SOME IDENTITIES RELATING TO EULERIAN POLYNOMIALS AND INVOLVING STIRLING NUMBERS

FENG QI

Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

DONGKYU LIM

School of Mathematical Sciences, Nankai University, Tianjin Ciy, 300071, China

BAI-NI GUO

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

Abstract. In the paper, the authors establish two identities, which can be regarded as nonlinear differential equations, for the generating function of Eulerian polynomials, find two identities for the Stirling numbers of the second kind, and present two identities for Eulerian polynomials and higher order Eulerian polynomials, pose two open problems about summability of two finite sums involving the Stirling numbers of the second kind. Some of these conclusions meaningfully and significantly simplify several known results.

1. Motivations

In [6, 7], Kims stated that Eulerian polynomials $A_n(t)$ for $n \ge 0$ can be generated by

$$\frac{1-t}{e^{x(t-1)}-t} = \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}, \quad t \ne 1$$

E-mail addresses: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com, dgrim84@gmail.com, bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com.

2010 *Mathematics Subject Classification.* Primary 11B73; Secondary 05A15, 11B68, 11B83, 11C08, 33B10.

Key words and phrases. identity; generating function; Eulerian polynomial; higher order Eulerian polynomial; Stirling number; open problem.

This paper was typeset using $\mathcal{AMS}$-LaTeX.

Peer-reviewed version available at *Applicable Analysis and Discrete Mathematics* **2018**, *12*, 1-14; doi:10.2298/AADM171008014Q

and that higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ for integers $n \ge 0$ and real numbers $\alpha > 0$ can be generated by

$$\left[\frac{1-t}{e^{x(t-1)}-t}\right]^\alpha = \sum_{n=0}^\infty A_n^{(\alpha)}(t)\frac{x^n}{n!}, \quad t \ne 1.$$

This generation of $A_n(t)$ is same as the one in [1, p. 2], but different from the one

$$\frac{1-u}{e^{t(u-1)}-u} = 1 + \sum_{n=1}^\infty \frac{A_n(u)}{u}\frac{t^n}{n!}.$$

in [2, p. 244, Eq. [5j]].

In [7, Theorem 1], Kims established inductively and recurrently that the generating function

$$F(t,x) = \frac{1}{e^{x(t-1)}-t}, \quad t \ne 1 \tag{1}$$

satisfies the nonlinear ordinary differential equation

$$\frac{\partial^n F(t,x)}{\partial x^n} = (1-t)^n \sum_{i=1}^{n+1} a_{i-1}(n,t) F^i(t,x), \quad n \in \{0\} \cup \mathbb{N}, \tag{2}$$

where

$$a_0(n,t) = a_0(n-1,t) = \cdots = a_0(1,t) = a_0(0,t) = 1 \tag{3}$$

and

$$a_i(n,t) = it\sum_{j=0}^{n-i}(i+1)^j a_{i-1}(n-j-1,t), \quad 1 \le j \le n. \tag{4}$$

In [7, Theorems 2 and 3], Kims presented that

$$A_{k+n}(t) = (1-t)^{n+1}\sum_{i=1}^{n+1} a_{i-1}(n,t)\frac{A_k^{(i)}(t)}{(1-t)^i}$$

and

$$\sum_{j=0}^\infty t^j(j+1)^{k+n} = \frac{1}{(1-t)^k}\sum_{i=1}^{n+1} a_{i-1}(n,t)\frac{A_k^{(i)}(t)}{(1-t)^i}$$

for $k,n \in \{0\} \cup \mathbb{N}$. From (3) and (4), Kims derived inductively that

$$\begin{aligned} a_i(n,t) = i!t^i \sum_{j_{i-1}=0}^{n-i} \sum_{j_{i-2}=0}^{n-j_{i-1}-i} \cdots \sum_{j_1=0}^{n-j_{i-1}-\cdots-j_2-i} (i+1)^{j_{i-1}} \\ \times i^{j_{i-2}} \cdots 3^{j_1}\left(2^{n-j_{i-1}-j_{i-2}-\cdots-j_1-i+1}-1\right) \end{aligned} \tag{5}$$

for $1 \le i \le n$.

It is clear that the above formulas (4) and (5) for $a_i(n,t)$ cannot be computed easily either by hand or by computer software. Can one find a simple expression for the quantities $a_i(n,t)$? For supplying a solution to this problem, the first and third authors obtained in [20] and its preprint [19] the following three theorems.

