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Article

The Class of (p, q)-spherical Distributions

With an extension of the sector and circle number functions

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Abstract: For evaluating probabilities of arbitrary random events with respect to a given multivariate probability distribution, specific techniques are of great interest. An important two-dimensional high risk limit law is the Gauss-exponential distribution whose probabilities can be dealt with based upon the Gauss-Laplace law. The latter will be considered here as an element of the newly introduced family of (p, q)-spherical distributions. Based upon a suitably defined non-Euclidean arc-length measure on (p, q)-circles we prove geometric and stochastic representations of these distributions and correspondingly distributed random vectors, respectively. These representations allow dealing with the new probability measures similarly like with elliptically contoured distributions and more general homogeneous star-shaped ones. This is demonstrated at hand of a generalization of the Box-Muller simulation method. En passant, we prove an extension of the sector and circle number functions.

Keywords: Gauss-exponential distribution; Gauss-Laplace distribution; stochastic vector representation; geometric measure representation; (p, q)-generalized polar coordinates; (p, q)-arc length; dynamic intersection proportion function; (p, q)-generalized Box-Muller simulation method; (p, q)-spherical uniform distribution; dynamic geometric disintegration

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1. Introduction

The Gauss-exponential distribution plays an important role as a high risk limit law, see Sections 8 and 9 of the lectures presented in [1] on high risk scenarios and extremes. Needless to recall here the numerous different fields where quantitative risk management applies. The Gauss-exponential distribution can be considered as a particular asymmetric derivation of the Gauss-Laplace law. In particular, Gauss-exponential probabilities of arbitrary events can be dealt with by considering the corresponding Gauss-Laplace probabilities. Density level sets of the standard Gauss-Laplace distribution are topological boundaries of star bodies centered at the origin. The Minkowski functionals of the corresponding star bodies, however, are not homogeneous functions of order one as it is often assumed in the literature on star-shaped distributions. Instead, the bodies corresponding to different density levels reflect different geometric properties and are typically directed to different directions. The aim of the present paper is to model (p, q)-spherical generalizations of the two-dimensional Gauss-Laplace distribution. We prove geometric and stochastic representations which can be considered as standard tools for dealing later on with the present distributions in a way being similar to that one has already successfully been dealing for a long time with elliptically contoured and, since more recently, even with more general homogeneous star-shaped distributions. This will be shortly indicated here by generalizing the Box-Muller simulation method.

It is well known from two-dimensional spherical distribution theory that a random vector X following such distribution allows a stochastic representation $X \stackrel{d}{=} R \cdot U$ with independent non-negative random variable R and singular (with respect to the Lebesgue measure in $\mathbb{R}^2$) random vector U being uniformly distributed on the Euclidean unit circle. Understanding the suitable way of generalizing the latter distribution needs the most efforts in general homogeneous star-shaped distribution theory. The corresponding singular distribution is dealt with by several authors (even in higher dimensions) by considering densities of marginal variables (vectors), see [23] and [20] for

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the particular case of l_p-spherical distributions. Studying a disintegration formula for this case, it is proved in [8] that the geometric surface measure cannot coincide with their uniform distribution if $p \notin \{1, 2, \infty\}$. Moreover, the authors mention that 'its treatment seems to need a completely different proof than the proof for the uniform distribution given in this (their) paper.' In [19] the authors exploit properties of the distribution being called the uniform distribution on l_p-spheres in [8], by making use of a representation of it being called later a cone measure representation, see [2]. An insightful Kepler law interpretation of this measure is discussed in [22]. A coordinate based approach to describing uniform distributions on l_p-spheres is given in [21] where it is inter alia said that 'it seems that the usage of the word *uniform*' ... 'does not refer to the real, geometrical uniformity of the probability mass on the surface of the unit sphere in n-dimensional L_α'. The differential geometric explanation of the generalized uniform distribution given for the particular case of $l_{n,p}$-spheres for arbitrary finite dimension in [10] (and for dimension two already in two earlier papers on the circle number function mentioned there) provided a qualitatively new approach to this long standing measure theoretical problem. More general homogeneous star-shaped distributions are studied analogously in [11–17] based upon the consideration of suitably introduced non-Euclidean geometries.

Much efforts are expected to be necessary to give suitable explanation of the geometric nature of a uniform distribution in the present case that the Minkowski functional of the density contour sets defining star body is not homogeneous of order one. In going all the necessary steps to reach such interpretation, we will be confronted with generalizing the notions of circle and its radius as well as its circumference, and with an extension of the sector and circle number functions. In the homogeneous case, that is, if the mentioned Minkowski functional is homogeneous of order one, the analogous steps can be observed in [9] and in author's series of papers mentioned before. Even more information on the history of the mentioned measure theoretical problem can be found there.

Among others, a technical key role will be played here by the suitable choice of coordinates for describing generalized uniform distributions. Moreover, we go a next basic step of extending the sector and circle number functions to classes of generalized circles having different geometric properties for different values of their generalized radii.

The density of the standard Gauss-Laplace law $\Phi^*_{G,L}$ in $\mathbb{R}^2$ is given by

$$\Phi^*_{G,L}(d(x,y)) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{2} - |y|} d(x,y).$$

Let us consider the (p,q)-spherical generalized normal density

$$\Phi_{p,q}(d(x,y)) = C_p C_q e^{-\frac{|x|^p}{p} - \frac{|y|^q}{q}} d(x,y)$$

where $p > 0$, $q > 0$ and $C_p = p^{1-1/p}/(2\Gamma(1/p))$. We note that $\Phi_{2,1} = \Phi^*_{G,L}$ and $\Phi_{2,2}$ and $\Phi_{1,1}$ are two-dimensional standard Gauss and Laplace distribution laws, respectively. Figure 1 shows (p,q)-spherical densities for different choices of (p,q).

