Peer-reviewed version available at *Tatra Mountains Mathematical Publications* **2018**, *72*, 67-76; doi:10.2478/tmmp-2018-0022

SIMPLIFICATION OF COEFFICIENTS IN DIFFERENTIAL EQUATIONS ASSOCIATED WITH HIGHER ORDER FROBENIUS–EULER NUMBERS

FENG QI

Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

DA-WEI NIU

Department of Mathematics, East China Normal University, Shanghai City, 200241, China

BAI-NI GUO

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

Abstract. In the paper, by virtue of the Faà di Bruno formula, some properties of the Bell polynomials of the second kind, and the inversion formulas of binomial numbers and the Stirling numbers of the first and second kinds, the authors simplify meaningfully and significantly coefficients in two families of ordinary differential equations associated with higher order Frobenius–Euler numbers.

E-mail addresses: `qifeng618@gmail.com`, `qifeng618@hotmail.com`, `qifeng618@qq.com`, `nnddww@gmail.com`, `bai.ni.guo@gmail.com`, `bai.ni.guo@hotmail.com`.

2010 *Mathematics Subject Classification.* Primary 34A05; Secondary 05A16, 11A25, 11B37, 11B68, 11B73, 11B83, 33B10, 34A34.

Key words and phrases. simplification; coefficient; ordinary differential equation; higher order Frobenius–Euler number; Faà di Bruno formula; Bell polynomial of the second kind; inversion formula.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Peer-reviewed version available at *Tatra Mountains Mathematical Publications* **2018**, *72*, 67-76; doi:10.2478/tmmp-2018-0022

1. Motivations

In [4, Theorem 2.2], it was inductively and recursively established that the family of differential equations

$$F^{(n)}(t)=\left[\sum_{i=0}^{n}a_i(n)\left(\frac{u}{e^t-u}\right)^i\right]F(t) \tag{1}$$

for $n\ge0$, $r\in\mathbb{N}$, and $u\in\mathbb{C}\setminus\{1\}$ has a solution

$$F(t)=F(t;r,u)=\left(\frac{1}{e^t-u}\right)^r, \tag{2}$$

where $a_0(n)=(-r)^n$,

$$a_i(n)=(-1)^n(r+i-1)_i\sum_{k_i=0}^{n-i}\sum_{k_{i-1}=0}^{n-i-k_i}\cdots\sum_{k_1=0}^{n-i-k_i-\cdots k_2}r^{n-i-\sum_{\ell=1}^{i}k_\ell}\prod_{\ell=1}^{i}(r+\ell)^{k_\ell} \tag{3}$$

for $1\le i\le n$, and

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)(x+2)\cdots(x+n-1), & n\ge1\\ 1, & n=0\end{cases}$$

is the rising factorial. Hereafter, the following results were deduced.

(1) For $k,n\ge0$, we have

$$H_{k+n}^{(r)}(u)=\sum_{i=0}^{n}a_i(n)\left(\frac{u}{1-u}\right)^iH_k^{(r+i)}(u),$$

where $H_k^{(r)}$, which can be generated by

$$\left(\frac{1-u}{e^t-u}\right)^r=\sum_{k=0}^{\infty}H_k^{(r)}(u)\frac{t^k}{k!},$$

stand for the Frobenius–Euler numbers of order r. See [4, Theorem 2.3].

(2) For $k,n\ge0$, we have

$$E_{k+n}^{(r)}=\sum_{i=0}^{n}\left(-\frac{1}{2}\right)^ia_i(n)E_k^{(r+i)},$$

where $E_k^{(r)}$, which can be generated by

$$\left(\frac{2}{e^t+1}\right)^r=\sum_{k=0}^{\infty}E_k^{(r)}\frac{t^k}{k!},$$

stand for the Euler numbers of order r. See [4, Corollary 2.5].

