SOME IDENTITIES RELATING TO EULERIAN POLYNOMIALS 
AND INVOLVING STIRLING NUMBERS

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Abstract. In the paper, the authors establish two identities, which can be 
regarded as nonlinear differential equations, for the generating function of 
Eulerian polynomials, find two identities for the Stirling numbers of the sec-
ond kind, and present two identities for Eulerian polynomials and higher order 
Eulerian polynomials, pose two open problems about summability of two fi-
nite sums involving the Stirling numbers of the second kind. Some of these 
conclusions meaningfully and significantly simplify several known results.

1. Motivations

In [6,7], Kims stated that Eulerian polynomials $A_n(t)$ for $n \geq 0$ can be generated by

$$\frac{1 - t}{e^{x(t-1)} - t} = \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!}, \quad t \neq 1$$

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and that higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ for integers $n \geq 0$ and real numbers $\alpha > 0$ can be generated by

$$\left[ \frac{1 - t}{e^{(t-1) - t}} \right]^\alpha = \sum_{n=0}^{\infty} A_n^{(\alpha)}(t) \frac{x^n}{n!}, \quad t \neq 1.$$  

This generation of $A_n(t)$ is same as the one in [11, p. 244, Eq. [5j]], but different from the one

$$\frac{1 - u}{e^{(u-1) - u}} = 1 + \sum_{n=1}^{\infty} A_n(u) \frac{t^n}{n!},$$

in [22, p. 244, Eq. [5j]].

In [7, Theorem 1], Kims established inductively and recurrently that the generating function

$$F(t, x) = \frac{1}{e^{(t-1) - t}}, \quad t \neq 1$$

satisfies the nonlinear ordinary differential equation

$$\frac{\partial^n F(t, x)}{\partial x^n} = (1 - t)^n \sum_{i=1}^{n+1} a_{i-1}(n, t) F^i(t, x), \quad n \in \{0\} \cup \mathbb{N},$$

where

$$a_0(n, t) = a_0(n - 1, t) = \cdots = a_0(1, t) = a_0(0, t) = 1$$

and

$$a_i(n, t) = i t \sum_{j=0}^{n-i} (i + 1)^j a_{i-1}(n - j - 1, t), \quad 1 \leq j \leq n.$$  

In [7] Theorems 2 and 3, Kims presented that

$$A_{k+n}(t) = (1 - t)^{n+1} \sum_{i=1}^{n+1} a_{i-1}(n, t) \frac{A_k^{(i)}(t)}{(1 - t)^i}$$

and

$$\sum_{j=0}^{\infty} t^j (j + 1)^{k+n} = \frac{1}{(1 - t)^k} \sum_{i=1}^{n+1} a_{i-1}(n, t) \frac{A_k^{(i)}(t)}{(1 - t)^i},$$

for $k, n \in \{0\} \cup \mathbb{N}$. From [3] and [4], Kims derived inductively that

$$a_i(n, t) = i t^i \sum_{j_i=0}^{n-i} \sum_{j_{i-1}=0}^{n-j_i-1} \cdots \sum_{j_1=0}^{n-j_{i-1}-\cdots-j_2-1} (i + 1)^{j_{i-1}}$$

$$\times j^{i-2} \cdots 2^{j_1} \left( 2^{n-j_i-j_{i-1}-\cdots-j_2} - 1 \right)$$

for $1 \leq i \leq n$.

It is clear that the above formulas [4] and [5] for $a_i(n, t)$ cannot be computed easily either by hand or by computer software. Can one find a simple expression for the quantities $a_i(n, t)$? For supplying a solution to this problem, the first and third authors obtained in [20] and its preprint [19] the following three theorems.