The case $p = q$ has been considered elsewhere, and we emphasize that this case will not be dealt with in the present paper. Thus, $p \neq q$ is assumed here. Let

$$|(x,y)|_{p,q} = \frac{|x|^p}{p} + \frac{|x|^q}{q}, (x,y)^T \in \mathbb{R}^2$$

denote the functional generating the density level sets or contour lines

$$S_{p,q}(r) = \{(x,y)^T \in \mathbb{R}^2 : |(x,y)|_{p,q} = r\}, r>0.$$

Such level set can be generated from the (p,q)-generalized unit circle $S_{p,q} = S_{p,q}(1)$ by matrix multiplication

$$S_{p,q}(r) = D_{p,q}(r) S_{p,q}$$

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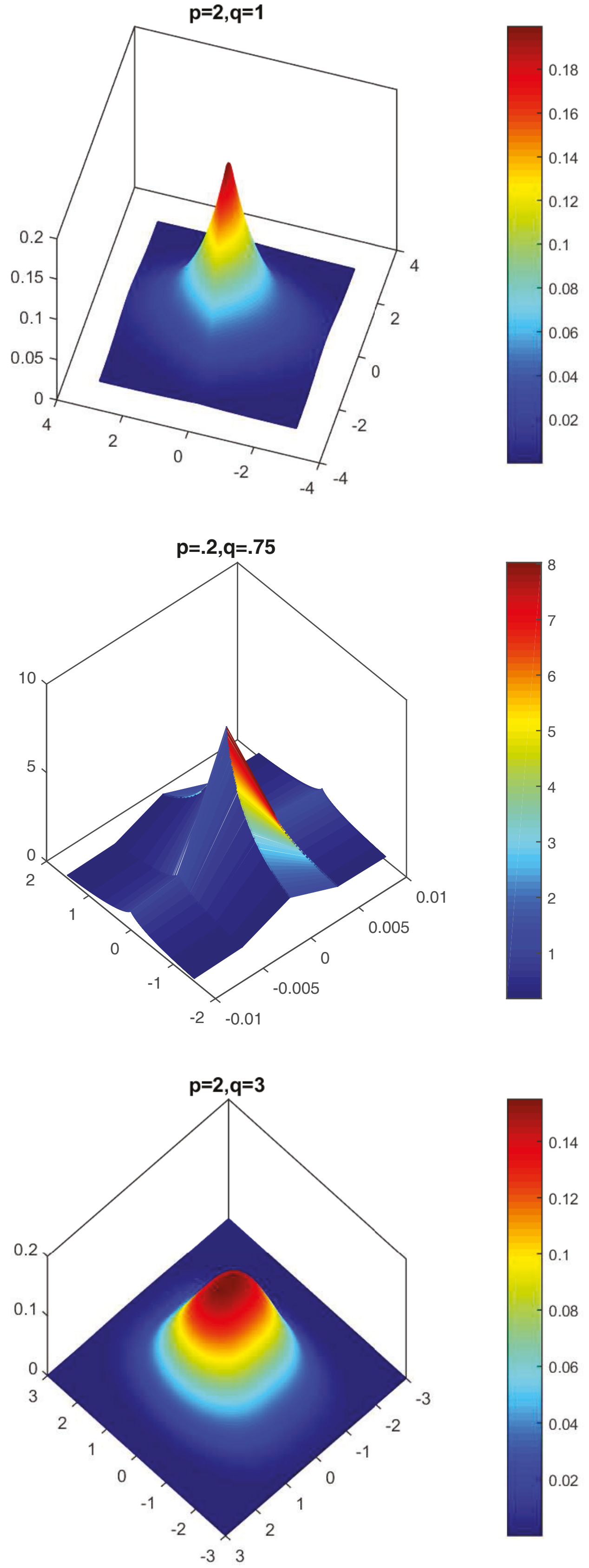

Figure 1. (p, q)-spherical densities

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and will be called for short the (p,q)-circle of (p,q)-radius r. Here, $D_{p,q}(r) = diag(r^{1/p}, r^{1/q})$ with $diag(a,b)$ denoting the diagonal matrix with diagonal entries a and b. The star body

$$K_{p,q}(r) = \{(x,y)^T \in \mathbb{R}^2 : |(x,y)|_{p,q} \leq r\}$$

having the origin in its interior and $S_{p,q}(r)$ as its topological boundary will be called the (p,q)-circle disc of (p,q)-radius r. It satisfies the subset relation

$$K_{p,q}(r_1) \subset K_{p,q}(r_2) \text{ if } r_1 < r_2$$

and allows the representation

$$K_{p,q}(r) = \bigcup_{\rho=0}^{r} S_{p,q}(\rho).$$

Such disc is convex if $p \geq 1$ and $q \geq 1$, and is radially concave if $0 < p \leq 1$ and $0 < q \leq 1$. For the latter notion we refer to [7]. Figure 2 (drawn with Matlab, as Figure 1) shows $(2,1)$-circles $S_{2,1}(r)$ according to different values of the $(2,1)$-generalized radius r, starting from a local (central) and turning to more global views. Roughly spoken, the impression of main orientation of density level lines is changing within two steps from 'west-east' to 'south-north'.

The paper is organized as follows. Section 2 presents preliminary material on (p,q)-generalizations of the common polar coordinates, the notions of arc-length and geometric disintegration, the sector and circle number functions and the generalized uniform distribution on a generalized circle. After further developing the general methodology from the theory of homogeneous star-shaped distributions, we are in a position to formally introduce the family of (p,q)-spherical distributions in Section 3 including the (p,q)-generalized normal distributions as particular cases of such distributions having a density. Section 4 deals with a geometric generalization of the asymmetric Gauss-exponential law, and Section 5 is devoted to some aspects of simulation. The final discussion in Section 6 delivers looks back and ahead onto the present distribution theory.