(3) When $0\le k\le r-1$ and $k\ge r+n$,

$$B_k^{(r)}=\frac{1}{(k-r)_n}\sum_{i=\max\{n-k,0\}}^{n}a_i(n)B_{k+i-n}^{(r+i)}(k)_{n-i};$$

when $r\le k\le r-1+n$,

$$\sum_{i=\max\{n-k,0\}}^{n}a_i(n)B_{k+i-n}^{(r+i)}(k)_{n-i}=0;$$

Peer-reviewed version available at Tatra Mountains Mathematical Publications 2018, 72, 67-76; doi:10.2478/tmmp-2018-0022

where $B_k^{(r)}$, which can be generated by

$$\left(\frac{t}{e^t-1}\right)^r=\sum_{k=0}^{\infty}B_k^{(r)}\frac{t^k}{k!},$$

stand for the Bernoulli numbers of order r. See [4, Theorem 2.7].

In [5, Theorem 2.1], it was inductively and recursively proved that the family of differential equations

$$(-1)^{n-1}(r)_n\left(\frac{u}{e^t-u}\right)^n F(t)=\sum_{i=0}^{n}b_i(n)F^{(i)}(t) \tag{4}$$

for $u\in\mathbb{C}$ and $r\in\mathbb{N}$ has a solution $F(t)$ defined in (2), where $b_0(n)=-\langle r+n-1\rangle_n$,

$$\begin{aligned}b_i(n)=-\sum_{k_i=0}^{n-i}\sum_{k_{i-1}=0}^{n-i-k_i}\cdots\sum_{k_1=0}^{n-i-k_i-\dots-k_2}\prod_{\ell=1}^{i}\left\langle r+n-i-1-\sum_{j=\ell+1}^{i}k_j+\ell\right\rangle_{k_\ell}\\ \times\left\langle r+n-i-1-\sum_{j=1}^{i}k_j\right\rangle_{n-i-\sum_{j=1}^{i}k_j},\quad 1\le i\le n,\end{aligned} \tag{5}$$

and

$$\langle x\rangle_n=\prod_{\ell=0}^{n-1}(x-\ell)=\begin{cases}x(x-1)(x-2)\cdots(x-n+1), & n\ge1\\ 1, & n=0\end{cases}$$

is the falling factorial of $x\in\mathbb{R}$ for $n\in\{0\}\cup\mathbb{N}$. Hereafter, the following conclusions were derived.

(1) For $k,n\ge0$, we have

$$(-1)^{n-1}(r)_n\left(\frac{u}{1-u}\right)^n H_k^{(r+n)}(u)=\sum_{i=0}^{n}b_i(n)H_{k+i}^{(r)}(u). \tag{6}$$

See [5, Theorem 3.1]. In particular, taking $u=-1$ in (6) leads to

$$(-1)^{n-1}(r)_n\left(-\frac{1}{2}\right)^n E_k^{(r+n)}=\sum_{i=0}^{n}b_i(n)E_{k+i}^{(r)}.$$

See [5, Corollary 3.3].

(2) When $0\le k\le n+r-1$, we have

$$B_k^{(r+n)}=(-1)^{n-1}\frac{1}{(r)_n}\sum_{i=\max\{n-k,0\}}^{\min\{r+n-1-k,n\}}b_i(n)B_{k+i-n}^{(r)}\frac{\langle k+i-n-r\rangle_i k!}{(k+i-n)!};$$

when $k\ge n+r$, we have

$$B_k^{(r+n)}=(-1)^{n-1}\frac{1}{(r)_n}\sum_{i=0}^{n}b_i(n)B_{k+i-n}^{(r)}\frac{\langle k+i-n-r\rangle_i k!}{(k+i-n)!}.$$

See [5, Theorem 3.4].

(3) The matrices $\big(a_i(j)\big)_{0\le i,j\le n}$ and $\left(\frac{b_i(j)}{(-1)^{j-1}(r)_j}\right)_{0\le i,j\le n}$ are inverse to each other for all n. See [5, Remark 3.2].