**Theorem 1.1** ([20, Theorem 1]). Eulerian polynomials $A_n(t)$ and higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ can be computed by

$$A_n(t) = \sum_{k=0}^{n} k! S(n, k) (t - 1)^{n-k}$$
and
\[
A_n^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \Gamma(k + \alpha) S(n, k) (t - 1)^{n-k},
\]
where \( n \geq 0 \) is an integer, \( \alpha > 0 \) is a real number, \( S(n, k) \), which can be generated by the exponential function
\[
\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}
\]
and can be computed by the explicit formula
\[
S(n, k) = \frac{1}{k!} \sum_{\ell=1}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n,
\]
stand for the Stirling numbers of the second kind, and \( \Gamma(z) \) denotes the classical gamma function which can be defined by
\[
\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{\prod_{k=0}^{n} (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}
\]
or
\[
\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0.
\]

**Theorem 1.2** ([20, Theorem 2]). The generating function \( F(t, x) \) satisfies the nonlinear ordinary differential equations
\[
\frac{\partial^n F(t, x)}{\partial x^n} = (t - 1)^n \sum_{i=0}^{n} \binom{n}{i} (-1)^{k-i} \binom{k}{i} \Gamma(k + \alpha) S(n, k) \left( \frac{k}{i} \right) t^{i} F^{i+1}(t, x)
\]
and, generally,
\[
\frac{\partial^n F^{(\alpha)}(t, x)}{\partial x^n} = \frac{(t - 1)^n}{\Gamma(\alpha)} \sum_{i=0}^{n} \binom{n}{i} (-1)^{k-i} \Gamma(k + \alpha) S(n, k) \left( \frac{k}{i} \right) t^{i} F^{i+\alpha}(t, x),
\]
where \( n \geq 0 \) is an integer and \( \alpha > 0 \) is a real number.

**Theorem 1.3** ([20, Theorem 3]). For \( n \in \mathbb{N} \) and \( \alpha > 0 \), the higher order Eulerian polynomials \( A_n^{(\alpha)}(t) \) satisfy the recurrence relation
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{\ell=0}^{n-k} S(n - k, \ell) \left( \frac{\alpha}{1-t} \right)^\ell A_k(t) \left( \frac{(1-t)^k}{(t-1)^k} \right) = 0,
\]
where
\[
(\alpha)_n = \prod_{k=0}^{n-1} (\alpha - k) = \begin{cases} \alpha(\alpha-1)\cdots(\alpha-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}
\]
is called the falling factorial. In particular, when \( \alpha = 1 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[ S(n-k, 0) + S(n-k, 1) \right] A_k(t) = 0.
\]
It is easy to see that the equation (8) simplifies the one (2). As stated in [20, Remarks 1 and 2], comparing the equation (2) with (8) reveals that
\[ a_i(n,t) = \left[ \sum_{k=i}^{n} (-1)^k k! S(n,k) \binom{k}{i} \right] t^i, \quad 0 \leq i \leq n. \tag{11} \]
This expression is simpler, more significant, more meaningful than the one in (5). It is easier to compute the quantities in the brackets of the nonlinear ordinary differential equations (8) and (9) than to compute the quantity \( a_i(n,t) \) in [7].

In [6, Theorem 1], it was obtained inductively and recursively that the nonlinear differential equations
\[ n! t^n (1-t)^n F^{n+1}(t,x) = \sum_{i=0}^{n} a_i(n)(1-t)^{n-i} \frac{\partial^i F(t,x)}{\partial x^i}, \quad n \in \mathbb{N} \tag{12} \]
have a solution \( F(t,x) \) defined by (1) for \( t \neq 1 \), where \( a_0(n) = (-1)^n n! \),
\[ a_i(n) = (-1)^{n-i} n! H_{n,i}, \quad 1 \leq i \leq n, \tag{13} \]
with \( H_{n,0} = 1 \) for \( n \in \mathbb{N} \), \( H_{n,1} = H_n = \sum_{k=1}^{n} \frac{1}{k} \) for \( n \in \mathbb{N} \), and
\[ H_{n,i} = \sum_{k=i}^{n} \frac{H_{k-1,i-1}}{k}, \quad 2 \leq i \leq n. \]

