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A NEW CLASS OF INTEGRALS INVOLVING EXTENDED MITTAG-LEFFLER FUNCTIONS

G. RAHMAN, A. GHAFFAR, K.S. NISAR* AND S. MUBEEN

ABSTRACT. The main aim of this paper is to establish two generalized integral formulas involving the extended Mittag-Leffler function based on the well known Lavoie and Trottier integral formula and the obtain results are express in term of extended Wright-type function. Also, we establish certain special cases of our main result.

1. INTRODUCTION

The Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are defined by the following series:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}, \; \Re(\alpha) > 0, \tag{1.1}$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z, \beta \in \mathbb{C}, \; \Re(\alpha) > 0, \tag{1.2}$$

respectively. For the generalizations and applications of Mittag-Leffler function, the readers can refer to the recent work of researchers ([4, 5, 6, 7, 8, 10, 20, 22, 30]), Kilbas et al. ([11], Chapter 1) and Saigo and Kilbas [12]. In recent years, the Mittag-Leffler function and some of its variety generalizations have been numerically established in the complex plane ([9, 24]). A new generalization of the Mittag-Leffler functions$E_{\alpha,\beta}(z)$ of (1.2) have defined by Prabhakar [21] as follows:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad z, \beta \in \mathbb{C}, \; \Re(\alpha) > 0, \tag{1.3}$$

where $(\gamma)_n$ denote the well known Pochhammer Symbol which is defined by:

$$(\gamma)_n = \begin{cases} 1, & (n = 0, \; \gamma \in \mathbb{C}), \\ \gamma(\gamma+1)(\gamma+2)...(\gamma+n-1) & (n \in N, \; \gamma \in \mathbb{C}). \end{cases}$$

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* Corresponding author.

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In fact, the following special cases are satisfied:

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), E^{1}_{\alpha,1}(z) = E_{\alpha,\beta}(z). \tag{1.4}$$

Recently, many researchers have established the significance and great consideration of Mittag-Leffler functions in the theory of special functions for exploring the generalizations and some applications. Various extensions for these functions are found in ([8],[25, 26, 27]). Srivastava and Tomovski [29] have defined further generalization of the Mittag-Leffler function $E^{1}_{\alpha,\beta}(z)$ of (3), which is defined as:

$$E^{\gamma,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{1.5}$$

where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > \max\{0, \Re(k)-1\}; \Re(k) > 0$. Very recently, Ozarslan and Yilmaz [19] have investigated an extended Mittag-Leffler function $E^{\gamma;c}_{\alpha,\beta}(z : p)$ which is defined as:

$$E^{\gamma;c}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n, c-\gamma)(c)_n}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{1.6}$$

where $p > 0$, $\Re(c) > \Re(\gamma) > 0$ and $B_p(x, y)$ is extended beta function defined in Mainardi as follows:

$$B_p(x, y) = \int_0^t t^{(x-1)}(1-t)^{(y-1)} e^{-(p/t(1-t))} dt, \tag{1.7}$$

where $\Re(p) > 0, \Re(x) > 0$ and $\Re(y) > 0$. If $p = 0$, then $B_p(x, y)$ reduces to the following beta function:

$$B(x, y) = \int_0^t t^{(x-1)}(1-t)^{(y-1)} dt, \tag{1.8}$$

The gamma function is defined by:

$$\Gamma(z) = \int_0^t t^{(z-1)} e^{(-t)} dt; \quad \Re(z) > 0. \tag{1.9}$$

By inspection, we conclude the following relation

$$\Gamma(z+1) = z\Gamma(z). \tag{1.10}$$

The generalized hypergeometric function ${}_pF_q(z)$ is defined in [6] as:

$$\begin{aligned} {}_pF_q(z) &= {}_pF_q\left[\begin{array}{cc} (a_1), (a_2), ..., (a_p); & \\ & z, p \\ (b_1), (b_2), ..., (b_q); & \end{array}\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_q)_n z^n}{(b_1)_n (b_2)_n ... (b_p)_n n!} \end{aligned} \tag{1.11}$$

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where $\alpha_i, \beta_i \in \mathbb{C}; i = 1, 2, ..., p; j = 1, 2, ..., q$and $b_j \neq 0, -1, -2, ...$ and $(z)_n$ is the Pochhammer symbols.