Theorem 1.1 ([20, Theorem 1]). *Eulerian polynomials $A_n(t)$ and higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ can be computed by*

$$A_n(t) = \sum_{k=0}^n k!S(n,k)(t-1)^{n-k} \tag{6}$$

and

$$A_n^{(\alpha)}(t)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}\Gamma(k+\alpha)S(n,k)(t-1)^{n-k}, \tag{7}$$

where $n\ge 0$ is an integer, $\alpha>0$ is a real number, $S(n,k)$, which can be generated by the exponential function

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}$$

and can be computed by the explicit formula

$$S(n,k)=\frac{1}{k!}\sum_{\ell=1}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n,$$

stand for the Stirling numbers of the second kind, and $\Gamma(z)$ denotes the classical gamma function which can be defined by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$$

or

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t, \quad \Re(z)>0.$$

Theorem 1.2 ([20, Theorem 2])**.** *The generating function $F(t,x)$ satisfies the nonlinear ordinary differential equations*

$$\frac{\partial^n F(t,x)}{\partial x^n}=(t-1)^n\sum_{i=0}^{n}\left[\sum_{k=i}^{n}(-1)^k k!S(n,k)\binom{k}{i}\right]t^iF^{i+1}(t,x) \tag{8}$$

and, generally,

$$\frac{\partial^n F^\alpha(t,x)}{\partial x^n}=\frac{(t-1)^n}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\sum_{k=i}^{n}(-1)^k\Gamma(k+\alpha)S(n,k)\binom{k}{i}\right]t^iF^{\alpha+i}(t,x), \tag{9}$$

where $n\ge 0$ is an integer and $\alpha>0$ is a real number.

Theorem 1.3 ([20, Theorem 3])**.** *For $n\in\mathbb{N}$ and $\alpha>0$, the higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ satisfy the recurrence relation*

$$\sum_{k=0}^{n}\binom{n}{k}\left[\sum_{\ell=0}^{n-k}S(n-k,\ell)\frac{\langle\alpha\rangle_\ell}{(1-t)^\ell}\right]\frac{A_k^{(\alpha)}(t)}{(t-1)^k}=0, \tag{10}$$

where

$$\langle\alpha\rangle_n=\prod_{k=0}^{n-1}(\alpha-k)=\begin{cases}\alpha(\alpha-1)\cdots(\alpha-n+1), & n\ge 1\\ 1, & n=0\end{cases}$$

is called the falling factorial. In particular, when $\alpha=1$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\frac{S(n-k,0)}{(1-t)^{k-1}}+\frac{S(n-k,1)}{(1-t)^k}\right]A_k(t)=0.$$

Peer-reviewed version available at *Applicable Analysis and Discrete Mathematics* **2018**, *12*, 1-14; doi:10.2298/AADM171008014Q

It is easy to see that the equation (8) simplifies the one (2). As stated in [20, Remarks 1 and 2], comparing the equation (2) with (8) reveals that

$$a_i(n,t)=\left[\sum_{k=i}^{n}(-1)^k k!S(n,k)\binom{k}{i}\right]t^i, \quad 0\le i\le n. \tag{11}$$

This expression is simpler, more significant, more meaningful than the one in (5). It is easier to compute the quantities in the brackets of the nonlinear ordinary differential equations (8) and (9) than to compute the quantity $a_i(n,t)$ in [7].

In [6, Theorem 1], it was obtained inductively and recursively that the nonlinear differential equations

$$n!t^n(1-t)^nF^{n+1}(t,x)=\sum_{i=0}^{n}a_i(n)(1-t)^{n-i}\frac{\partial^i F(t,x)}{\partial x^i}, \quad n\in\mathbb{N} \tag{12}$$

have a solution $F(t,x)$ defined by (1) for $t\ne 1$, where $a_0(n)=(-1)^n n!$,

$$a_i(n)=(-1)^{n-i}n!H_{n,i}, \quad 1\le i\le n, \tag{13}$$

with $H_{n,0}=1$ for $n\in\mathbb{N}$, $H_{n,1}=H_n=\sum_{k=1}^{n}\frac{1}{k}$ for $n\in\mathbb{N}$, and

$$H_{n,i}=\sum_{k=i}^{n}\frac{H_{k-1,i-1}}{k}, \quad 2\le i\le n.$$

For more information on $H_{n,i}$, please refer to [30, Remark 1], [32, Remark 1], and closely related references therein. Therefore, the following results were derived in [6, Theorems 2 and 3]:

