SOME PROPERTIES OF THE FUSS–CATALAN NUMBERS

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ABSTRACT. In the paper, the authors express the Fuss–Catalan numbers as several forms in terms of the Catalan–Qi function, find some analytic properties, including the monotonicity, logarithmic convexity, complete monotonicity, and minimality, of the Fuss–Catalan numbers, and derive a double inequality for bounding the Fuss–Catalan numbers.

1. INTRODUCTION AND MAIN RESULTS

It is stated in [18] that the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n$$
$$= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + \cdots.$$

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Two explicit formulas for C_n with $n\ge0$ read that

$$C_n=\frac{(2n)!}{n!(n+1)!}=\frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \tag{1.1}$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

is the classical Euler gamma function. In [13, 18, 53], it was mentioned that there exists an asymptotic expansion

$$C_x\sim\frac{4^x}{\sqrt{\pi}}\biggl(\frac{1}{x^{3/2}}-\frac98\frac{1}{x^{5/2}}+\frac{145}{128}\frac{1}{x^{7/2}}+\dotsm\biggr) \tag{1.2}$$

for the Catalan function

$$C_x=\frac{4^x\Gamma(x+1/2)}{\sqrt{\pi}\,\Gamma(x+2)},\quad x\ge0. \tag{1.3}$$

In the newly published papers [32, 34, 36, 48], there are some new results on the Catalan numbers C_n and others.

In [47], an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced as

$$C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\biggl(\frac{b}{a}\biggr)^z\frac{\Gamma(z+a)}{\Gamma(z+b)},\quad \Re(a),\Re(b)>0,\quad \Re(z)\ge0. \tag{1.4}$$

For uniqueness and convenience of referring to the quantity $C(a,b;x)$, we call $C(a,b;x)$ the Catalan–Qi function and, when taking $x=n\ge0$, call $C(a,b;n)$ the Catalan–Qi numbers. Comparing with the second formula in (1.3), it is clear that

$$C\biggl(\frac12,2;x\biggr)=C_x,\quad x\ge0. \tag{1.5}$$

By the definition (1.4), we easily see that

$$C(a,b;x)C(b,c;x)=C(a,c;x),\quad a,b,c>0,\quad x\ge0.$$

In the recent papers [21, 23, 37, 38, 39, 44, 45, 46, 47, 50], among other things, some analytic properties, including the general expression and a generalization of the asymptotic expansion (1.2), the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities, of the Catalan numbers C_n, the Catalan function C_x, and the Catalan–Qi function $C(a,b;x)$ were established.

In combinatorial mathematics and statistics, the Fuss–Catalan numbers $A_n(p,r)$ are defined in [11] as numbers of the form

$$A_n(p,r)=\frac{r}{np+r}\binom{np+r}{n}=r\frac{\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}. \tag{1.6}$$

Comparing with the first formula in (1.3), it is obvious that

$$A_n(2,1)=C_n,\quad n\ge0. \tag{1.7}$$

A generalization of the Catalan numbers C_n was defined in [16, 17, 24] by

$${}_pd_n=\frac1n\binom{pn}{n-1}=\frac{1}{(p-1)n+1}\binom{pn}{n}$$

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for $n \ge 1$. The usual Catalan numbers $C_n = {}_2d_n$ are a special case with $p = 2$. It is immediate that

$$A_{n-1}(p,p) = {}_pd_n$$

for $n \ge 1$. There exists some literature such as [2, 5, 7, 12, 20, 22, 26, 27, 28, 51, 52] devoted to the investigation of the Fuss–Catalan numbers $A_n(p,r)$.

Considering the relations (1.5) and (1.7), one may ask a question: what is the relation between the Catalan–Qi numbers $C(a,b;n)$ and the Fuss–Catalan numbers $A_n(p,r)$? This question is answered by Theorem 1.1 below.

Theorem 1.1. *For $n, r \ge 0$ and $p > 1$, we have*

$$A_n(p,r) = r^n \frac{\prod_{k=1}^{p} C\left(\frac{k+r-1}{p}, 1; n\right)}{\prod_{k=1}^{p-1} C\left(\frac{k+r}{p-1}, 1; n\right)}. \tag{1.8}$$

For $n, p \in \mathbb{N}$ and $r \ge 0$, we have

$$A_n(p,r) = \frac{r}{nB(n(p-1)+1,n)} \frac{(np)^{r-1}}{[n(p-1)+1]^r} C(np, n(p-1)+1; r), \tag{1.9}$$

where $B(x,y)$ denotes the classical beta function.