2. Preliminaries

2.1. A class of (p,q)-generalized polar coordinates

Let $N_p(\phi) = (|\cos\phi|^p + |\sin\phi|^p)^{1/p}$. The p-generalized cosine and sine functions are defined according to [9] by

$$\cos_p(\phi) = \frac{\cos\phi}{N_p(\phi)} \text{ and } \sin_p(\phi) = \frac{\sin\phi}{N_p(\phi)}.$$

The (p,q)-generalized polar, or (p,q)-spherical, coordinate transformation

$$Pol_{p,q} : (0,\infty) \times [0,2\pi) \to \mathbb{R}^2 \setminus \{(0,0)^T\}$$

is defined by $(x,y)^T = Pol_{p,q}(r,\phi)$ where

$$\begin{aligned}
x &= (pr)^{1/p}(\cos_{pq}(\phi))^q, \quad && 0 \leq \phi < \pi/2, 3\pi/2 \leq \phi < 2\pi \\
x &= -(pr)^{1/p}(-\cos_{pq}(\phi))^q, \quad && \pi/2 \leq \phi < 3\pi/2 \\
y &= (qr)^{1/q}(\sin_{pq}(\phi))^p, \quad && 0 \leq \phi < \pi \\
y &= -(qr)^{1/q}(-\sin_{pq}(\phi))^p, \quad && \pi \leq \phi < 2\pi.
\end{aligned}$$

The absolute value of the Jacobian of this transformation is

$$J(Pol_{p,q})(r,\phi) = r^{1/p+1/q-1} J^*(\phi)$$

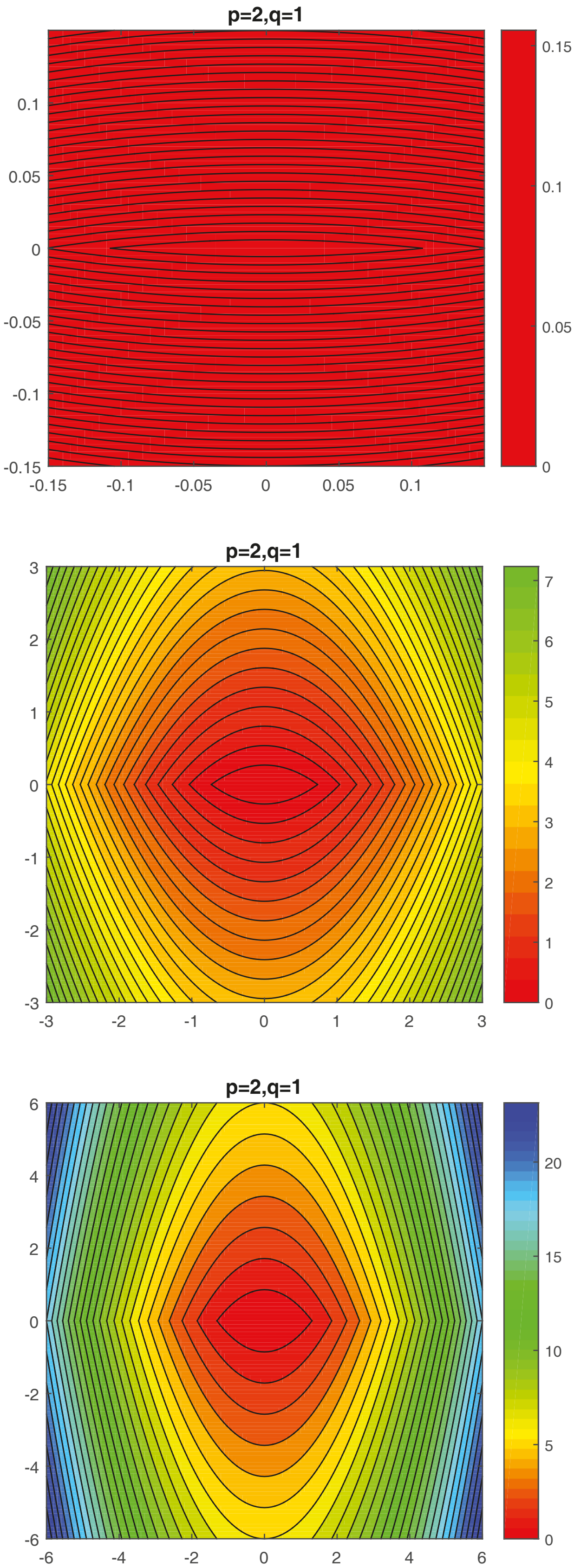

Figure 2. Gauss-Laplace density level sets: from local (central) to global view

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where
$$J^*(\phi) = p^{1/p}q^{1/q}|\cos_{pq}(\phi)|^{q-1}|\sin_{pq}(\phi)|^{p-1}N_{pq}(\phi)^{-2}. \quad (1)$$

The proof of this lemma makes use of results in [9]. Moreover, the inverse of the coordinate transformation $Pol_{p,q}$ is given by
$$r = |(x,y)|_{p,q},$$
$$\phi = \begin{cases} \frac{\pi}{2} & \text{if} \quad y > 0, x = 0 \\ \arctan\left((p/q)^{1/(pq)}\frac{y^{1/p}}{x^{1/q}}\right) & \text{if} \quad y > 0, x \neq 0 \end{cases}$$
and
$$\phi = \begin{cases} \frac{3}{2}\pi & \text{if} \quad y < 0, x = 0 \\ \arctan\left(-(p/q)^{1/(pq)}\frac{(-y)^{1/p}}{x^{1/q}}\right) & \text{if} \quad y < 0, x \neq 0. \end{cases}$$

2.2. The (p,q)-arc length measure and dynamic geometric disintegration of the Lebesgue measure

In this section, first the (p,q)-generalized arc length and the area content of (p,q)-circles and -circle discs are introduced, respectively. Then, a new type of geometric disintegration of the Lebesgue measure will be established. In the next section, extensions of the sector and circle number functions are considered. To start with, we recall that the area content of the (p,q)-circle disc with (p,q)-generalized radius ρ is

$$\mu(K_{p,q}(\rho)) = \int_{K_{p,q}(\rho)} d(x,y) = \int_0^{\rho}\int_0^{2\pi} |J(Pol_{p,q})(r,\phi)|d\phi dr.$$