(1) For $n,k\ge 0$,

$$n!t^n A_k^{(n+1)}(t)=\sum_{i=0}^{n}a_i(n)(1-t)^{n-i}A_{k+i}(t).$$

(2) For $k\ge 1$,

$$n!t^n A_k^{(n+1)}(t)=\sum_{m=0}^{k+n-1}\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^\ell\binom{n+k+1}{\ell}(m-\ell+1)^{k+i}a_i(n)t^m$$

and

$$\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^\ell\binom{n+k+1}{\ell}(m-\ell+1)^{k+i}a_i(n)=0$$

for $m\ge k+n$.

(3) For $k=0$,

$$n!t^n A_0^{(n+1)}(t)=\sum_{m=0}^{n}\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^\ell\binom{n+1}{\ell}(m-\ell+1)^{i}a_i(n)t^m$$

and

$$\sum_{i=0}^{n}\sum_{\ell=0}^{m}(-1)^\ell\binom{n+1}{\ell}(m-\ell+1)^{i}a_i(n)=0$$

for $m\ge n+1$.

We observe that the equations (2) and (12) can be rewritten respectively as

$$\frac{1}{(1-t)^n}\frac{\partial^n F(t,x)}{\partial x^n}=\sum_{i=0}^{n}a_i(n,t)F^{i+1}(t,x), \quad n\in\{0\}\cup\mathbb{N}$$

and

$$n!t^nF^{n+1}(t,x)=\sum_{i=0}^{n}a_i(n)\frac{1}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i}, \quad n\in\{0\}\cup\mathbb{N}. \tag{14}$$

Consequently, the equations (2) and (12) are essentially inversive each other. This motivates us to consider two questions:

(1) can one simplify the expression of the quantities $a_i(n)$ significantly and meaningfully?
(2) what are the inversive ones of the equations (8) and (9)?

2. Main results and their proofs

Now we are in a position to state and prove our main results.

Theorem 2.1. *For $n\ge0$, the function $F(t,x)$ defined by* (1) *satisfies nonlinear differential equations*

$$F^{n+1}(t,x)=\frac{1}{n!t^n}\sum_{i=0}^{n}\frac{s(n+1,i+1)}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i}, \tag{15}$$

where $s(n,k)$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!}, \quad |x|<1,$$

stand for the Stirling numbers of the first kind.

Proof. In [24, 25] and [30, 32], it was obtained that

$$(-1)^{n+k}s(n,k)=(n-1)!H_{n-1,k-1}, \quad n\ge k\ge1. \tag{16}$$

Substututing the relations (13) and (16) into (12) or (14) yields

$$\begin{aligned} n!t^nF^{n+1}(t,x)&=\sum_{i=0}^{n}(-1)^{n-i}n!H_{n,i}\frac{1}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i}\\ &=\sum_{i=0}^{n}s(n+1,i+1)\frac{1}{(1-t)^i}\frac{\partial^i F(t,x)}{\partial x^i}\end{aligned}$$

for $n\ge0$. The proof of Theorem 2.1 is complete. □

Theorem 2.2. *For $n\ge0$, the function $F(t,x)$ defined by* (1) *satisfies nonlinear differential equations*

$$\frac{\partial^n F(t,x)}{\partial x^n}=(1-t)^n\sum_{k=0}^{n}S(n+1,k+1)k!t^kF^{k+1}(t,x). \tag{17}$$

Consequently, identities

$$\sum_{\ell=k}^{n}(-1)^\ell\binom{\ell}{k}\ell!S(n,\ell)=(-1)^nk!S(n+1,k+1), \quad n\ge k\ge0 \tag{18}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2017 doi:10.20944/preprints201708.0004.v1

Peer-reviewed version available at *Applicable Analysis and Discrete Mathematics* **2018**, *12*, 1-14; doi:10.2298/AADM171008014Q

and

$$\sum_{\ell=k}^{n}(-1)^{\ell}\binom{\ell-1}{k}\ell!S(n,\ell)=(-1)^{n}(k+1)!S(n,k+1),\quad n>k\ge0 \tag{19}$$

are true under conventions $\binom{p}{q}=0$ *for* $q>p$ *and* $S(p,q)=0$ *for* $q>p\ge0$.