For $r+1 > n > 0$ and $p \ge 0$, we have

$$A_n(p,r) = \frac{1}{nB(n, r-n+1)} \left(\frac{r}{r-n+1}\right)^{np} C(r, r-n+1; np). \tag{1.10}$$

When $r+1 > n \ge 1$ and $p \ge 0$, we have

$$A_n(p,r) = \frac{1}{n} \frac{[B(r+1-n,n)]^{p-1}}{[B(r,n)]^p} \prod_{k=0}^{n-1} C\left(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\right). \tag{1.11}$$

For $n \ge 2$, $r+1 > n$, and $p \in \mathbb{N}$, we have

$$\begin{aligned} A_n(p,r) = rp^{1/2} B(n-1,2) \frac{[B(r+1-n,n-1)]^{n-1}}{[B(r+1-n+p,n-1)]^n} \\ \times \prod_{k=0}^{p-1} C\left(\frac{r+k}{p}, \frac{r+k+1-n}{p}; n\right). \end{aligned} \tag{1.12}$$

Recall from [25, pp. 372–373] and [54, p. 108, Definition 4] that a sequence $\{\mu_n\}_{0 \le n \le \infty}$ is said to be completely monotonic if its elements are non-negative and its successive differences alternate sign, that is,

$$(-1)^k \Delta^k \mu_n \ge 0$$

for $n, k \ge 0$, where

$$\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m}.$$

Further recall from [54, p. 163, Definition 14a] that a completely monotonic sequence $\{a_n\}_{n \ge 0}$ is minimal if it ceases to be completely monotonic when a_0 is decreased.

Applying the identity (1.8) and several analytic properties of the Catalan–Qi function $C(a,b;x)$, we find several analytic properties, including monotonicity, logarithmic convexity, complete monotonicity, and minimality, of the Fuss–Catalan numbers $A_n(p,r)$.

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Theorem 1.2. *When $p \ge r > 0$,*

(1) *the sequence $\{\mathcal{A}_n(p,r)\}_{n\ge0}$,*

$$\mathcal{A}_n(p,r) = \begin{cases} 1, & n = 0, \\ \dfrac{1}{\sqrt[n]{A_n(p,r)}}, & n \in \mathbb{N}, \end{cases} \tag{1.13}$$

is logarithmically convex, completely monotonic, and minimal;

(2) *the sequence of the Fuss–Catalan numbers $\{A_n(p,r)\}_{n\ge0}$ is increasing and logarithmically convex.*

Finally, by applying a double inequality of the beta function in the papers [3] and [9, pp. 78–81, Section 3], we derive a double inequality for the Fuss–Catalan numbers $A_n(p,r)$.

Theorem 1.3. *For $n \ge 2$ and $p, r \in \mathbb{N}$, we have*

$$r\left(p+\frac{r}{n}\right) \le A_n(p,r) \le r\left(p+\frac{r}{n}\right)\frac{1}{1-\min\{D(n-1)D(np+r), b_A(n-1)(np+r)\}}, \tag{1.14}$$

where

$$D(x) = \frac{x-1}{\sqrt{2x-1}} \quad \text{and} \quad b_A = \max_{x\ge1}\left[\frac{1}{x^2} - \frac{\Gamma^2(x)}{\Gamma(2x)}\right] = 0.08731\dots. \tag{1.15}$$

2. Lemmas

In order to prove Theorems 1.2 and 1.3, we need the following notion and lemmas.

Recall from [4, 35, 49] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ holds on I for all $k \in \mathbb{N}$.

Lemma 2.1 ([44, Theorem 6])**.** *The function*

$$\mathcal{C}^{\pm1}(a,b;x) = \begin{cases} 1, & x = 0 \\ [C(a,b;x)]^{\pm1/x}, & x > 0 \end{cases}$$

is logarithmically completely monotonic on $[0,\infty)$ if and only if $a \gtreqless b$.

