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SIMPLIFYING COEFFICIENTS IN DIFFERENTIAL EQUATIONS ASSOCIATED WITH HIGHER ORDER BERNOULLI NUMBERS OF THE SECOND KIND

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Abstract. In the paper, by virtue of the Faà di Bruno formula, some properties of the Bell polynomials of the second kind, and an inversion formula for the Stirling numbers of the first and second kinds, the authors establish meaningfully and significantly two identities which simplify coefficients in a family of ordinary differential equations associated with higher order Bernoulli numbers of the second kind.

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1. Motivations

In [2, Theorem 1], it was inductively and recursively established that the family of differential equations

$$(-1)^n(r)_nF(t)=[\ln(1+t)]^n\sum_{i=1}^{n}a_i(n)(1+t)^iF^{(i)}(t),\quad n\in\mathbb{N} \tag{1}$$

has a solution

$$F(t)=F(t,r)=\left[\frac{1}{\ln(1+t)}\right]^r,\quad r\in\mathbb{N}, \tag{2}$$

where $a_1(n)=1$ and

$$a_i(n)=\sum_{k_{i-1}=0}^{n-i}\sum_{k_{i-2}=0}^{n-i-k_{i-1}}\cdots\sum_{k_1=0}^{n-i-k_{i-1}-\dots-k_2}\prod_{\ell=2}^{i}\ell^{k_{\ell-1}},\quad 2\le i\le n. \tag{3}$$

Let

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)(x+2)\cdots(x+n-1), & n\ge1\\ 1, & n=0\end{cases}$$

and

$$\langle x\rangle_n=\prod_{\ell=0}^{n-1}(x-\ell)=\begin{cases}x(x-1)(x-2)\cdots(x-n+1), & n\ge1\\ 1, & n=0\end{cases}$$

be the rising and falling factorials of $x\in\mathbb{R}$ for $n\in\{0\}\cup\mathbb{N}$. Let $b_n^{(r)}$ for $r\in\mathbb{N}$, which can be generated by

$$\left[\frac{t}{\ln(1+t)}\right]^r=\sum_{n=0}^{\infty}b_n^{(r)}\frac{t^n}{n!},$$

stand for the Bernoulli numbers of the second kind with order r. Theorem 2 in [2] reads that, if $n=0,1,2,\dots$ and $N=1,2,3,\dots$, then

(1) for $0\le n<N+r$,

$$(-1)^N(r)_Nb_n^{(r+N)}=\sum_{i=0}^{\min\{N-1,n\}}\sum_{\ell=\max\{i,n-r+1\}}^{n}\binom{N-i}{\ell-i}\langle n-\ell-r\rangle_{N-i}\langle n\rangle_\ell a_{N-i}(N)b_{n-\ell}^{(r)};$$

(2) for $n\ge N+r$,

$$(-1)^N(r)_Nb_n^{(r+N)}=\left(\sum_{i=0}^{\min\{n,N-1\}}\sum_{\ell=\max\{i,n-r+1\}}^{n}+\sum_{i=0}^{N-1}\sum_{\ell=i}^{n-N-r+i}\right)\binom{N-i}{\ell-i}\times\langle n-\ell-r\rangle_{N-i}\langle n\rangle_\ell a_{N-i}(N)b_{n-\ell}^{(r)}.$$

It is not difficult to see that the expression (3) of the quantity $a_i(n)$ is too complicated to be computed by hand and computer software. Can one find a simple, meaningful, and significant expression for the quantity $a_i(n)$ in (3)?

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2. Lemmas

For answering the above question and proving our main results, we need the following lemmas.

Lemma 1 ([1, p. 134, Theorem A] and [1, p. 139, Theorem C]). *For $n \ge k \ge 0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{\ell_1,\ell_2,\dots,\ell_n\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n} i\ell_i=n\\ \sum_{i=1}^{n} \ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!} \prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} f\circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big). \tag{4}$$

Lemma 2 ([1, p. 135]). *For $n \ge k \ge 0$, we have*

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{5}$$

and

$$\mathrm{B}_{n,k}(0!, 1!, 2!, \dots, (n-k)!) = (-1)^{n-k} s(n,k), \tag{6}$$

where a and b are any complex numbers and $s(n,k)$ for $n \ge k \ge 0$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x| < 1,$$

stand for the Stirling numbers of the first kind.