Proof. Rewriting the equations in (15) as

$$\begin{pmatrix} F(t,x)\\ tF^2(t,x)\\ 2!t^2F^3(t,x)\\ \vdots\\ n!t^nF^{n+1}(t,x)\end{pmatrix}=M_{(n+1)\times(n+1)}\begin{pmatrix} F(t,x)\\ \frac{1}{1-t}\frac{\partial F(t,x)}{\partial x}\\ \frac{1}{(1-t)^2}\frac{\partial^2 F(t,x)}{\partial x^2}\\ \vdots\\ \frac{1}{(1-t)^n}\frac{\partial^n F(t,x)}{\partial x^n}\end{pmatrix},$$

where the $(n+1)\times(n+1)$ matrix

$$M_{(n+1)\times(n+1)}=\begin{pmatrix} s(1,1) & 0 & 0 & \cdots & 0\\ s(2,1) & s(2,2) & 0 & \cdots & 0\\ s(3,1) & s(3,2) & s(3,3) & \cdots & 0\\ \cdots & \cdots & \cdots & \ddots & \cdots\\ s(n+1,1) & s(n+1,2) & s(n+1,3) & \cdots & s(n+1,n+1)\end{pmatrix}.$$

Since the inverse matrix

$$M_{(n+1)\times(n+1)}^{-1}=\begin{pmatrix} S(1,1) & 0 & 0 & \cdots & 0\\ S(2,1) & S(2,2) & 0 & \cdots & 0\\ S(3,1) & S(3,2) & S(3,3) & \cdots & 0\\ \cdots & \cdots & \cdots & \ddots & \cdots\\ S(n+1,1) & S(n+1,2) & S(n+1,3) & \cdots & S(n+1,n+1)\end{pmatrix},$$

see [2, p. 213, eq. [5c]], it follows immediately that

$$\begin{pmatrix} F(t,x)\\ \frac{1}{1-t}\frac{\partial F(t,x)}{\partial x}\\ \frac{1}{(1-t)^2}\frac{\partial^2 F(t,x)}{\partial x^2}\\ \vdots\\ \frac{1}{(1-t)^n}\frac{\partial^n F(t,x)}{\partial x^n}\end{pmatrix}=M_{(n+1)\times(n+1)}^{-1}\begin{pmatrix} F(t,x)\\ tF^2(t,x)\\ 2!t^2F^3(t,x)\\ \vdots\\ n!t^nF^{n+1}(t,x)\end{pmatrix},$$

that is,

$$\frac{1}{(1-t)^n}\frac{\partial^n F(t,x)}{\partial x^n}=\sum_{k=1}^{n+1}S(n+1,k)(k-1)!t^{k-1}F^k(t,x),\quad n\ge0.$$

This is equivalent to (17).

Comparing (17) with (8) leads to (18).

In [34, p. 118, Eq. (9.18)], it was proved that

$$\sum_{j=\alpha}^{n}(-1)^{n-j}\binom{j-1}{\alpha-1}j!S(n,j)=\alpha!S(n,\alpha),\quad n\ge\alpha\ge0. \tag{20}$$

Peer-reviewed version available at *Applicable Analysis and Discrete Mathematics* **2018**, *12*, 1-14; doi:10.2298/AADM171008014Q

By virtue of (18) and (20), we obtain

$$\sum_{\ell=k}^{n}(-1)^{\ell}\left[\binom{\ell}{k}-\binom{\ell-1}{k-1}\right]\ell!S(n,\ell)=(-1)^{n}k![S(n+1,k+1)-S(n,k)].$$

Since the recurrence relations

$$\binom{x}{j}=\binom{x-1}{j}+\binom{x-1}{j-1},\quad x\in\mathbb{C},\quad j\in\{0\}\cup\mathbb{N}$$

and

$$S(n+1,k)=kS(n,k)+S(n,k-1),\quad 0\le k-1\le n,$$

see [34, p. 8, Eq. (1.27)] and [34, p. 114, Eq. (9.1)], we arrive at the identity (19) straightforwardly. The proof of Theorem 2.2 is complete. □