Lemma 2.2 ([44, Theorem 7])**.** *Let $a, b > 0$ and $x \ge 0$. Then*

(1) *the unique zero x_0 of the equation*

$$\frac{\psi(x+b)-\psi(x+a)}{\ln b - \ln a} = 1$$

satisfies $x_0 \in \left(0, \frac{1}{2}\right)$, where ψ is the logarithmic derivative of the gamma function Γ;

(2) *when $b > a$, the function $C(a,b;x)$ is decreasing in $x \in [0,x_0)$, increasing in $x \in (x_0,\infty)$, and logarithmically convex in $x \in [0,\infty)$;*

(3) *when $b < a$, the function $C(a,b;x)$ is increasing in $x \in [0,x_0)$, decreasing in $x \in (x_0,\infty)$, and logarithmically concave in $x \in [0,\infty)$.*

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Lemma 2.3 ([3] and [9, pp. 78–81, Section 3])**.** *For $x, y > 1$, we have*

$$0 \le \frac{1}{xy} - B(x,y) \le \min\left\{\frac{D(x)D(y)}{xy}, b_A\right\}, \tag{2.1}$$

where $D(x)$ and b_A are defined as in (1.15).

3. Proofs of Theorems 1.1 to 1.3

We now start out to prove our theorems.

Proof of Theorem 1.1. By virtue of the Gauss multiplication formula

$$\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) \tag{3.1}$$

in [1, p. 256, 6.1.20], the Fuss–Catalan numbers $A_n(p,r)$ defined by the second expression in (1.6) can also be rewritten as

$$\begin{aligned}
A_n(p,r) &= \frac{r\Gamma(p(n+r/p))}{\Gamma(n+1)\Gamma((p-1)(n+(r+1)/(p-1)))} \\
&= \frac{r\dfrac{p^{np+r-1/2}}{(2\pi)^{(p-1)/2}}\displaystyle\prod_{k=0}^{p-1}\Gamma\left(n+\frac{k+r}{p}\right)}{\Gamma(n+1)\dfrac{(p-1)^{n(p-1)+r+1/2}}{(2\pi)^{(p-2)/2}}\displaystyle\prod_{k=0}^{p-2}\Gamma\left(n+\frac{k+r+1}{p-1}\right)} \\
&= \frac{1}{\sqrt{2\pi}}\frac{rp^{np+r-1/2}}{(p-1)^{n(p-1)+r+1/2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(n+\frac{k+r}{p}\right)}{\Gamma(n+1)\prod_{k=0}^{p-2}\Gamma\left(n+\frac{k+r+1}{p-1}\right)} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{r}{[p(p-1)]^{1/2}}\left[\frac{p^p}{(p-1)^{p-1}}\right]^n\frac{\Gamma\left(n+\frac{r}{p}\right)\prod_{k=1}^{p-1}\Gamma\left(n+\frac{k+r}{p}\right)}{\Gamma(n+1)\prod_{k=1}^{p-1}\Gamma\left(n+\frac{k+r}{p-1}\right)} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{r}{[p(p-1)]^{1/2}}\left[\frac{p^p}{(p-1)^{p-1}}\right]^n\frac{\Gamma\left(n+\frac{r}{p}\right)}{\Gamma(n+1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(n+\frac{k+r}{p}\right)}{\Gamma\left(n+\frac{k+r}{p-1}\right)} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{r}{[p(p-1)]^{1/2}}\left[\frac{p^p}{(p-1)^{p-1}}\right]^n \\
&\quad\times\frac{\Gamma(1)}{\Gamma\left(\frac{r}{p}\right)}\left(\frac{1}{r/p}\right)^n\frac{\Gamma\left(n+\frac{r}{p}\right)}{\Gamma(n+1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p-1}\right)}{\Gamma\left(\frac{k+r}{p}\right)}\left(\frac{\frac{k+r}{p-1}}{\frac{k+r}{p}}\right)^n\frac{\Gamma\left(n+\frac{k+r}{p}\right)}{\Gamma\left(n+\frac{k+r}{p-1}\right)} \\
&\quad\times\frac{\Gamma\left(\frac{r}{p}\right)}{\Gamma(1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p}\right)}{\Gamma\left(\frac{k+r}{p-1}\right)}\times\left(\frac{r/p}{1}\right)^n\prod_{k=1}^{p-1}\left(\frac{\frac{k+r}{p}}{\frac{k+r}{p-1}}\right)^n \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{r}{[p(p-1)]^{1/2}}\left[\frac{p^p}{(p-1)^{p-1}}\right]^n
\end{aligned}$$