Sharma and Devi [23] defined extended Wright type function in the following series:

$$
\begin{aligned}
{}_{p+1}F_{q+1}(z;p) &= {}_{p+1}F_{q+1}\left[\begin{array}{lr} (a_i, A_i)_{1,p}, (\gamma, 1); & \\ & z \\ (b_j, \beta_j)_{1,p}, (c, 1); & \end{array}\right] \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + A_1 n)...\Gamma(a_p + A_p n)\mathbf{B}_p(\gamma + n, c - \gamma) z^n}{\Gamma(b_1 + \beta_1 n)...\Gamma(b_q + \beta_q n)\Gamma(c - \gamma) n!}
\end{aligned} \tag{1.12}
$$

where $i = 1, 2, ..., p, j = 1, 2, ..., q, \Re(p) > 0, \Re(c) > \Re(\gamma) > 0$ and $p \geq 0$. Recently, Mittal et al. [15] defined an extended generalized Mittag-Leffer function as:

$$
E_{\alpha,\beta}^{\gamma,q;c}(z;p) = \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma + nq, c - \gamma)(c)_{nq}}{\mathbf{B}(\gamma, c - \gamma)\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.13}
$$

where $\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q > 0$ and $B_p(x, y)$ is an extended beta function defined in (1.7). In this paper, we derive new integral formulas involving the Mittag-Leffler function (1.6) and (1.13). For the present investigation, we need the following result of Lavoie and Trottier [13]

$$
\int_0^1 x^{(a-1)}(1-x)^{(2b-1)} \left(1 - \frac{x}{3}\right)^{(2a-1)} \left(1 - \frac{x}{4}\right)^{(b-1)} dx = \left(\frac{2}{3}\right)^{2a} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{1.14}
$$

where $\Re(a) > 0, \Re(b) > 0$.

For more details about the recent works in the field of dynamical systems theory, stochastic systems, non-equilibrium statistical mechanics and quantum mechanics, the readers may refer to the recent work of the researchers[1, 2, 3, 14, 16, 17, 18] and the references cited therein.

2. Main Result

In this section, the generalized integral formulas involving the Mittag-Leffler function (1.6) and (1.13), are established here by inserting with the suitable argument in the integrand of (1.14).

Theorem 1. *Let* $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; *with* $\Re(\alpha) > \max\{0, \Re(c) - 1\}; \Re(c) > 0$. $\Re(\gamma) > 0, \Re(2\rho + j) > 0$ *and* $x > 0$. *Then the following formula holds true:*

$$
\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)} \left(1 - \frac{x}{3}\right)^{(2(\rho+j)-1)} \left(1 - \frac{x}{4}\right)^{(\rho-1)} E_{\alpha,\beta}^{\gamma;c}\left(y\left(1 - \frac{x}{4}\right)(1-x)^2\right) dx
$$

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$$= \left(\frac{2}{3}\right)^{2(\rho+j)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} {}_3\psi_3 \left[\begin{array}{c} (\rho,1),(c,1),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1),(c,1); \end{array} |y;p \right]. \quad (2.1)$$

Proof. Let S_1 be the left-hand side of (2.1). Now applying the series representation of the Mittag-Leffler function (1.6) to the integrand of (1.14) and by interchanging the order of integration and summation under the given condition in Theorem (1), we have

$$\begin{aligned} S_1 &= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)} \left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)} \left(1-\frac{x}{4}\right)^{(\rho-1)} E_{\alpha,\beta}^{\gamma;c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right) dx \\ &= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)} \left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)} \left(1-\frac{x}{4}\right)^{(\rho-1)} \\ &\quad \times \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)} \frac{\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)^n}{n!} dx \\ &= \int_0^1 x^{(\rho+j-1)}(1-x)^{2(\rho+n)-1} \left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)} \left(1-\frac{x}{4}\right)^{(\rho+n-1)} \\ &\quad \times \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)} \frac{y^n}{n!} dx. \end{aligned}$$