Theorem 2.3. *For $n\ge 0$, Eulerian polynomials $A_n(t)$ and higher order Eulerian polynomials $A_k^{(\alpha)}(t)$ satisfy*

$$\sum_{k=0}^{n}\frac{s(n,k)}{(t-1)^{k}}A_k(t)=\frac{n!}{(t-1)^{n}}\tag{21}$$

and

$$\sum_{k=0}^{n}\frac{s(n,k)}{(t-1)^{k}}A_k^{(\alpha)}(t)=\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}\frac{1}{(t-1)^{n}}.\tag{22}$$

Proof. Theorem 12.1 in [34, p. 171] reads that, if b_α and a_k are a collection of constants independent of n, then

$$a_n=\sum_{\alpha=0}^{n}S(n,\alpha)b_\alpha\quad\text{if and only if}\quad b_n=\sum_{k=0}^{n}s(n,k)a_k.$$

The identity (6) can be rearranged as

$$\frac{A_n(t)}{(t-1)^{n}}=\sum_{k=0}^{n}S(n,k)\frac{k!}{(t-1)^{k}}.$$

Consequently, it follows that

$$\frac{n!}{(t-1)^{n}}=\sum_{k=0}^{n}s(n,k)\frac{A_k(t)}{(t-1)^{k}}$$

which can be rewritten as (21).

Similarly, the identity (7) can also be reformulated as

$$\frac{\Gamma(\alpha)A_n^{(\alpha)}(t)}{(t-1)^{n}}=\sum_{k=0}^{n}S(n,k)\frac{\Gamma(k+\alpha)}{(t-1)^{k}}$$

and, consequently,

$$\frac{\Gamma(n+\alpha)}{(t-1)^{n}}=\sum_{k=0}^{n}s(n,k)\frac{\Gamma(\alpha)A_k^{(\alpha)}(t)}{(t-1)^{k}}$$

which can be rearranged as (22). □

Peer-reviewed version available at *Applicable Analysis and Discrete Mathematics* **2018**, *12*, 1-14; doi:10.2298/AADM171008014Q

3. Remarks

In this section, among other things, we give several remarks about our main results and pose two open problems.

Remark 3.1. The expressions (11) and (13) for $a_i(n,t)$ and $a_i(n)$ can be meaningfully and significantly simplified as

$$a_i(n,t) = (-1)^n i! S(n+1,i+1)t^i, \quad 0 \le i \le n$$

and

$$a_i(n) = s(n+1,i+1).$$

As a result, those main results in [6, 7], mentioned in the first section, can be meaningfully and significantly simplified. For the sake of saving the space and shortening the length of this paper, we do not rewrite them in details here. This answers the first question posed in the first section of this paper.

Remark 3.2. The identity (15) in Theorem 2.1 partially answers the second question posed in the first section of this paper.

Remark 3.3. Theorem 1.3 is a simplified and reformulated version of Theorem 3 in [20].

Remark 3.4. The identity (17) in Theorem 2.2 simplifies the one (8) in Theorem 1.2.

Remark 3.5. To the best of our knowledge, identities (18) and (19) are new.

Remark 3.6. Motivated by identities (18), (19), and (20), we naturally pose another open problem: for $n \ge k \ge 0$ and $\alpha > 0$, is the finite sum

$$\sum_{k=i}^{n} (-1)^k \binom{k}{i} \Gamma(k+\alpha) S(n,k)$$

in the bracket of the identity (9) summable? The solution of this problem can be used to partially answer the second question posed in the first section of this paper.

Remark 3.7. Theorem 12.2 in [34, p. 171] states that, if b_j and a_k are a collection of constants which are independent of n and if α is a nonnegative integer such that $\alpha \ge n$, then

$$a_n = \sum_{j=0}^{\alpha} S(j,n) b_j \quad \text{if and only if} \quad b_n = \sum_{k=0}^{\alpha} s(k,n) a_k.$$

Motivated by the above mentioned Theorems 12.1 and 12.2 in [34, p. 171] and by identities (18), (19), and (20), we naturally pose an open problem: is the finite sum

$$\sum_{\ell=0}^{n-k} S(n-k,\ell) \frac{\langle\alpha\rangle_\ell}{(1-t)^\ell}$$

in the bracket of the identity (10), or equivalently,

$$\sum_{\ell=0}^{n} S(n,\ell) \frac{\langle\alpha\rangle_\ell}{(1-t)^\ell},$$

summable?