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$$
\begin{aligned}
&\times C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right)\\
&\times\frac{\Gamma\left(\frac{r}{p}\right)}{\Gamma(1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p}\right)}{\Gamma\left(\frac{k+r}{p-1}\right)}\times\left(\frac{r}{p}\right)^n\left(\frac{p-1}{p}\right)^{n(p-1)}\\
={}&\frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{r}{[p(p-1)]^{1/2}}\left[\frac{p^p}{(p-1)^{p-1}}\right]^n\left(\frac{r}{p}\right)^n\left(\frac{p-1}{p}\right)^{n(p-1)}\\
&\times\frac{\Gamma\left(\frac{r}{p}\right)}{\Gamma(1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p}\right)}{\Gamma\left(\frac{k+r}{p-1}\right)}\times C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right)\\
={}&\frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{r^{n+1}}{[p(p-1)]^{1/2}}\frac{\Gamma\left(\frac{r}{p}\right)}{\Gamma(1)}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p}\right)}{\Gamma\left(\frac{k+r}{p-1}\right)}\\
&\times C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right)\\
={}&\frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{\Gamma\left(\frac{r}{p}\right)}{[p(p-1)]^{1/2}}\prod_{k=1}^{p-1}\frac{\Gamma\left(\frac{k+r}{p}\right)}{\Gamma\left(\frac{k+r}{p-1}\right)}\\
&\times r^{n+1}C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right)\\
={}&\frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{1}{[p(p-1)]^{1/2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{r}{p}+\frac{k}{p}\right)}{\prod_{k=0}^{p-2}\Gamma\left(\frac{r+1}{p-1}+\frac{k}{p-1}\right)}\\
&\times r^{n+1}C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right)\\
={}&\frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^r\frac{1}{[p(p-1)]^{1/2}}\frac{\Gamma(r)(2\pi)^{(p-1)/2}/p^{r-1/2}}{\Gamma(r+1)(2\pi)^{(p-2)/2}/(p-1)^{r+1/2}}\\
&\times r^{n+1}C\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right)\\
={}&r^nC\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},\frac{k+r}{p-1};n\right).
\end{aligned}
$$

Further making use of

$$
C(a,b;x)=\frac{1}{C(b,a;x)},\quad a,b>0,\quad x\ge 0
$$

can rearrange the above result as

$$
\begin{aligned}
A_n(p,r)&=r^nC\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}\left[C\left(\frac{k+r}{p},1;n\right)C\left(1,\frac{k+r}{p-1};n\right)\right]\\
&=r^nC\left(\frac{r}{p},1;n\right)\prod_{k=1}^{p-1}\frac{C\left(\frac{k+r}{p},1;n\right)}{C\left(\frac{k+r}{p-1},1;n\right)}=r^nC\left(\frac{r}{p},1;n\right)\frac{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p},1;n\right)}{\prod_{k=1}^{p-1}C\left(\frac{k+r}{p-1},1;n\right)}
\end{aligned}
$$

$$= r^n \frac{\prod_{k=0}^{p-1} C\bigl(\frac{k+r}{p}, 1; n\bigr)}{\prod_{k=1}^{p-1} C\bigl(\frac{k+r}{p-1}, 1; n\bigr)} = r^n \frac{\prod_{k=1}^{p} C\bigl(\frac{k+r-1}{p}, 1; n\bigr)}{\prod_{k=1}^{p-1} C\bigl(\frac{k+r}{p-1}, 1; n\bigr)}.$$

The identity (1.8) thus follows.