Successively changing variables $x = \cos_{pq}(\phi)$ and $y = x^{pq}$ shows that

$$\mu(K_{p,q}(\rho)) = 4B(\frac{1}{p},\frac{1}{q})\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}}{p+q}\rho^{\frac{1}{p}+\frac{1}{q}}. \quad (2)$$

We note that the power exponent of ρ reflects a certain aspect of the change of shape of $K_{p,q}(\rho)$ when ρ varies. Moreover, we emphasize the remarkable differences between convex and radially concave cases. In the sequel, in a certain analogy to what was done in [9–12] for various situations, we define the (p,q)-generalized arc length measure $AL_{p,q}$ on the Borel σ-field $\mathcal{B}(S_{p,q})$ of $S_{p,q}$. To this end, for an arbitrary set $A \in \mathcal{B}(S_{p,q})$, let us call

$$CPC_{p,q}(A) = \{D_{p,q}(r)(x,y)^T : (x,y)^T \in A, \, r > 0\}$$

a nonlinear matrix transformed central projection cone. Note that $|x|^p/p + |y|^q/q = 1$ if $(x,y)^T \in A$ and that $D_{p,q}(r_1)(x,y)^T \neq D_{p,q}(r_2)(x,y)^T$ if $r_1 \neq r_2$. The set $CPC_{p,q}(A)$ can be considered as a union of pairwise disjoint sets,
$$CPC_{p,q}(A) = \bigcup_{r>0}[D_{p,q}(r)A].$$
Here, multiplication of a set by a matrix is defined in the common pointwise sense. Furthermore, a (p,q)-sector of $K_{p,q}(r)$ having (p,q)-generalized radius r {is defined by

$$Se_{p,q}(A,r) = CPC_{p,q}(A) \cap K_{p,q}(r),$$

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andwe consider the (p,q)-radius dependent area content function

$$f(\rho) = \mu(Se_{p,q}(A,\rho)) = \int\limits_0^{\rho} \int\limits_{Pol_{p,q}^{*-1}(A)} |J(Pol_{p,q})(r,\phi)| d(r,\phi), \rho \geq 0. \tag{3}$$

Here, $Pol^*_{p,q}(\phi) = Pol_{p,q}(1,\phi)$ and $Pol^{*-1}_{p,q}$ denotes the inverse function of $Pol^*_{p,q}$.

The (p,q)-arc length measure $AL_{p,q}(A)$ is defined for arbitrary $A \in \mathcal{B}(S_{p,q})$ by

$$AL_{p,q}(A) = f'(1).$$

It follows from the definition of f that

$$AL_{p,q}(A) = \int\limits_{Pol_{p,q}^{*-1}(A)} |J^*(\phi)| d\phi, A \in \mathcal{B}(S_{p,q}) \tag{4}$$

and

$$AL_{p,q}(D_{p,q}(r)A) = r^{1/p+1/q-1} AL_{p,q}(A). \tag{5}$$

In particular,

$$AL_{p,q}(S_{p,q}(r)) = 4B(\frac{1}{p},\frac{1}{q}) p^{1/p-1} q^{1/q-1} r^{1/p+1/q-1}. \tag{6}$$

If $B \in \mathcal{B}^2$ has finite area content then

$$\mu(B) = \int\limits_B dx = \int\limits_0^{\infty} \int\limits_0^{2\pi} I_B(Pol_{p,q}(r,\phi)) |J(r,\phi)| d\phi dr$$

$$= \int\limits_0^{\infty} \left(r^{1/p+1/q-1} \int\limits_0^{2\pi} I_B(Pol_{p,q}(r,\phi)) |J^*(\phi)| d\phi \right) dr.$$

Because $I_B(Pol_{p,q}(r,\phi)) = 1$ if and only if $Pol_{p,q}(r,\phi) \in S_{p,q}(r) \cap B$ it follows that

$$\mu(B) = AL_{p,q}(S_{p,q}) \int\limits_0^{\infty} r^{1/p+1/q-1} \mathcal{F}_{p,q}(B,r) dr \tag{7}$$

where

$$\mathcal{F}_{p,q}(B,r) = AL_{p,q}([D_{p,q}(r^{-1})B] \cap S_{p,q}) / AL_{p,q}(S_{p,q}), r > 0 \tag{8}$$

denotes the (p,q)-spherical intersection proportion function of the set B. By similar argumentation the following theorem is proved. We note that (7) represents a new non-Euclidean type of geometric disintegration of the Lebesgue measure in $\mathbb{R}^2$ which due to the effects of action $D_{p,q}$ will be called a dynamic disintegration. For this reason, the function in (8) will be called a dynamic intersection proportion function, and the integration method in (7) will be called dynamic geometric disintegration of the Lebesgue measure. If h is integrable over B then the dynamic geometric disintegration

$$\int\limits_B h(x) dx = \int\limits_0^{\infty} \left(r^{1/p+1/q-1} \int\limits_{B^*(r)} h(Pol_{p,q}(r,\phi)) |J^*(\phi)| d\phi \right) dr$$

is valid where

$$B^*(r) = \{\phi \in [0,2\pi) : Pol_{p,q}(1,\phi) \in [D_{p,q}(r^{-1})B] \cap S_{p,q}\}.$$

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2.3. The (p,q)-circle and sector number functions

Circle numbers of star discs are studied in [11] and for particular cases in earlier papers cited therein. Similarly, we define the (p,q)-circle number function to assign the number $\pi_{p,q}$ to any (p,q)-circle of (p,q)-radius r where for any $r > 0$

$$\frac{\mu(K_{p,q}(r))}{r^{1/p+1/q}} = \pi_{p,q} = \frac{AL_{p,q}(S_{p,q}(r))}{(1/p+1/q)r^{1/p+1/q-1}}.$$