Remark 3.8. For some new results on the gamma function $\Gamma(z)$, please refer to [8, 16, 18, 21, 27] and closely-related references therein.

Peer-reviewed version available at *Applicable Analysis and Discrete Mathematics* **2018**, *12*, 1-14; doi:10.2298/AADM171008014Q

Remark 3.9. For some recent development of the Stirling numbers of the first and send kinds, please refer to [3, 4, 5, 9, 10, 11, 12, 13, 15, 17, 28] and closely related references therein.

Remark 3.10. The motivation of this paper is same as the one in [14, 22, 23, 25, 26, 29, 31, 33, 35] and closely related references therein.

References

[1] P. Barry, *Eulerian polynomials as moments, via exponential Riordan arrays*, J. Integer Seq. **14** (2011), no. 9, Article 11.9.5, 14 pages.

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.

[3] B.-N. Guo, I. Mező, and F. Qi, *An explicit formula for the Bernoulli polynomials in terms of the r-Stirling numbers of the second kind*, Rocky Mountain J. Math. **46** (2016), no. 6, 1919–1923; Available online at `http://dx.doi.org/10.1216/RMJ-2016-46-6-1919`.

[4] B.-N. Guo and F. Qi, *Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind*, J. Comput. Appl. Math. **272** (2014), 251–257; Available online at `http://dx.doi.org/10.1016/j.cam.2014.05.018`.

[5] B.-N. Guo and F. Qi, *Some identities and an explicit formula for Bernoulli and Stirling numbers*, J. Comput. Appl. Math. **255** (2014), 568–579; Available online at `http://dx.doi.org/10.1016/j.cam.2013.06.020`.

[6] D. S. Kim and T. Kim, *Revisit nonlinear differential equations associate with Eulerian polynomials*, Bull. Korean Math. Soc. **54** (2017), no. 4, 1185–1194; Available online at `https://doi.org/10.4134/BKMS.b160276`.

[7] T. Kim and D. S. Kim, *Some identities of Eulerian polynomials arising from nonlinear differential equations*, Iran. J. Sci. Technol. Trans. Sci. (2016), in press; Available online at `http://dx.doi.org/10.1007/s40995-016-0073-0`.

[8] F. Qi, *A completely monotonic function involving the gamma and trigamma functions*, Kuwait J. Sci. **43** (2016), no. 3, 32–40.

[9] F. Qi, *A new formula for the Bernoulli numbers of the second kind in terms of the Stirling numbers of the first kind*, Publ. Inst. Math. (Beograd) (N.S.) **100 (114)** (2016), 243–249; Available online at `http://dx.doi.org/10.2298/PIM150501028Q`.

[10] F. Qi, *An explicit formula for the Bell numbers in terms of the Lah and Stirling numbers*, Mediterr. J. Math. **13** (2016), no. 5, 2795–2800; Available online at `http://dx.doi.org/10.1007/s00009-015-0655-7`.

[11] F. Qi, *Diagonal recurrence relations, inequalities, and monotonicity related to the Stirling numbers of the second kind*, Math. Inequal. Appl. **19** (2016), no. 1, 313–323; Available online at `http://dx.doi.org/10.7153/mia-19-23`.

[12] F. Qi, *Diagonal recurrence relations for the Stirling numbers of the first kind*, Contrib. Discrete Math. **11** (2016), no. 1, 22–30; Available online at `http://hdl.handle.net/10515/sy5wh2dx6` and `http://dx.doi.org/10515/sy5wh2dx6`.

[13] F. Qi, *Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind*, Filomat **28** (2014), no. 2, 319–327; Available online at `http://dx.doi.org/10.2298/FIL1402319O`.

[14] F. Qi, *Explicit formulas for the convolved Fibonacci numbers*, ResearchGate Working Paper (2016), available online at `http://dx.doi.org/10.13140/RG.2.2.36768.17927`.

[15] F. Qi, *Integral representations and properties of Stirling numbers of the first kind*, J. Number Theory **133** (2013), no. 7, 2307–2319; Available online at `http://dx.doi.org/10.1016/j.jnt.2012.12.015`.

[16] F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604; Available online at `http://dx.doi.org/10.2298/FIL1304601Q`.

[17] F. Qi and B.-N. Guo, *A closed form for the Stirling polynomials in terms of the Stirling numbers*, Preprints **2017**, 2017030055, 4 pages; Available online at `http://dx.doi.org/10.20944/preprints201703.0055.v1`.