For more information on \( H_{n,i} \), please refer to [30, Remark 1], [32, Remark 1], and closely related references therein. Therefore, the following results were derived in [6, Theorems 2 and 3]:
(1) For \( n, k \geq 0 \),
\[ n! t^n A^{(n+1)}_k(t) = \sum_{i=0}^{n} a_i(n)(1-t)^{n-i} A_{k+i}(t). \]
(2) For \( k \geq 1 \),
\[ n! t^n A^{(n+1)}_k(t) = \sum_{m=0}^{k+n-1} \sum_{i=0}^{n} (1)^\ell \binom{n+k+1}{\ell} (m-\ell+1)^{k+i} a_i(n) t^m \]
and
\[ \sum_{i=0}^{n} \sum_{\ell=0}^{m} (1)^\ell \binom{n+k+1}{\ell} (m-\ell+1)^{k+i} a_i(n) = 0 \]
for \( m \geq k+n \).
(3) For \( k = 0 \),
\[ n! t^n A^{(n+1)}_0(t) = \sum_{m=0}^{n} \sum_{i=0}^{n} \sum_{\ell=0}^{m} (1)^\ell \binom{n+1}{\ell} (m-\ell+1)^{i} a_i(n) t^m \]
and
\[ \sum_{i=0}^{n} \sum_{\ell=0}^{m} (1)^\ell \binom{n+1}{\ell} (m-\ell+1)^{i} a_i(n) = 0 \]
for \( m \geq n+1 \).
We observe that the equations (2) and (12) can be rewritten respectively as
\[
\frac{1}{(1-t)^n} \frac{\partial^n F(t,x)}{\partial x^n} = \sum_{i=0}^{n} a_i(n,t) F^{i+1}(t,x), \quad n \in \{0\} \cup \mathbb{N}
\]
and
\[
n! t^n F^{n+1}(t,x) = \sum_{i=0}^{n} a_i(n) \frac{1}{(1-t)^i} \frac{\partial^i F(t,x)}{\partial x^i}, \quad n \in \{0\} \cup \mathbb{N}.
\]
Consequently, the equations (2) and (12) are essentially inversive each other. This motivates us to consider two questions:

1) can one simplify the expression of the quantities \(a_i(n)\) significantly and meaningfully?

2) what are the inversive ones of the equations (8) and (9)?

2. Main results and their proofs

Now we are in a position to state and prove our main results.

**Theorem 2.1.** For \(n \geq 0\), the function \(F(t,x)\) defined by (1) satisfies nonlinear differential equations
\[
F^{n+1}(t,x) = \frac{1}{n! t^n} \sum_{i=0}^{n} s(n+1,i+1) \frac{\partial^i F(t,x)}{\partial x^i}, \quad (15)
\]
where \(s(n,k)\), which can be generated by
\[
\left[\ln(1+x)\right]^k \frac{x^n}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1,
\]
stand for the Stirling numbers of the first kind.

*Proof.* In [24, 25] and [30, 32], it was obtained that
\[
(-1)^{n+k} s(n,k) = (n-1)! H_{n-1,k-1}, \quad n \geq k \geq 1.
\]
Substituting the relations (13) and (16) into (12) or (14) yields
\[
n! t^n F^{n+1}(t,x) = \sum_{i=0}^{n} (-1)^{n-i} n! H_{n,i} \frac{1}{(1-t)^i} \frac{\partial^i F(t,x)}{\partial x^i}
\]
\[
= \sum_{i=0}^{n} s(n+1,i+1) \frac{1}{(1-t)^i} \frac{\partial^i F(t,x)}{\partial x^i}
\]
for \(n \geq 0\). The proof of Theorem 2.1 is complete. \(\square\)

**Theorem 2.2.** For \(n \geq 0\), the function \(F(t,x)\) defined by (1) satisfies nonlinear differential equations
\[
\frac{\partial^n F(t,x)}{\partial x^n} = (1-t)^n \sum_{k=0}^{n} S(n+1,k+1) k! t^k F^{k+1}(t,x). \quad (17)
\]
Consequently, identities
\[
\sum_{\ell=k}^{n} (-1)^{\ell} \binom{\ell}{k} \ell! S(n,\ell) = (-1)^n k! S(n+1,k+1), \quad n \geq k \geq 0
\]
and
\[ \sum_{\ell=k}^{n} (-1)^\ell \binom{\ell - 1}{k} \ell! S(n, \ell) = (-1)^n (k + 1)! S(n, k + 1), \quad n > k \geq 0 \quad (19) \]
are true under conventions \( \binom{\ell}{p} = 0 \) for \( q > p \) and \( S(p, q) = 0 \) for \( q > p \geq 0 \).