As for any $A \in \mathcal{B}(S_{p,q})$,

$$f'(r) = AL_{p,q}(D_{p,q}(r)A),$$

we can also define the (p,q)-sector number function to assign the number $\pi_{p,q}(A)$ to any (p,q)-sector $Se_{p,q}(A,r)$ of (p,q)-radius r where

$$\frac{\mu(Se_{p,q}(A,r))}{r^{1/p+1/q}} = \pi_{p,q}(A) = \frac{AL_{p,q}(D_{p,q}(r)A)}{(1/p+1/q)r^{1/p+1/q-1}}, \forall r > 0$$

and

$$\pi_{p,q}(A) = \frac{p^{1+1/p}q^{1+1/q}}{p+q} \int\limits_{Pol_{p,q}^{*-1}(A)} |J^*(\phi)| d\phi. \tag{9}$$

In particular, the (p,q)-circle number $\pi_{p,q}$ allows the following integral representations

$$\pi_{p,q} = \frac{p^{1+1/p}q^{1+1/q}}{p+q} \int\limits_0^{2\pi} |\cos_{pq}(\phi)|^{q-1} |\sin_{pq}(\phi)|^{p-1} \frac{d\phi}{N_{p,q}^2(\phi)},$$

$$\pi_{p,q} = 4\frac{p^{1/p}q^{1/q}}{p+q} \int\limits_0^1 x^{1/p-1}(1-x)^{1/q-1} dx,$$

thus

$$\pi_{p,q} = 4\frac{p^{1/p}q^{1/q}}{p+q} \frac{\Gamma(1/p)\Gamma(1/q)}{\Gamma(1/p+1/q)}. \tag{10}$$

Moreover,

$$\pi_{p,q} = \frac{1}{\Gamma(1/p+1/q+1)} \int\limits_{\mathbb{R}^2} e^{-\frac{|x|^p}{p} - \frac{|x|^q}{q}} d(x,y). \tag{11}$$

As to summarize some results from this and the last sections for the case $(p,q) = (2,1)$, that is for the case being of particular interest when considering the Gauss-Laplace law, we have seen that

$$\mu(K_{2,1}(r)) = \pi_{2,1} r^{3/2} \quad \text{and} \quad AL_{2,1}(S_{2,1}(r)) = \frac{3}{2}\pi_{2,1} r^{1/2} \quad \text{where} \quad \pi_{2,1} = \frac{8\sqrt{2}}{3}.$$

2.4. The (p,q)-spherical uniform distribution

The (p,q)-arc length measure introduced in Section 2.2 will be used now to define the generalized (non-Euclidean) uniform distribution on a (p,q)-circle. The (p,q)-spherical uniform probability law on the Borel σ-field $\mathcal{B}(S_{p,q})$ is defined by

$$\omega_{p,q}(A) = AL_{p,q}(A)/AL_{p,q}(S_{p,q}).$$

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This relative arc-length measure appears to be quite natural. To see this, let $(\Omega, \mathcal{A}, P)$ be a probability space, a random vector $X = (X_1, X_2)^T : \Omega \to \mathbb{R}^2$ follow the common uniform distribution on the (p,q)-circle disc $K_{p,q}$,

$$P(X \in M) = \frac{\mu(M)}{\mu(K_{p,q})}, \; M \in \mathcal{B}(K_{p,q}),$$

and $R = |(X_1, X_2)^T|_{p,q}$ be the (p,q)-radius of X. {Excepting the origin, every point from the interior of $K_{p,q}$ belongs just to one of the (p,q)-circles $S_{p,q}(r)$, $r \in (0,1)$, thus for every $\omega \in \Omega$ there is a uniquely defined $r \geq 0$ such that

$$X(\omega) \in D_{p,q}(r) S_{p,q}.$$

The random vector $U_{p,q} = D_{p,q}(R^{-1})X$ follows the (p,q)-spherical uniform distribution on $S_{p,q}$, is independent of the random variable R, and R has the following density with respect to the Lebesgue measure on the real line

$$(1/p + 1/q) r^{1/p+1/q-1} I_{[0,1)}(r) dr. \tag{12}$$

Proof. The cumulative distribution function of R is

$$F_R(\rho) = P(X \in \bigcup_{r \leq \rho} D_{p,q}(r) S_{p,q}) = P(X \in K_{p,q}(\rho)) = \mu(K_{p,q}(\rho))/\mu(K_{p,q})$$

$$= \rho^{1/p+1/q} I_{(0,1)}(\rho) + I_{[1,\infty)}(\rho),$$

thus the density of R is given by

$$F_R(d\rho) = I_{(0,1)}(\rho)(1/p + 1/q)\rho^{1/p+1/q-1} d\rho.$$

Now, let $A \in \mathcal{B}(S_{p,q})$, then

$$P(U_{p,q} \in A) = P(X \in Se_{p,q}(A,1)) = \frac{\mu(Se_{p,q}(A,1))}{\mu(K_{p,q})}.$$

Because

$$\mu(Se_{p,q}(A,r)) = \frac{pq}{p+q} r AL_{p,q}(D_{p,q}(r)A),$$

it follows that

$$P(U_{p,q} \in A) = \frac{AL_{p,q}(A)}{\mu(K_{p,q})(1/p + 1/q)} = \frac{AL_{p,q}(A)}{AL_{p,q}(S_{p,q})} = \omega_{p,q}(A).$$