Now using (1.6) and (1.14), we get

$$\begin{aligned} S_1 &= \left(\frac{2}{3}\right)^{2(\rho+j)} \frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)} \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)} \frac{y^n}{n!} \\ &= \left(\frac{2}{3}\right)^{(2\rho+j)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\mathbf{B}_p(\gamma+n,c-\gamma)\Gamma(\rho+n)\Gamma(c+n)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)} \frac{y^n}{n!}, \end{aligned}$$

which upon using (1.12), we get

$$S_1 = \left(\frac{2}{3}\right)^{2(\rho+j)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} {}_3\psi_3 \left[\begin{array}{c} (\rho,1),(c,1),(\gamma,1) \\ \\ (\beta,\alpha),(2\rho+j,1),(c,1) \end{array} ;|y;p \right].$$

□

Theorem 2. *Let* $y,c,\rho,j,\alpha,\beta,\gamma \in \mathbb{C}$; *with* $\Re(\alpha) > \max\{0,\Re(c)-1\}; \Re(c) > 0$. $\Re(\gamma) > 0, \Re(2\rho+j) > 0$ *and* $x > 0$. *Then the following formula holds true:*

$$\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1} \left(1-\frac{x}{3}\right)^{(2\rho-1)} \left(1-\frac{x}{4}\right)^{(\rho+j-1)} E_{\alpha,\beta}^{\gamma;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right) dx$$

$$= \left(\frac{2}{3}\right)^{2(\rho)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} {}_3\psi_3 \left[\begin{array}{c} (\rho,1),(c,1),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1),(c,1); \end{array} ;|\tfrac{4}{9}y;p \right]. \quad (2.2)$$

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Proof. Let S_2 be the left-hand side of (2.2). Now applying the series representation of the Mittag-Leffler function (1.6) to the integrand of (1.14), we have

$$\begin{aligned} S_2 &= \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} E_{\alpha,\beta}^{\gamma;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right)dx \\ &= \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\ &\quad\times\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{\left(y\left(1-\frac{x}{3}\right)^2\right)^n}{n!}dx \\ &= \int_0^1 x^{(\rho+n-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2(\rho+n)-1}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} \\ &\quad\times\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}dx. \end{aligned}$$

Now using (1.6) and (1.14), we get

$$\begin{aligned} S_2 &= \left(\frac{2}{3}\right)^{2(\rho+n)}\frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)(c)_n}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!} \\ &= \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n,c-\gamma)\Gamma(\rho+n)\Gamma(c+n)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)}\frac{(\frac{4}{9}y)^n}{n!}, \end{aligned}$$

which upon using (1.12), we get

$$S_2 = \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\,{}_3\psi_3\left[\begin{matrix}(\rho,1),(c,1),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1),(c,1);\end{matrix}\;;|\tfrac{4}{9}y;p\right].$$

□

Theorem 3. *Let $y,c,\rho,j,\alpha,\beta,\gamma\in\mathbb{C}$; with $\Re(\alpha)>\max\{0,\Re(c)-1\};\Re(c)>0$. $\Re(\gamma)>0,\Re(2\rho+j)>0$ and $x>0$. Then the following formula holds true:*

$$\begin{aligned} &\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)}E_{\alpha,\beta}^{\gamma,q;c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2;p\right)dx \\ &\quad= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\,{}_3\psi_3\left[\begin{matrix}(\rho,1),(c,q),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1),(c,1);\end{matrix}\;;|y;p\right] \end{aligned} \tag{2.3}$$

Proof. Let S_3 be the left-hand side of (2.3). Now applying the series representation of the Mittag-Leffler function (1.6) to the integrand of (1.14) and by interchanging the order of

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integration and summation under the given condition in Theorem (1), we have

$$\begin{aligned} S_3 &= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} E_{\alpha,\beta}^{\gamma,q;c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2;p\right)dx \\ &= \int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} \\ &\times \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)^n}{n!}dx \\ &= \int_0^1 x^{(\rho+j-1)}(1-x)^{2(\rho+n)-1}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho+n-1)} \\ &\times \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}dx. \end{aligned}$$