It is straightforward that

$$\begin{aligned}
A_n(p,r) &= \frac{r}{\Gamma(n+1)} \frac{\Gamma(r+np)}{\Gamma(r+np-n+1)} \\
&= \frac{r}{\Gamma(n+1)} \frac{\Gamma(np)}{\Gamma(np-n+1)} \left(\frac{np}{np-n+1}\right)^r C(np, np-n+1; r) \\
&= \frac{r\Gamma(n)}{np\Gamma(n+1)} \frac{\Gamma(np+1)}{\Gamma(np-n+1)\Gamma(n)} \left(\frac{np}{np-n+1}\right)^r C(np, np-n+1; r) \\
&= \frac{r}{nB(np-n+1,n)} \frac{(np)^{r-1}}{(np-n+1)^r} C(np, np-n+1; r)
\end{aligned}$$

and, interchanging the role of r by np,

$$\begin{aligned}
A_n(p,r) &= \frac{r}{\Gamma(n+1)} \frac{\Gamma(np+r)}{\Gamma(np+r-n+1)} \\
&= \frac{r}{\Gamma(n+1)} \frac{\Gamma(r)}{\Gamma(r-n+1)} \left(\frac{r}{r-n+1}\right)^{np} C(r, r-n+1; np) \\
&= \frac{1}{n} \frac{\Gamma(r+1)}{\Gamma(n)\Gamma(r-n+1)} \left(\frac{r}{r-n+1}\right)^{np} C(r, r-n+1; np) \\
&= \frac{1}{nB(n, r-n+1)} \left(\frac{r}{r-n+1}\right)^{np} C(r, r-n+1; np).
\end{aligned}$$

Therefore, the identities (1.9) and (1.10) follow.

By the Gauss multiplication formula (3.1) again, when $r+1 > n$, the Fuss–Catalan numbers $A_n(p,r)$ defined by the second expression in (1.6) can be rearranged as

$$\begin{aligned}
A_n(p,r) &= r \frac{\Gamma\bigl(n\bigl(p+\frac{r}{n}\bigr)\bigr)}{\Gamma(n+1)\Gamma\bigl(n\bigl(p+\frac{r-n+1}{n}\bigr)\bigr)} \\
&= \frac{r}{\Gamma(n+1)} \frac{\dfrac{n^{np+r-1/2}}{(2\pi)^{(n-1)/2}} \displaystyle\prod_{k=0}^{n-1} \Gamma\left(p+\frac{r}{n}+\frac{k}{n}\right)}{\dfrac{n^{n(p-1)+r+1/2}}{(2\pi)^{(n-1)/2}} \displaystyle\prod_{k=0}^{n-1} \Gamma\left(p+\frac{r-n+1}{n}+\frac{k}{n}\right)} \\
&= \frac{n^{n-1}r}{\Gamma(n+1)} \prod_{k=0}^{n-1} \frac{\Gamma\bigl(p+\frac{r+k}{n}\bigr)}{\Gamma\bigl(p+\frac{r-n+k+1}{n}\bigr)} \\
&= \frac{n^{n-1}r}{\Gamma(n+1)} \prod_{k=0}^{n-1} \frac{\Gamma\bigl(\frac{r+k}{n}\bigr)}{\Gamma\bigl(\frac{r-n+k+1}{n}\bigr)} \left(\frac{r+k}{r-n+k+1}\right)^p \\
&\quad \times \prod_{k=0}^{n-1} \frac{\Gamma\bigl(\frac{r-n+k+1}{n}\bigr)}{\Gamma\bigl(\frac{r+k}{n}\bigr)} \left(\frac{r-n+k+1}{r+k}\right)^p \frac{\Gamma\bigl(p+\frac{r+k}{n}\bigr)}{\Gamma\bigl(p+\frac{r-n+k+1}{n}\bigr)} \\
&= \frac{n^{n-1}r}{\Gamma(n+1)} \prod_{k=0}^{n-1} \frac{\Gamma\bigl(\frac{r}{n}+\frac{k}{n}\bigr)}{\Gamma\bigl(\frac{r-n+1}{n}+\frac{k}{n}\bigr)} \left(\prod_{k=0}^{n-1} \frac{r+k}{r-n+k+1}\right)^p
\end{aligned}$$