Finally,

$$P(R < \rho, U_{p,q} \in A) = P(X \in Se_{p,q}(A,\rho)) = \frac{\mu(Se_{p,q}(A,\rho))}{\mu(K_{p,q})}$$

$$= \frac{1}{\mu(K_{p,q})} \int_0^{\rho} \int_{Pol_{p,q}^{*-1}(A)} |J(Pol_{p,q})(r,\phi)| d(r,\phi) = \frac{\rho^{1/p+1/q}}{1/p + 1/q} \frac{AL_{p,q}(A)}{\mu(K_{p,q})}$$

$$= \rho^{1/p+1/q} \frac{AL_{p,q}(A)}{AL_{p,q}(S_{p,q})} = F_R(\rho)\omega_{p,q}(A).$$

□

If Y is a uniformly on $(0,1)$-distributed random variable then $Y^{pq/(p+q)}$ follows the distribution having the density in (12). On the one hand, the (p,q)-spherical uniform probability distribution is singular with respect to the Lebesgue measure in the two-dimensional Euclidean space $\mathbb{R}^2$, but, on

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the other hand, it is absolutely continuous with respect to the Lebesgue measure on the real line. Its density is given there for $\phi \in (0, 2\pi)$ by

$$\omega_{p,q}(d\phi) = \frac{AL_{p,q}(d\phi)}{AL_{p,q}(S_{p,q})} = \frac{|J^*(\phi)|d\phi}{\pi_{p,q}(1/p+1/q)},$$

that is

$$\omega_{p,q}(d\phi) = \frac{p^{1/p+1}q^{1/q+1}}{\pi_{p,q}(p+q)} |\cos_{pq}(\phi)|^{q-1} |\sin_{pq}(\phi)|^{p-1} \frac{d\phi}{N^2_{pq}(\phi)}. \tag{13}$$

Now, let us define the (p,q)-spherical sector measure on the Borel σ-field of $S_{p,q}$ as

$$sm_{p,q}(A) = \mu(Se_{p,q}(A,1))/\mu(K_{p,q}).$$

According to our consideration in Section 2.3, the (p,q)-spherical uniform distribution allows the (p,q)-spherical sector measure and the (p,q)-arc length representations

$$\omega_{p,q}(A) = sm_{p,q}(A) = \frac{AL_{p,q}(A)}{(1/p+1/q)\pi_{p,q}}, A \in \mathcal{B}(S_{p,q}),$$

respectively. An additional nonlinear matrix transformed cone measure representation of the (p,q)-spherical uniform distribution is given by

$$\omega_{p,q}(A) = \Phi_{p,q}(CPC_{p,q}(A)),\ A \in \mathcal{B}(S_{p,q}).$$

3. On the (p,q)-spherical generalization of the Gauss-Laplace law

The following analogue to Theorem 2 is our starting point for introducing here the new general family of (p,q)-spherical distributions.

Let a random vector $X = (X_1, X_2)^T : \Omega \to \mathbb{R}^2$ follow the (p,q)-spherical distribution law $\Phi_{p,q}$ and put $R = |(X_1, X_2|^T_{p,q}$. The random vector $U_{p,q} = D_{p,q}(R)^{-1}X$ follows the (p,q)-generalized uniform distribution on $S_{p,q}$, is independent of the random variable R, and R has the density with respect to the Lebesgue measure on the real line

$$\frac{1}{\Gamma(1/p+1/q)} r^{1/p+1/q-1} e^{-r} I_{(0,\infty)}(r)dr. \tag{14}$$

Proof. The cumulative distribution function of R is

$$F_R(\rho) = P(X \in K_{p,q}(\rho)) = C_pC_q \int\limits_{\{(x,y)^T \in \mathcal{R}^2 : |(x,y)^T|_{p,q} \le \rho\}} e^{-|(x,y)^T|_{p,q}} d(x,y)$$

$$= C_pC_q \int_0^\rho \left(\int_0^{2\pi} r^{1/p+1/q-1} e^{-r} |J^*(\phi)| d\phi \right) dr = C_pC_q \int_0^\rho r^{1/p+1/q-1} e^{-r} dr AL_{p,q}(S_{p,q}).$$

Because of

$$C_pC_q AL_{p,q}(S_{p,q})\Gamma(1/p+1/q) = 1, \tag{15}$$

it follows that

$$F_R(d\rho) = \frac{1}{\Gamma(1/p+1/q)} \rho^{1/p+1/q-1} e^{-\rho} d\rho.$$

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Furthermore, for $A \in \mathcal{B}(S_{p,q})$,

$$P(U_{p,q} \in A) = P(X \in CPC_{p,q}(A)) = C_pC_q \int_0^\infty r^{1/p+1/q-1}e^{-r}dr \int_{Pol^{*-1}_{p,q}(A)} |J^*(\phi)|d\phi$$

$$= \frac{AL_{p,q}(A)}{AL_{p,q}(S_{p,q})} = \omega_{p,q}(A)$$

and

$$P(R < \rho, U_{p,q} \in A) = P(X \in Se_{p,q}(A,\rho)) = C_pC_q \int_0^\rho \left(\int_{Pol^{*-1}_{p,q}(A)} |J^*(\phi)|d\phi \right) r^{1/p+1/q-1}e^{-r}dr$$

$$= \frac{\int_0^\rho r^{1/p+1/q-1}e^{-r}dr}{\Gamma(1/p+1/q)} \frac{AL_{p,q}(A)}{AL_{p,q}(S_{p,q})} = F_R(\rho)P(U_{p,q} \in A).$$

□

According to Theorem 3, if $X \sim \Phi_{p,q}$ then X allows the stochastic representation

$$X \stackrel{d}{=} D_{p,q}(R) \cdot U_{p,q} where$$

$U_{p,q}$ follows the (p,q)-spherical uniform distribution, $U_{p,q} \sim \omega_{p,q}$, and R has the density (14) and is independent of $U_{p,q}$.