Peer-reviewed version available at Tatra Mountains Mathematical Publications 2018, 72, 67-76; doi:10.2478/tmmp-2018-0022

It is easy to see that expressions (3) and (5) of the quantities $a_i(n)$ and $b_i(n)$ are too complicated to be understood by common man's brain or to be computed by hand and computer software. Can one find simple, meaningful, and significant expressions for the quantities (3) and (5)?

2. Lemmas

For answering the above question and proving our main results, we need the following lemmas.

Lemma 2.1 ([1, p. 134, Theorem A] and [1, p. 139, Theorem C]). *For $n\ge k\ge0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{\ell_1,\ell_2,\dots,\ell_n\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n \ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\biggl(\frac{x_i}{i!}\biggr)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^n f^{(k)}(h(t))\mathrm{B}_{n,k}\bigl(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\bigr). \tag{7}$$

Lemma 2.2 ([1, p. 135]). *For $n\ge k\ge0$, we have*

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{8}$$

and

$$\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k), \tag{9}$$

where a and b are any complex numbers.

Lemma 2.3 ([17, p. 171, Theorem 12.1]). *If b_α and a_k are a collection of constants independent of n, then*

$$a_n=\sum_{\alpha=0}^n S(n,\alpha)b_\alpha\quad\textit{if and only if}\quad b_n=\sum_{k=0}^n s(n,k)a_k.$$

Lemma 2.4 ([17, p. 83, Eq. (7.12)]). *If a_k and b_k for $k\ge0$ are a collection of constants independent of n, then*

$$a(n)=\sum_{k=0}^n(-1)^k\binom{n}{k}b(k)\quad\textit{if and only if}\quad b(n)=\sum_{k=0}^n(-1)^k\binom{n}{k}a(k).$$

3. Main results and their proofs

Now we are in a position to answer the above question and to state and prove our main results.

Theorem 3.1. *For $n\ge0$, $r\in\mathbb{R}$, and $u\in\mathbb{C}$, the function $F(t)$ defined by (2) satisfies*

$$F^{(n)}(t)=\sum_{\ell=0}^n\Biggl[\sum_{k=\ell}^n\binom{k}{\ell}S(n,k)\langle-r\rangle_k\Biggr]\biggl(\frac{u}{e^t-u}\biggr)^\ell F(t) \tag{10}$$

and

$$\sum_{k=0}^{n}\left[\sum_{\ell=k}^{n}(-1)^{\ell}\binom{n}{\ell}\frac{s(\ell,k)}{\langle -r\rangle_{\ell}}\right]F^{(k)}(t)=(-1)^{n}\left(\frac{u}{e^{t}-u}\right)^{n}F(t). \tag{11}$$

Proof. Let $F(t)=\frac{1}{w^r}$ and $w=w(t)=w(t;u)=e^t-u$. Then, by the Faà di Bruno formula (7) and the identities (8) and (9) in sequence,

$$\begin{aligned}
F^{(n)}(t)&=\sum_{k=0}^{n}\left(\frac{1}{w^r}\right)^{(k)}\mathrm{B}_{n,k}(e^t,e^t,\dots,e^t)\\
&=\sum_{k=0}^{n}\frac{\langle -r\rangle_k}{w^{r+k}}e^{kt}\mathrm{B}_{n,k}(1,1,\dots,1)\\
&=\sum_{k=0}^{n}\frac{\langle -r\rangle_k}{(e^t-u)^{r+k}}e^{kt}S(n,k)\\
&=\frac{1}{(e^t-u)^r}\sum_{k=0}^{n}\frac{\langle -r\rangle_k}{(e^t-u)^{k}}e^{kt}S(n,k)\\
&=F(t)\sum_{k=0}^{n}\langle -r\rangle_k S(n,k)\left(\frac{e^t}{e^t-u}\right)^k\\
&=F(t)\sum_{k=0}^{n}\langle -r\rangle_k S(n,k)\left(1+\frac{u}{e^t-u}\right)^k\\
&=F(t)\sum_{k=0}^{n}\langle -r\rangle_k S(n,k)\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{u}{e^t-u}\right)^\ell\\
&=F(t)\sum_{\ell=0}^{n}\left[\sum_{k=\ell}^{n}\langle -r\rangle_k S(n,k)\binom{k}{\ell}\right]\left(\frac{u}{e^t-u}\right)^\ell.
\end{aligned}$$