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$$
\begin{aligned}
&\times \prod_{k=0}^{n-1} C\biggl(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\biggr) \\
&= \frac{n^{n-1}r}{\Gamma(n+1)} \frac{\prod_{k=0}^{n-1}\Gamma\bigl(\frac{r}{n}+\frac{k}{n}\bigr)}{\prod_{k=0}^{n-1}\Gamma\bigl(\frac{r-n+1}{n}+\frac{k}{n}\bigr)} \biggl[\frac{\Gamma(r+n)\Gamma(r+1-n)}{\Gamma(r)\Gamma(r+1)}\biggr]^p \\
&\times \prod_{k=0}^{n-1} C\biggl(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\biggr) \\
&= \frac{n^{n-1}r}{\Gamma(n+1)} \frac{\frac{(2\pi)^{(n-1)/2}}{n^{r-1/2}}\Gamma(r)}{\frac{(2\pi)^{(n-1)/2}}{n^{r-n+1/2}}\Gamma(r-n+1)} \biggl[\frac{\Gamma(r+n)\Gamma(r+1-n)}{\Gamma(r)\Gamma(r+1)}\biggr]^p \\
&\times \prod_{k=0}^{n-1} C\biggl(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\biggr) \\
&= \frac{r}{\Gamma(n+1)} \frac{\Gamma(r)}{\Gamma(r-n+1)} \biggl[\frac{\Gamma(r+n)\Gamma(r+1-n)}{\Gamma(r)\Gamma(r+1)}\biggr]^p \\
&\times \prod_{k=0}^{n-1} C\biggl(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\biggr) \\
&= \frac{1}{\Gamma(n+1)} \biggl[\frac{\Gamma(r+n)}{\Gamma(r)}\biggr]^p \biggl[\frac{\Gamma(r+1-n)}{\Gamma(r+1)}\biggr]^{p-1} \\
&\times \prod_{k=0}^{n-1} C\biggl(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\biggr) \\
&= \frac{\Gamma(n)}{\Gamma(n+1)} \biggl[\frac{\Gamma(r+n)}{\Gamma(r)\Gamma(n)}\biggr]^p \biggl[\frac{\Gamma(r+1-n)\Gamma(n)}{\Gamma(r+1)}\biggr]^{p-1} \\
&\times \prod_{k=0}^{n-1} C\biggl(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\biggr) \\
&= \frac{1}{n} \frac{[B(r+1-n,n)]^{p-1}}{[B(r,n)]^p} \prod_{k=0}^{n-1} C\biggl(\frac{r+k}{n}, \frac{r-n+k+1}{n}; p\biggr).
\end{aligned}
$$

The identity (1.11) is thus proved.

Similarly, by virtue of (3.1) once again, we have

$$
\begin{aligned}
A_n(p,r) &= \frac{r\Gamma(p(n+r/p))}{\Gamma(n+1)\Gamma(p(n+(r+1-n)/p))} \\
&= \frac{rp^{n-1}}{\Gamma(n+1)} \prod_{k=0}^{p-1} \frac{\Gamma\bigl(n+\frac{r+k}{p}\bigr)}{\Gamma\bigl(n+\frac{r+k+1-n}{p}\bigr)} \\
&= \frac{rp^{n-1}}{\Gamma(n+1)} \prod_{k=0}^{p-1} \frac{\Gamma\bigl(\frac{r}{p}+\frac{k}{p}\bigr)}{\Gamma\bigl(\frac{r+1-n}{p}+\frac{k}{p}\bigr)} \biggl(\frac{r+k}{r+k+1-n}\biggr)^n
\end{aligned}
$$

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$$
\begin{aligned}
&\times\prod_{k=0}^{p-1}C\biggl(\frac{r+k}{p},\frac{r+k+1-n}{p};n\biggr)\\
&=\frac{rp^{n-1}}{\Gamma(n+1)}\frac{\prod_{k=0}^{p-1}\Gamma\bigl(\frac{r}{p}+\frac{k}{p}\bigr)}{\prod_{k=0}^{p-1}\Gamma\bigl(\frac{r+1-n}{p}+\frac{k}{p}\bigr)}\Biggl(\prod_{k=0}^{p-1}\frac{r+k}{r+k+1-n}\Biggr)^{n}\\
&\times\prod_{k=0}^{p-1}C\biggl(\frac{r+k}{p},\frac{r+k+1-n}{p};n\biggr)\\
&=\frac{rp^{n-1}}{\Gamma(n+1)}\frac{p^{r+1-n}\Gamma(r)}{p^{r-1/2}\Gamma(r+1-n)}\biggl[\frac{\Gamma(r+p)}{\Gamma(r)}\frac{\Gamma(r+1-n)}{\Gamma(r+1-n+p)}\biggr]^{n}\\
&\times\prod_{k=0}^{p-1}C\biggl(\frac{r+k}{p},\frac{r+k+1-n}{p};n\biggr)\\
&=\frac{rp^{1/2}\Gamma(n-1)}{\Gamma(n+1)}\biggl[\frac{\Gamma(r+1-n)\Gamma(n-1)}{\Gamma(r)}\biggr]^{n-1}\biggl[\frac{\Gamma(r+p)}{\Gamma(r+1-n+p)\Gamma(n-1)}\biggr]^{n}\\
&\times\prod_{k=0}^{p-1}C\biggl(\frac{r+k}{p},\frac{r+k+1-n}{p};n\biggr)\\
&=rp^{1/2}B(n-1,2)\frac{[B(r+1-n,n-1)]^{n-1}}{[B(r+1-n+p,n-1)]^{n}}\\
&\times\prod_{k=0}^{p-1}C\biggl(\frac{r+k}{p},\frac{r+k+1-n}{p};n\biggr).
\end{aligned}
$$