The following definition is well motivated by Remark 2. (a) Let $U_{p,q}$ follow the (p,q)-spherical uniform distribution on the Borel σ-field of the (p,q)-unit circle $S_{p,q}$, $U_{p,q} \sim \omega_{p,q}$, and R a non-negative random variable having cumulative distribution function F and characteristic function ϕ, and being independent of $U_{p,q}$, then

$$X = D_{p,q}(R)U_{p,q} \tag{16}$$

is said to follow the (p,q)-spherical distribution $\Phi^{cdf(F)}_{p,q} = \Phi^{cf(\phi)}_{p,q}$. The vector $U_{p,q}$ is called the (p,q)-spherical uniform basis and R the generating variate of X. The distribution of X will alternatively be denoted $\Phi^{df(f)}_{p,q}$ if R has density function f.

(b) An arbitrary random vector X taking values in $\mathbb{R}^2$ is called (p,q)-spherically distributed if there exist a nonnegative random variable R being independent of a (p,q)-spherical uniformly distributed random vector U such that $X \stackrel{d}{=} D_{p,q}(R)U$. Here, $Y \stackrel{d}{=} Z$ means that random vector Y is distributed as random vector Z.

The characteristic function of a (p,q)-spherically distributed random vector X satisfying the representation $X \stackrel{d}{=} D_{p,q}(R)U$ can be written as

$$\phi_X(t) = \int_0^\infty \phi_U(D_{p,q}(r)t)P^R(dr), t \in \mathbb{R}^2$$

where P^R and Φ_U denote the distribution law induced by the random variable R and the characteristic function of the (p,q)-spherical uniform distribution, respectively.

Proof. By definition,

$$\phi_X(t) = \mathbb{E}\exp\{it^TX\} = \mathbb{E}\exp\{it^T(D_{p,q}(R)U)\},$$

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thus

$$\phi_X(t) = \mathbb{E}\exp\{i(t_1 R^{1/p} U_1 + t_2 R^{1/q} U_2)\}, t \in \mathbb{R}^2.$$

Because R and U are independent,

$$\phi_X(t) = \int\limits_{(0,\infty)\times S_{p,q}} \exp\{i(t_1 r^{1/p}, t_2 r^{1/q})u\}(P^R \times P^U)(dr \times du),$$

and by Fubini's theorem,

$$\phi_X(t) = \int\limits_0^\infty \mathbb{E}\exp\{i(t_1 r^{1/p}, t_2 r^{1/q})U\} P^R(dr).$$

□

a) The distribution of a (p,q)-spherically distributed random vector X is uniquely determined by the distribution of its generating variate R.

b) If a (p,q)-spherically distributed random vector X has a density then it is of the form $f_X = \varphi_{g;p,q}$,

$$\varphi_{g;p,q}(x) = C(g;p,q)g(|x|_{p,q}),\ x \in \mathbb{R}^2,$$

where $g : [0,\infty) \to [0,\infty)$ is a density generating function (dgf) satisfying

$$0 < I(g;p,q) = \int\limits_0^\infty r^{1/p+1/q-1} g(r)dr < \infty,$$

and the normalizing constant allows the factorization

$$1/C(g;p,q) = I(g;p,q)AL_{p,q}(S_{p,q}).$$

Proof. a) If $X \stackrel{d}{=} D_{p,q}(R_1)U$, $Y \stackrel{d}{=} D_{p,q}(R_2)U$ with R_1 and R_2 being independent of U, and $R_1 \stackrel{d}{=} R_2$ then, by Theorem 4, $\phi_X = \phi_Y$.

b) Because the distribution of a (p,q)-spherically distributed random vector $X \stackrel{d}{=} D_{p,q}(R)U$ with nonnegative R being independent of U, $U \sim \omega_{p,q}$, is already determined by the distribution of R, the density of X is already determined by the density of R. By Fubini's theorem,

$$1 = C(g;p,q)\int\limits_0^\infty \left(\int\limits_0^{2\pi} r^{1/p+1/q-1} g(r)|J^*(\phi)|d\phi\right) dr = C(g;p,q)I(g;p,q)AL_{p,q}(S_{p,q}).$$

□

In what follows, we denote the distribution law of a (p,q)-spherically distributed random vector having dgf g by $\Phi_{g;p,q}$. The following theorem deals with a geometric representation of such measures. For every $B \in \mathcal{B}(R^2)$, $\Phi_{g;p,q}(B) = \frac{1}{I(g;p,q)} \int\limits_0^\infty r^{1/p+1/q-1} g(r)\mathcal{F}_{p,q}(B,r)dr$.

Proof. Because of $\Phi_{g;p,q}(B) = C(g;p,q)\int\limits_B g(|x|_{p,q})dx$,

$$\Phi_{g;p,q}(B) = C(g;p,q)\int\limits_0^\infty \left(r^{1/p+1/q-1} g(r)\int\limits_0^{2\pi} I_B(Pol_{p,q}(r,\phi))|J^*(\phi)|d\phi\right) dr.$$

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Since $I_B(Pol_{p,q}(r,\phi)) = 1$ if and only if $Pol_{p,q}(r,\phi) \in S_{p,q}(r) \cap B$, it follows that

$$AL_{p,q}(S_{p,q})\mathcal{F}_{p,q}(B,r) = AL_{p,q}([D_{p,q}(r)B] \cap S_{p,q}) = \int_0^{2\pi} I_B(Pol_{p,q}(r,\phi))|J^*(\phi)|d\phi dr.$$

□

The geometric measure representation in Theorem 5 will be called a dynamic geometric disintegration of the (p,q)-spherical measure $\Phi_{g;,p,q}$. Let $X \sim \Phi_{g;p,q}$ and $R = |X|_{p,q}$. Then R follows the density

$$f_R(\rho) = I(g;p,q)^{-1}\rho^{1/p+1/q-1}g(\rho)I_{(0,\infty)}(\rho).$$

Proof. Let $B = K_{p,q}(\rho)$. Theorem 5 applies with $\mathcal{F}_{p,q}(K_{p,q}(\rho),r) = I_{(0,\rho]}(r)$, and R follows the density

$$f_R(\rho) = \frac{d}{d\rho}\Phi_{g;p,q}(K_{p,q}(\rho)).$$

□

Finally, we note that Corollary 2 generalizes formula (14). That is, if $g(r) = e^{-r}$ in Corollary 2 then $I(g;p,q) = \Gamma(1/p + 1/q)$ and $\Phi_{g;p,q} = \Phi_{p,q}$.