**Proof.** Rewriting the equations in [15] as
\[
\begin{pmatrix}
F(t, x) \\
tF^2(t, x) \\
2t^2F^3(t, x) \\
\vdots \\
n!t^nF^{n+1}(t, x)
\end{pmatrix} = M_{(n+1) \times (n+1)} \begin{pmatrix}
\frac{1}{1-t} \frac{\partial F(t, x)}{\partial x} \\
\frac{1}{1-t} \frac{\partial^2 F(t, x)}{\partial x^2} \\
\vdots \\
\frac{1}{1-t} \frac{\partial^n F(t, x)}{\partial x^n}
\end{pmatrix},
\]
where the \((n+1) \times (n+1)\) matrix
\[
M_{(n+1) \times (n+1)} = \begin{pmatrix}
s(1, 1) & 0 & 0 & \cdots & 0 \\
s(2, 1) & s(2, 2) & 0 & \cdots & 0 \\
s(3, 1) & s(3, 2) & s(3, 3) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s(n+1, 1) & s(n+1, 2) & s(n+1, 3) & \cdots & s(n+1, n+1)
\end{pmatrix},
\]
Since the inverse matrix
\[
M^{-1}_{(n+1) \times (n+1)} = \begin{pmatrix}
S(1, 1) & 0 & 0 & \cdots & 0 \\
S(2, 1) & S(2, 2) & 0 & \cdots & 0 \\
S(3, 1) & S(3, 2) & S(3, 3) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S(n+1, 1) & S(n+1, 2) & S(n+1, 3) & \cdots & S(n+1, n+1)
\end{pmatrix},
\]
see [2] p. 213, eq. [5c]], it follows immediately that
\[
\begin{pmatrix}
\frac{F(t, x)}{1-t} \frac{\partial F(t, x)}{\partial x} \\
\frac{1}{1-t} \frac{\partial^2 F(t, x)}{\partial x^2} \\
\vdots \\
\frac{1}{1-t} \frac{\partial^n F(t, x)}{\partial x^n}
\end{pmatrix} = M^{-1}_{(n+1) \times (n+1)} \begin{pmatrix}
F(t, x) \\
tF^2(t, x) \\
2t^2F^3(t, x) \\
\vdots \\
n!t^nF^{n+1}(t, x)
\end{pmatrix},
\]
that is,
\[
\frac{1}{(1-t)^n} \frac{\partial^n F(t, x)}{\partial x^n} = \sum_{k=1}^{n+1} S(n+1, k)(k-1)!t^{k-1} F^k(t, x), \quad n \geq 0.
\]
This is equivalent to [17].

Comparing [17] with (8) leads to [18].

In [34] p. 118, Eq. (9.18)], it was proved that
\[
\sum_{j=\alpha}^{n} (-1)^{n-j} \binom{j-1}{\alpha-1} j! S(n, j) = \alpha! S(n, \alpha), \quad n \geq \alpha \geq 0.
\quad (20)
By virtue of \((18)\) and \((20)\), we obtain
\[
\sum_{\ell=k}^{n} (-1)^{\ell} \left[ \binom{\ell}{k} - \binom{\ell - 1}{k - 1} \right] \ell! S(n, \ell) = (-1)^n k! [S(n + 1, k + 1) - S(n, k)].
\]

Since the recurrence relations
\[
\begin{align*}
(x_j) &= x_{j-1} + (x_{j-1})_j, & x \in \mathbb{C}, & j \in \{0\} \cup \mathbb{N}
\end{align*}
\]
and
\[
S(n + 1, k) = kS(n, k) + S(n, k - 1), & 0 \leq k - 1 \leq n,
\]
see \[34, p. 8, Eq. (1.27)\] and \[34, p. 114, Eq. (9.1)\], we arrive at the identity \((19)\) straightforwardly. The proof of Theorem 2.2 is complete. \(\square\)

**Theorem 2.3.** For \(n \geq 0\), Eulerian polynomials \(A_n(t)\) and higher order Eulerian polynomials \(A_k^{(\alpha)}(t)\) satisfy
\[
\sum_{k=0}^{n} s(n,k) \frac{A_k(t)}{(t-1)^k} = \frac{n!}{(t-1)^n}.
\] \((21)\)

and
\[
\sum_{k=0}^{n} s(n,k) \frac{A_k^{(\alpha)}(t)}{(t-1)^k} = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)} \frac{1}{(t-1)^n}.
\] \((22)\)