The identity (1.12) is demonstrated. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Recall from [25, Chapter XIII], [49, Chapter 1], and [54, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0\le(-1)^kf^{(k)}(x)<\infty$ on I for all $k\ge0$. The inclusions

$$\mathcal{L}[I]\subset\mathcal{C}[I]\quad\text{and}\quad\mathcal{S}\setminus\{0\}\subset\mathcal{L}[(0,\infty)] \tag{3.2}$$

were discovered in [6, 15, 35], where $\mathcal{L}[I]$, $\mathcal{C}[I]$, and $\mathcal{S}$ denote respectively the set of all logarithmically completely monotonic functions on an interval I, the set of all completely monotonic functions on I, and the set of all Stieltjes transforms. See also the monograph [49] and plenty of references therein.

By Lemma 2.1, since $\frac{r}{p}<1$ and $\frac{k+r}{p}<\frac{k+r}{p-1}$ for all $p>1$, $p>r>0$, and $1\le k\le p-1$, the functions

$$\frac{1}{\mathcal{C}(r/p,1;x)}\quad\text{and}\quad\frac{1}{\mathcal{C}((k+r)/p,(k+r)/(p-1);x)}$$

are logarithmically completely monotonic on $[0,\infty)$. It is easy to see that the product of finitely many logarithmically completely monotonic functions is still logarithmically completely monotonic. Hence, the function

$$\frac{1}{r\mathcal{C}(r/p,1;x)\prod_{k=1}^{p-1}\mathcal{C}((k+r)/p,(k+r)/(p-1);x)} \tag{3.3}$$

is logarithmically completely monotonic on the interval $[0,\infty)$. A a result, the sequence (1.13) is decreasing and logarithmically convex.

Further, by virtue of the first inclusion in (3.2) and the logarithmically complete monotonicity of the function (3.3), the function (3.3) is also completely monotonic on $[0,\infty)$. From [54, p. 164, Theorem 14b] which reads that a necessary and sufficient condition that there should exist a completely monotonic function $f(x)$ in $0\le x<\infty$ such that $f(n)=a_n$ for $n\ge0$ is that $\{a_n\}_0^\infty$ should be a minimal completely monotonic sequence, we arrive at the complete monotonicity and minimality of the sequence $\mathcal{A}_n(p,r)$ defined by (1.13).

Similarly, by Lemma 2.2, the identity (1.8), and $A_0(p,r)=1$ for all $p>1$ and $r>0$, we conclude that the sequence of the Fuss–Catalan numbers $\{A_n(p,r)\}_{n\ge0}$ is increasing and logarithmically convex. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. Applying the inequality (2.1) to

$$A_n(p,r)=\frac{r}{n(n-1)}\frac{1}{B(n-1,np+r)}$$

results in

$$\begin{aligned}\frac{r(np+r)}{n}&\le A_n(p,r)\\&\le\frac{r(np+r)}{n}\frac{1}{1-(n-1)(np+r)\min\{C(n-1)C(np+r),b_A\}}\end{aligned}$$

which can be rearranged as (1.14). The proof of Theorem 1.3 is complete. □

4. Remarks

Finally, we give several remarks on our main results.