Therefore, the identity (10) follows immediately.

From the above proof of the identity (10), it can be deduced that

$$F^{(n)}(t)=F(t)\sum_{k=0}^{n}S(n,k)\langle -r\rangle_k\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{u}{e^t-u}\right)^\ell,\quad n\ge 0.$$

Utilizing Lemma 2.3 arrives at

$$F(t)\langle -r\rangle_n\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\frac{u}{e^t-u}\right)^\ell=\sum_{k=0}^{n}s(n,k)F^{(k)}(t),\quad n\ge 0$$

which can be rearranged as

$$\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}\left(-\frac{u}{e^t-u}\right)^\ell=\frac{1}{F(t)\langle -r\rangle_n}\sum_{k=0}^{n}s(n,k)F^{(k)}(t),\quad n\ge 0.$$

Further making use of Lemma 2.4 derives

$$\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}\frac{1}{F(t)\langle -r\rangle_\ell}\sum_{k=0}^{\ell}s(\ell,k)F^{(k)}(t)=\left(-\frac{u}{e^t-u}\right)^n,\quad n\ge 0$$

which can be rewritten as (11). The required proof is complete. $\square$

4. Remarks

In this section, we give several remarks and some explanation about our main results.

Remark 4.1. Theorem 3.1 extends the range of r from $\mathbb{N}$ to $\mathbb{R}$.

Remark 4.2. Comparing (1) with (10) reveals that

$$a_i(n) = \sum_{k=i}^{n} \binom{k}{i} S(n,k)\langle -r\rangle_k, \quad 0 \le i \le n. \tag{12}$$

This implies that the identity (10) is more meaningful, more significant, more computable than the one (1).

Remark 4.3. It is not difficult to see that

$$a_0(n) = \sum_{k=0}^{n} S(n,k)\langle -r\rangle_k = (-r)^n, \quad n \ge 0.$$

Then it is natural to ask a question: is the finite sum

$$a_i(n) = \sum_{k=i}^{n} \binom{k}{i} S(n,k)\langle -r\rangle_k, \quad 1 \le i \le n$$

summarizable?

Remark 4.4. Comparing (4) with (11) discloses

$$b_i(n) = (r)_n \sum_{\ell=i}^{n} (-1)^{\ell+1} \binom{n}{\ell} \frac{s(\ell,i)}{\langle -r\rangle_\ell}, \quad n \ge i \ge 0. \tag{13}$$

This means that the identity (11) is more meaningful, more significant, more computable than (4).

Remark 4.5. By virtue of the expressions (12) and (13), all the above mentioned results in the papers [4, 5] can be reformulated simpler, more meaningfully, and more significantly. For the sake of saving the space and shortening the length of this paper, we do not rewrite them in details here.

Remark 4.6. Till now we can see that the method used in this paper is simpler, shorter, nicer, more meaningful, and more significant than the inductive and recursive method used in [4, 5].

Remark 4.7. In the papers and preprints [2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18], there are similar ideas, methods, techniques, and purposes to this paper.

Acknowledgements. The authors thank Professor Taekyun Kim at Kwangwoon University in South Korea for his supplying electronic versions of the papers [4, 5] through e-mail on 31 July 2017.

References

[1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.