Lemma 3 ([21, p. 171, Theorem 12.1]). *If b_α and a_k are a collection of constants independent of n, then*

$$a_n = \sum_{\alpha=0}^{n} S(n,\alpha) b_\alpha \quad \textit{if and only if} \quad b_n = \sum_{k=0}^{n} s(n,k) a_k,$$

where $S(n,k)$ for $n \ge k \ge 0$, which can be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!},$$

stand for the Stirling numbers of the second kind.

3. Main results and their proofs

Now we are in a position to answer the above question and to state and prove our main results.

Theorem 1. *For $n \ge 0$ and $r \in \mathbb{R}$, the function $F(t) = F(t,r)$ defined by (2) satisfies*

$$F^{(n)}(t) = \left(\frac{1}{1+t}\right)^n \left[\sum_{k=0}^{n} s(n,k)\frac{\langle -r\rangle_k}{[\ln(1+t)]^k}\right] F(t) \tag{7}$$

and

$$\sum_{k=0}^{n} S(n,k)(1+t)^k F^{(k)}(t) = \frac{\langle -r\rangle_n}{[\ln(1+t)]^n} F(t). \tag{8}$$

Proof. Let $u = u(t) = \ln(1+t)$ and $r \in \mathbb{R}$. Then, by virtue of the Faà di Bruno formula (4) and the identities (5) and (6) in sequence,

$$\begin{aligned} F^{(n)}(t) &= \sum_{k=0}^{n} \left(u^{-r}\right)^{(k)} \mathrm{B}_{n,k}\left(\frac{0!}{1+t}, -\frac{1!}{(1+t)^2}, \dots, (-1)^{n-k}\frac{(n-k)!}{(1+t)^{n-k+1}}\right) \\ &= \sum_{k=0}^{n} \frac{\langle -r\rangle_k}{u^{r+k}} \left(\frac{1}{1+t}\right)^n (-1)^{n+k} \mathrm{B}_{n,k}(0!, 1!, \dots, (n-k)!) \\ &= \sum_{k=0}^{n} \frac{\langle -r\rangle_k}{[\ln(1+t)]^{r+k}} \left(\frac{1}{1+t}\right)^n (-1)^{n+k}(-1)^{n-k} s(n,k) \\ &= \left(\frac{1}{1+t}\right)^n \frac{1}{[\ln(1+t)]^r} \sum_{k=0}^{n} \frac{\langle -r\rangle_k}{[\ln(1+t)]^k} s(n,k) \end{aligned}$$

for $n \ge 0$. Thus, the identity (7) is proved.

Applying Lemma 3 to (7) leads to

$$\frac{\langle -r\rangle_n}{[\ln(1+t)]^n} F(t) = \sum_{k=0}^{n} S(n,k)(1+t)^k F^{(k)}(t)$$

which can be rewritten as (8). The required proof is complete. □

4. Remarks

In this section, we give several remarks and some explanation about our main results.

Remark 1. Theorem 1 extends the range of r from $\mathbb{N}$ to $\mathbb{R}$.

Remark 2. Comparing (1) with (8) reveals that

$$a_i(n) = S(n,i), \quad n \ge i \ge 0. \tag{9}$$

This implies that the identity (8) is more meaningful, more significant, more computable than the one (1).

Remark 3. By virtue of the expression (9), all the above mentioned results in the paper [2] can be reformulated simpler, more meaningfully, and more significantly. For the sake of saving the space and shortening the length of this paper, we do not rewrite them in details here.

Remark 4. Currently we can see that the method used in this paper is simpler, shorter, nicer, more meaningful, and more significant than the inductive and recursive method used in [2] and closely related references therein.

Remark 5. In the papers [5, 6, 18, 19], there are some new results about the Bernoulli numbers of the second kind.

Remark 6. In the papers and preprints [3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 22], there are similar ideas, methods, techniques, and purposes to this paper.

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