4. Asymmetric (p,q)-spherical generalization of the Gauss-exponential law

The Gauss-exponential density $\varphi_{G,E}$ and the Gauss-Laplace density $\varphi_{G,L}$ are connected through the equation $\varphi_{G,E}(x,y) = 2I_{(0,\infty)}(y)\varphi_{G,L}(x,y)$. Accordingly, the Gauss-exponential probability of an arbitrary random event from $\mathcal{B}(R \times [0,\infty))$ can be dealt with by doubling the corresponding Gauss-Laplace probability. Therefore, the Gauss-exponential law can be considered as an asymmetric derivation of the Gauss-Laplace law.

In [1], the exponential component of the Gauss-exponential distribution law arises when a Gaussian vector is subject to a certain conditioning process assuming that this vector belongs to a half space having a positive distance from the origin. As one result of this conditioning process, the domain of definition of a multivariate distribution is restricted to a proper subset. If the conditioning process would be modified, other asymmetric distributions derived from the Gauss-Laplace law could be of interest.

The possible consequences on the geometric measure representation of a multivariate star-shaped distribution law caused by a restriction of the domain of definition to a proper subset are described in Remark 1 in [15]. A similar approach can be of interest, here.

The two-dimensional exponential distribution may be considered as a restriction of the distribution $\Phi_{p,q}$ with $p = q = 1$ (not considered here) to the domain of definition $[0,\infty) \times [0,\infty)$. For a geometric generalization of the multivariate exponential law we refer to the class of regular simplicially contoured or l_1-norm symmetric distributions studied in [4]. Further results for p-spherical distributions with p from $\{1,2,\infty\}$ can be found in [8], [6] and [18].

5. Simulation

First of all, we recall that the radius component of a uniformly on $K_{p,q}$ distributed random vector can be simulated according to Remark 1. Numerous others generalized radius distributions may be simulated using various particular methods.

The well known simulation method in [3] was extended to the p-generalized normal distribution in [5]. Similarly, here we establish a simulation method for arbitrary (p,q)-spherically distributed vectors. According to Theorem 2, the vector $\mathcal{U}_{p,q}$ follows independently of the variable R the (p,q)-spherical uniform distribution on $S_{p,q}$,

$$(p^{1/p}\cos_{pq}^{q}(\Phi),\, q^{1/q}\sin_{pq}^{p}(\Phi))^T \sim \omega_{p,q}. \tag{17}$$

By (13), the density of angle Φ is given as

$$f_\Phi(\phi) = \frac{pq}{B(1/p,\,1/q)}\frac{|\cos_{pq}(\phi)|^{q-1}|\sin_{pq}(\phi)|^{p-1}}{N_{pq}^2(\phi)}, 0 \le \phi < 2\pi. \tag{18}$$

Starting from this representation, one can proceed as described in [5] and [14], Example 9(b) or any of the standard monographs on simulation mentioned there.

6. Discussion

The way of probabilistic modeling developed in this paper is closely related to various challenging mathematical problems. It is well known from the results in [13,16,17] and the references given there that representations of star-shaped distributions whose contour defining star body has a homogeneous Minkowski functional of order one are closely related to suitably chosen non-Euclidean geometries. Here, we discover that there is again a need to go some steps beyond such geometries and realize a first of them.

Already since 17th century, basically starting from the work of Descartes, various coordinate systems play a fruitful role in geometric applications. Nevertheless, it seems that suitably chosen coordinates may serve even these days as a powerful tool for solving nontrivial problems in different areas of mathematics. In the present case, star bodies whose Minkowski functionals are not homogeneous functions of degree one are effectively described for the purposes of representing two-dimensional Gauss-Laplace laws and their (p,q)-spherical generalizations with the help of generalized polar coordinates based upon generalized sine and cosine functions.

Starting latest from the work of Leibniz and Newton who founded modern calculus, in many areas of mathematics one makes use of thin parallel layers when defining and studying certain basic notions. Here, however, small changes of a generalized radius variable related to such body generate thin layers close to the bodies boundary being nonparallel. To the best of author's knowledge, the fundamental measure theoretical problem of understanding the factorization components of cross sections or disintegrations of the present type seems to be approached here for the first time.

The present work extends the line of interchanging the role the notions of circle and distance play in comparison with Euclidean geometry, described inter alia in [11]. Here the 'circle' is given by a density level set modeling a 'contour line' of a sample cloud, and the understanding of what is a 'distance' leads to a directional dependent notion of radius being related to a matrix-vector multiplication. It remains, however, the question what is the differential geometric meaning of the newly introduced (p,q)-generalized arc length measure. Therefore, it is stated here as an open problem.

Finally, we remark that the results in Section 2 allow the following additional representations of the Lebesgue measure which may be useful in future applications of (p,q)-spherical distributions. For $r \in (0,1), \phi \in [0,2\pi)$,

$$\mu(dx) = (1/p+1/q)r^{1/p+1/q-1}dr\,\frac{p^{1/p}q^{1/q}}{1/p+1/q}|\cos_{pq}(\phi)|^{q-1}|\sin_{pq}(\phi)|^{p-1}\frac{d\phi}{N_{pq}^2(\phi)}$$

and, for $\rho \in (0,1]$,

$$\mu(Se_{p,q}(A,\rho)) = \int_0^\rho AL_{p,q}(D_{p,q}(r)A)dr.$$

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An alternative representation is given for $r \in (0,1), t \in (0,1)$ by

$$\mu(dx) = (1/p + 1/q)r^{1/p+1/q-1}dr\, t^{1/p-1}(1-t)^{1/q-1}dt.$$

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