[18] F. Qi and B.-N. Guo, *An inequality involving the gamma and digamma functions*, J. Appl. Anal. **22** (2016), no. 1, 49–54; Available online at `http://dx.doi.org/10.1515/jaa-2016-0005`.

[19] F. Qi and B.-N. Guo, *Explicit formulas and nonlinear ODEs of generating functions for Eulerian polynomials*, ResearchGate Working Paper (2016), available online at `http://dx.doi.org/10.13140/RG.2.2.23503.69288`.

[20] F. Qi and B.-N. Guo, *Explicit formulas and recurrence relations for higher order Eulerian polynomials*, Indag. Math. **28** (2017), no. 4, 884–891; Available online at `https://doi.org/10.1016/j.indag.2017.06.010`.

[21] F. Qi and B.-N. Guo, *Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **111** (2017), no. 2, 425–434; Available online at `http://dx.doi.org/10.1007/s13398-016-0302-6`.

[22] F. Qi and B.-N. Guo, *Some properties of a solution to a family of inhomogeneous linear ordinary differential equations*, Preprints **2016**, 2016110146, 11 pages; Available online at `http://dx.doi.org/10.20944/preprints201611.0146.v1`.

[23] F. Qi and B.-N. Guo, *Some properties of the Hermite polynomials and their squares and generating functions*, Preprints **2016**, 2016110145, 14 pages; Available online at `http://dx.doi.org/10.20944/preprints201611.0145.v1`.

[24] F. Qi and B.-N. Guo, *Viewing some nonlinear ODEs and their solutions from the angle of derivative polynomials*, ResearchGate Working Paper (2016), available online at `http://dx.doi.org/10.13140/RG.2.1.4593.1285`.

[25] F. Qi and B.-N. Guo, *Viewing some ordinary differential equations from the angle of derivative polynomials*, Preprints **2016**, 2016100043, 12 pages; Available online at `http://dx.doi.org/10.20944/preprints201610.0043.v1`.

[26] F. Qi, D. Lim, J.-L. Zhao, and B.-N. Guo, *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM (2018), in press. ResearchGate Working Paper (2016), available online at `http://dx.doi.org/10.13140/RG.2.2.32155.87843`.

[27] F. Qi and M. Mahmoud, *Bounding the gamma function in terms of the trigonometric and exponential functions*, Acta Sci. Math. (Szeged) **83** (2017), no. 1-2, 125–141; Available online at `http://dx.doi.org/10.14232/actasm-016-813-x`.

[28] F. Qi, X.-T. Shi, and F.-F. Liu, *Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers*, Acta Univ. Sapientiae Math. **8** (2016), no. 2, 282–297; Available online at `http://dx.doi.org/10.1515/ausm-2016-0019`.

[29] F. Qi, J.-L. Wang, and B.-N. Guo, *Notes on a family of inhomogeneous linear ordinary differential equations*, Preprints **2017**, 2017040026, 5 pages; Available online at `http://dx.doi.org/10.20944/preprints201704.0026.v1`.

[30] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying and finding nonlinear ordinary differential equations*, ResearchGate Working Paper (2017), available online at `http://dx.doi.org/10.13140/RG.2.2.28855.32166`.

[31] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying differential equations concerning degenerate Bernoulli and Euler numbers*, ResearchGate Working Paper (2017), available online at `http://dx.doi.org/10.13140/RG.2.2.12078.10566`.

[32] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying two families of nonlinear ordinary differential equations*, Preprints **2017**, 2017030192, 6 pages; Available online at `http://dx.doi.org/10.20944/preprints201703.0192.v1`.

[33] F. Qi, Q. Zou, and B.-N. Guo, *Identities of the Chebyshev polynomials, the inverse of a triangular matrix, and identities of the Catalan numbers*, ResearchGate Working Paper (2017), available online at `http://dx.doi.org/10.13140/RG.2.2.27344.71684`.

[34] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*. The unpublished notes of H. W. Gould. With a foreword by George E. Andrews. World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.

[35] J.-L. Zhao, J.-L. Wang, and F. Qi, *Derivative polynomials of a function related to the Apostol–Euler and Frobenius–Euler numbers*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1345–1349; Available online at `http://dx.doi.org/10.22436/jnsa.010.04.06`.