Remark 4.1. The Catalan–Qi function $C(a,b;z)$, the second expression in (1.6) of the Fuss–Catalan numbers $A_n(p,r)$, and all the identities from (1.8) to (1.12) in Theorem 1.1 can be extended to real values, even to complex values, that the variables a,b,x and n,p,r can take.

Remark 4.2. The identity (1.8) is more informative than the others in Theorem 1.1, because the form of the identity (1.8) is simpler, more regular, and with less restrictions to n, p, and r.

Remark 4.3. Theorem 1.1 means that the Fuss–Catalan numbers $A_n(p,r)$ can be represented in terms of the Catalan–Qi functions $C(a,b;x)$. This fact shows that introducing the Catalan–Qi function $C(a,b;x)$ in [47] is analytically significant. However, we need the combinatorialists to combinatorially interpret the Catalan–Qi function $C(a,b;x)$ or its special cases.

Remark 4.4. By the definition of the classical beta function, we easily see that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=\frac{B(b-a,z+a)}{\Gamma(b-a)}. \tag{4.1}$$

We can use the inequality (2.1) and the formula (4.1) to bound the Catalan numbers C_n, the Fuss–Catalan numbers $A_n(p,r)$, and the Catalan–Qi function $C(a,b;x)$.

There has been an amount of literature on the ratio of two gamma functions, see, for example, the expository and survey articles [10, 19, 30, 31, 41, 42, 43] and plenty of references therein. Applying results in these literature, we can estimate the Catalan numbers C_n, the Fuss–Catalan numbers $A_n(p,r)$, and the Catalan–Qi function $C(a,b;x)$ in terms of inequalities and asymptotic formulas of the gamma function $\Gamma(x)$. For example, the double inequality

$$\sqrt{\frac{b}{a}}\,[I(a,b)]^{a-b}\exp\left[\sum_{j=1}^{2m}\frac{B_{2j}}{2j(2j-1)}\left(\frac{1}{a^{2j-1}}-\frac{1}{b^{2j-1}}\right)\right]<\frac{\Gamma(a)}{\Gamma(b)}<\sqrt{\frac{b}{a}}\,[I(a,b)]^{a-b}\exp\left[\sum_{j=1}^{2m-1}\frac{B_{2j}}{2j(2j-1)}\left(\frac{1}{a^{2j-1}}-\frac{1}{b^{2j-1}}\right)\right]$$

for $m\in\mathbb{N}$ and $a,b>0$ was derived in [44, Theorem 11], where B_i for $i\ge 0$, which can be generated by

$$\frac{x}{e^x-1}=\sum_{i=0}^{\infty}B_i\frac{x^i}{i!}=1-\frac{x}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{x^{2j}}{(2j)!},\quad |x|<2\pi,$$

denote the classical Bernoulli numbers and

$$I(\alpha,\beta)=\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)}$$

for $\alpha,\beta>0$ with $\alpha\ne\beta$ stands for the identric mean [8]. One more example is the double inequality

$$e^{\psi(L(a,b))}<\left[\frac{\Gamma(a)}{\Gamma(b)}\right]^{1/(a-b)}<e^{\psi(I(a,b))},\quad a,b>0,\quad a\ne b$$

in [29, 40], where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and

$$L(a,b)=\frac{b-a}{\ln b-\ln a},\quad a,b>0,\quad a\ne b$$

is called the logarithmic mean [8].

In the paper [14], some inequalities for the beta function $B(x,y)$ are reviewed and surveyed. Hereafter, the main result in [14, Theorem] states that, for every $k\ge 1$ and every point $x,y>0$, the quantity $(-1)^{k-1}D_{x,y}^{(k)}(X,Y)$ is positive for $X,Y>0$ and, if k is even, positive definite in X,Y, where $D_{x,y}^{(k)}(X,Y)$ denotes the kth differential of $F(x+X,y+Y)$ in X and Y and

$$F(x,y)=\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)}\frac{(x+y)^{x+y}}{x^xy^y}.$$

As said in [14, p. 1430], this theorem can produce lower and upper bounds for $F(x,y)$ and every bound for $F(x,y)$ is actually a bound for $B(x,y)$. By inequalities for the beta function $B(x,y)$, we can derive more inequalities for the Fuss–Catalan numbers $A_n(p,r)$.

It seems that there are more identities than inequalities in combinatorics.

Remark 4.5. This paper is a revised and extended version of the preprint [33].

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