of the problem.

In Fig. 5 we compare the three efficiencies, where we see the effect of the degeneracy. One form to understand this behavior corresponds to the approximation that we show in Appendix A for the parametric solution, with the finality to "uncouple" the solution to magnetic field and the temperature for the adiabatic trajectory getting a function in the form

$$B(T) = \frac{k}{2C_1}\left(1 - \mathrm{e}^{-k\sqrt{\frac{0.64}{T^2}\left(1-\mathrm{e}^{\frac{T}{k}}\right)}}\right), \tag{35}$$

where we define $k = \mathrm{e}^{C_2}$. So for this exponential form for the field as a function of temperature, when we parametrize the efficiency vs. a function of the typical compression ratio (r), we obtain the behavior present in Fig. 5.

For the definition of works and its interpretation as $W = -\int MdB$, we study the magnetization along the cycle defined as

$$\mathcal{M} = \frac{e\hbar}{2m}\left(\frac{2}{\beta\hbar\omega_B} - \coth\left(\frac{\beta\hbar\omega_B}{2}\right)\right). \tag{36}$$

For the adiabatic trajectory, the temperature and the magnetic field change along the entire process, so we can use a contour plot to see the value of magnetization displayed in Fig. 6. Here, we clearly see

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that the values of magnetization are always negative and the same occurs for the different curves in the "isomagnetic" process, shown in Fig. 7, indicating that the response of the system is diamagnetic.

Our system was studied to prove a concept rather than a practical implementation protocol. However, we believe the readers will find attractive the study of the optimization of this cycle in degenerate conditions, following the work of Kosloff and Rezek [21] for the case of frictionless adiabats using the methods of shortcuts to adiabaticity [28].

5. Conclusions

In this work, we explored the possibility of constructing a single-particle quantum engine of the Landau problem. In particular, we found an analytical solution for the dependence of the magnetic field and temperature in the adiabatic trajectories. We use this relation to obtain the form of the efficiency showing a radically different behavior of the typical harmonic case and find that a major increase in the external magnetic field to reach the Carnot efficiency is necessary. We remark that the useful work of this engine, related to change in the magnetization along the process, can be used for example in the generation of induction current in other physics systems.

It is important to note that our one-particle approach must be refined to take into account a many electrons scenario, which yields more precise calculations. However, the one electron case is important due to simplicity and enrichment of physics for comparatives cases. For instance, we can work with a one-particle system combining the effects of a cylindrical potential well, which physically represents an accurate model for a semiconductor quantum dot, and an externally imposed magnetic field, where the number of electrons can be controlled without problems; thus, the same analysis presented in this work can be replicated.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

We need to solve the differential equation in the form

$$\frac{dB}{dT} = -\frac{C_1^2 \frac{B^2}{T^3} \mathrm{csch}^2\left(C_1 \frac{B}{T}\right)}{\frac{1}{B} - C_1^2 \frac{B}{T^2} \mathrm{csch}^2\left(C_1 \frac{B}{T}\right)}. \tag{A1}$$

We define the parameter $u = C_1 \left(\frac{B}{T}\right)$. So, differentiating respect to T, we obtain

$$\frac{du}{dT} = -C_1 \frac{B}{T^2} + \frac{C_1}{T} \frac{dB}{dT}. \tag{A2}$$

Collecting these two last equations, we obtain the first order differential equation in the u parameter in the form

$$\frac{du}{dT} = \frac{u}{T\left(u^2 \mathrm{csch}^2(u) - 1\right)}, \tag{A3}$$

which corresponds to a differential equation of separable variables that has a solution given by

$$u \coth(u) + \ln(u) - \ln\left[\sinh(u)\right] + \ln(T) = \mathcal{C}_2, \tag{A4}$$

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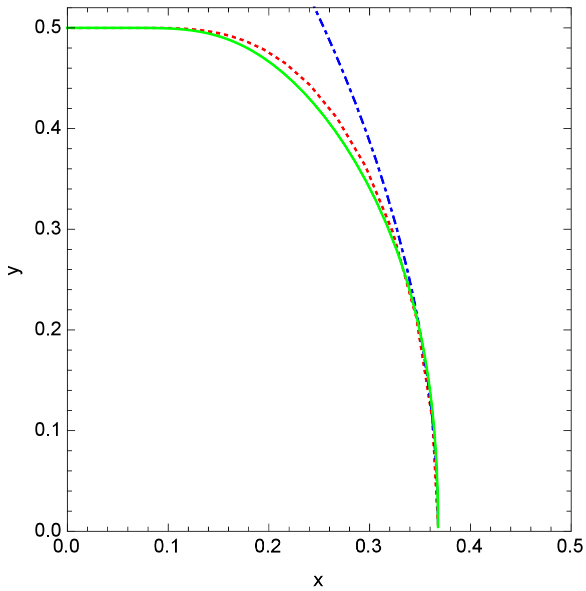

Figure A1. A parametric solution of the differential equation along the adiabatic trajectories for the Landau case. The dotted line represents the exact solution and the dot-dashed line the asymptotic case for $u << 1$. We can clearly see the constant value 0.5 for the solution in the case of $u >> 1$ from the dotted line in the figure. The solid line represents the proposal curve given by the Eq.(A13) showing a good fit for the problem under study.

where $\mathcal{C}_2$ is a constant of integration. We can compact this solution if we define the two variables

$$y = \frac{C_1 B}{k} \qquad x = \frac{T}{k}, \tag{A5}$$

where k is given by

$$k = \mathrm{e}^{\mathcal{C}_2}, \tag{A6}$$

with e as the Euler number. So, the solution takes the parametric form

$$y = \mathrm{e}^{-u \coth u} \sinh u \qquad x = \frac{1}{u}\mathrm{e}^{-u \coth u} \sinh u. \tag{A7}$$

The asymptotic behaviors of these solutions is very interesting. The expression in the case of $u << 1$, which corresponds to high-temperature or small magnetic field limit, takes the form

$$y = \frac{1}{\mathrm{e}}\sqrt{6\,(1 - \mathrm{e}x)}, \tag{A8}$$

so, we have a critical value, x_c, when $y \to 0$ given by

$$x_c = \frac{1}{\mathrm{e}}. \tag{A9}$$

It gives us a critical temperature T_c when the magnetic field goes to zero, and is strongly dependent on initial values of the problem under study, given by

$$T_c = \frac{k}{\mathrm{e}} \equiv \mathrm{e}^{(\mathcal{C}_2 - 1)}. \tag{A10}$$

In the other case, for $u >> 1$, which corresponds to low temperature or high magnetic field limit, we obtain

$$y_c = \frac{1}{2}, \tag{A11}$$

and, this represents a critical constant value for the magnetic field, given by

$$B_c = \frac{k}{2C_1} \equiv \frac{e^{C_2}}{2C_1}, \tag{A12}$$

Thus, it is important to keep in mind that, when we consider a variation of the magnetic field as the cause for effective work in the system, the limits discussed before impose physical variable restrictions to operate the quantum machine proposed in the text.

To understand the magnetic field behavior in an explicit form along the adiabatic process, we propose an approximated curve in the form

$$y = \frac{1}{2}\left(1 - e^{\frac{-\sqrt{0.64(1-ex)}}{x}}\right). \tag{A13}$$

The exact parametric solution, the asymptotic behavior for the limiting cases ($u >> 1$ and $u << 1$) and our proposal function are displayed in Fig.(A1).

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