[2] B.-N. Guo and F. Qi, *Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind*, J. Comput. Appl. Math. **272** (2014), 251–257; Available online at `http://dx.doi.org/10.1016/j.cam.2014.05.018`.

[3] B.-N. Guo and F. Qi, *Some identities and an explicit formula for Bernoulli and Stirling numbers*, J. Comput. Appl. Math. **255** (2014), 568–579; Available online at http://dx.doi.org/10.1016/j.cam.2013.06.020.
[4] D. S. Kim, T. Kim, J.-W. Park, and J.-J. Seo, *Differential equations associated with higher-order Frobenius-Euler numbers*, Glob. J. Pure Appl. Math. **12** (2016), no. 3, 2537–2547.
[5] T. Kim, D. S. Kim, L.-C. Jang, and H. I. Kwon, *Differential equations associated with higher-order Frobenius–Euler numbers revisited*, Differ. Equ. Dyn. Syst. (2017), in press; Available online at https://doi.org/10.1007/s12591-017-0380-8.
[6] F. Qi, *Explicit formulas for the convolved Fibonacci numbers*, ResearchGate Working Paper (2016), available online at http://dx.doi.org/10.13140/RG.2.2.36768.17927.
[7] F. Qi and B.-N. Guo, *Explicit formulas and recurrence relations for higher order Eulerian polynomials*, Indag. Math. **28** (2017), no. 4, 884–891; Available online at https://doi.org/10.1016/j.indag.2017.06.010.
[8] F. Qi and B.-N. Guo, *Some properties of a solution to a family of inhomogeneous linear ordinary differential equations*, Preprints **2016**, 2016110146, 11 pages; Available online at http://dx.doi.org/10.20944/preprints201611.0146.v1.
[9] F. Qi and B.-N. Guo, *Some properties of the Hermite polynomials and their squares and generating functions*, Preprints **2016**, 2016110145, 14 pages; Available online at http://dx.doi.org/10.20944/preprints201611.0145.v1.
[10] F. Qi and B.-N. Guo, *Viewing some ordinary differential equations from the angle of derivative polynomials*, Preprints **2016**, 2016100043, 12 pages; Available online at http://dx.doi.org/10.20944/preprints201610.0043.v1.
[11] F. Qi, D. Lim, and B.-N. Guo, *Some identities relating to Eulerian polynomials and involving Stirling numbers*, Preprints **2017**, 2017080004, 10 pages; Available online at http://dx.doi.org/10.20944/preprints201708.0004.v1.
[12] F. Qi, D. Lim, and B.-N. Guo, *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM (2018), in press.
[13] F. Qi, J.-L. Wang, and B.-N. Guo, *Notes on a family of inhomogeneous linear ordinary differential equations*, Preprints **2017**, 2017040026, 5 pages; Available online at http://dx.doi.org/10.20944/preprints201704.0026.v1.
[14] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying and finding nonlinear ordinary differential equations*, ResearchGate Working Paper (2017), available online at http://dx.doi.org/10.13140/RG.2.2.28855.32166.
[15] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying differential equations concerning degenerate Bernoulli and Euler numbers*, Trans. A. Razmadze Math. Inst. **171** (2017), no. 3, in press. ResearchGate Working Paper (2017), available online at http://dx.doi.org/10.13140/RG.2.2.12078.10566.
[16] F. Qi, Q. Zou, and B.-N. Guo, *Identities of the Chebyshev polynomials, the inverse of a triangular matrix, and identities of the Catalan numbers*, ResearchGate Working Paper (2017), available online at http://dx.doi.org/10.13140/RG.2.2.27344.71684.
[17] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*. The unpublished notes of H. W. Gould. With a foreword by George E. Andrews. World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.
[18] J.-L. Zhao, J.-L. Wang, and F. Qi, *Derivative polynomials of a function related to the Apostol–Euler and Frobenius–Euler numbers*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1345–1349; Available online at http://dx.doi.org/10.22436/jnsa.010.04.06.