**Proof.** Theorem 12.1 in \[34, p. 171\] reads that, if \(b_\alpha\) and \(a_k\) are a collection of constants independent of \(n\), then
\[
a_n = \sum_{\alpha=0}^{n} S(n,\alpha)b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^{n} s(n,k)a_k.
\]
The identity \((\square)\) can be rearranged as
\[
\frac{A_n(t)}{(t-1)^n} = \sum_{k=0}^{n} S(n,k) \frac{k!}{(t-1)^k}.
\]

Consequently, it follows that
\[
\frac{n!}{(t-1)^n} = \sum_{k=0}^{n} s(n,k) \frac{A_k(t)}{(t-1)^k}
\]
which can be rewritten as \((21)\).

Similarly, the identity \((7)\) can also be reformulated as
\[
\frac{\Gamma(\alpha)A_n^{(\alpha)}(t)}{(t-1)^n} = \sum_{k=0}^{n} S(n,k) \frac{\Gamma(k + \alpha)}{(t-1)^k}
\]
and, consequently,
\[
\frac{\Gamma(n + \alpha)}{(t-1)^n} = \sum_{k=0}^{n} s(n,k) \frac{\Gamma(\alpha)A_k^{(\alpha)}(t)}{(t-1)^k}
\]
which can be rearranged as \((22)\). \(\square\)
3. Remarks

In this section, among other things, we give several remarks about our main results and pose two open problems.

Remark 3.1. The expressions (11) and (13) for \( a_i(n,t) \) and \( a_i(n) \) can be meaningfully and significantly simplified as
\[
a_i(n,t) = (-1)^n t S(n+1,i+1), \quad 0 \leq i \leq n
\]
and
\[
a_i(n) = s(n+1,i+1).
\]
As a result, those main results in [6, 7], mentioned in the first section, can be meaningfully and significantly simplified. For the sake of saving the space and shortening the length of this paper, we do not rewrite them in details here. This answers the first question posed in the first section of this paper.

Remark 3.2. The identity (15) in Theorem 2.1 partially answers the second question posed in the first section of this paper.

Remark 3.3. Theorem 1.3 is a simplified and reformulated version of Theorem 3 in [20].

Remark 3.4. The identity (17) in Theorem 2.2 simplifies the one (8) in Theorem 1.2.

Remark 3.5. To the best of our knowledge, identities (18) and (19) are new.

Remark 3.6. Motivated by identities (18), (19), and (20), we naturally pose another open problem: for \( n \geq k \geq 0 \) and \( \alpha > 0 \), is the finite sum
\[
\sum_{k=1}^{n} (-1)^k \binom{k}{i} \Gamma(k + \alpha) S(n,k)
\]
in the bracket of the identity (9) summable? The solution of this problem can be used to partially answer the second question posed in the first section of this paper.

Remark 3.7. Theorem 12.2 in [34, p. 171] states that, if \( b_j \) and \( a_k \) are a collection of constants which are independent of \( n \) and if \( \alpha \) is a nonnegative integer such that \( \alpha \geq n \), then
\[
a_n = \sum_{j=0}^{\alpha} S(j,n) b_j \quad \text{if and only if} \quad b_n = \sum_{k=0}^{\alpha} s(k,n) a_k.
\]
Motivated by the above mentioned Theorems 12.1 and 12.2 in [34, p. 171] and by identities (18), (19), and (20), we naturally pose an open problem: is the finite sum
\[
\sum_{\ell=0}^{n-k} S(n-k,\ell) \frac{(\alpha)\ell}{(1-t)^{\ell}}
\]
in the bracket of the identity (10), or equivalently,
\[
\sum_{\ell=0}^{n} S(n,\ell) \frac{(\alpha)\ell}{(1-t)^{\ell}},
\]
summable?

Remark 3.8. For some new results on the gamma function \( \Gamma(z) \), please refer to [8, 16, 18, 21, 27] and closely-related references therein.
Remark 3.9. For some recent development of the Stirling numbers of the first and second kinds, please refer to [3, 4, 5, 9, 10, 11, 12, 13, 15, 17, 28] and closely related references therein.

Remark 3.10. The motivation of this paper is same as the one in [14, 22, 23, 25, 26, 29, 31, 33, 35] and closely related references therein.

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