Now using (1.6) and (1.14), we get

$$\begin{aligned} S_1 &= \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma,c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!} \\ &= \left(\frac{2}{3}\right)^{(2\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq,c-\gamma)\Gamma(\rho+n)\Gamma(c+nq)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)}\frac{y^n}{n!}, \end{aligned}$$

which upon using (1.12), we get

$$S_3 = \left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\, {}_3\psi_3\left[\begin{array}{c} (\rho,1),(c,1),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1),(c,1); \end{array} ;|y;p\right].$$

□

Theorem 4. *Let* $\rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; *with* $R(\alpha) > 0, \Re(c) > \Re(\gamma) > 0, \Re(2\rho+j) > 0$ *and* $x > 0$. *Then the following formula holds true:*

$$\begin{aligned} &\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} E_{\alpha,\beta}^{\gamma,q;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right)dx \\ &= \left(\frac{2}{3}\right)^{2(\rho)}\frac{\Gamma(\rho+j)}{\Gamma(\rho)}\, {}_3\psi_3\left[\begin{array}{c} (\rho,1),(c,q),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1),(c,1); \end{array} ;|\tfrac{4}{9}y;p\right]. \end{aligned} \tag{2.4}$$

Proof. Let S_4 be the left-hand side of (2.4). Now applying the series representation of the Mittag-Leffler function (1.6) to the integrand of (1.14), we have

$$S_4 = \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} E_{\alpha,\beta}^{\gamma,q;c}\left(yx\left(1-\frac{x}{3}\right)^2;p\right)dx$$

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$$= \int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)}$$
$$\times \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq, c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\alpha n+\beta)}\frac{\left(y\left(1-\frac{x}{3}\right)^2\right)^n}{n!}dx$$
$$= \int_0^1 x^{(\rho+n-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2(\rho+n)-1}\left(1-\frac{x}{4}\right)^{(\rho+j-1)}$$
$$\times \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq, c-\gamma)(c)_n}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}dx$$

Now using (1.13) and (1.14), we get

$$S_4 = \left(\frac{2}{3}\right)^{2(\rho+n)}\frac{\Gamma(\rho+j)\Gamma(\rho+n)}{\Gamma(2\rho+j+n)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq, c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\alpha n+\beta)}\frac{y^n}{n!}$$
$$= \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq, c-\gamma)\Gamma(\rho+n)\Gamma(c+nq)}{\Gamma(c-\gamma)\Gamma(\alpha n+\beta)\Gamma(2\rho+j+n)}\frac{(\frac{4}{9}y)^n}{n!},$$

which upon using (1.12), we get

$$S_4 = \left(\frac{2}{3}\right)^{2\rho}\frac{\Gamma(\rho+j)}{\Gamma(\gamma)}\ {}_3\psi_3\left[\begin{array}{ll}(\rho,1),(c,q),(\gamma,1); & \\ & |\frac{4}{9}y;p \\ (\beta,\alpha),(2\rho+j,1),(c,1); & \end{array}\right].$$

□

3. Particular cases

In this section, we obtained some particular cases of Theorems 1-4. Setting $p = 0$ in (1.6) and (1.13), then we get the following well known Mittag-Leffler functions.

$$E_{\alpha,\beta}^{\gamma,c}(z) = \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+n, c-\gamma)(c)_n}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{3.1}$$

and

$$E_{\alpha,\beta}^{\gamma,q;c}(z) = \sum_{n=0}^{\infty}\frac{\mathbf{B}_p(\gamma+nq, c-\gamma)(c)_{nq}}{\mathbf{B}(\gamma, c-\gamma)\Gamma(\alpha n+\beta)}\frac{z^n}{n!}. \tag{3.2}$$

Corollary 3.1. *Let* $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; *with* $\Re(\alpha) > \max\{0, \Re(c)-1\}; \Re(c) > 0$. $\Re(\gamma) > 0, \Re(2\rho+j) > 0$ *and* $x > 0$. *Then the following formula holds true:*

$$\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)}E_{\alpha,\beta}^{\gamma}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

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$$= \left(\frac{2}{3}\right)^{2(\rho+j)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} {}_2\psi_2 \left[\begin{array}{c} (\rho,1),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1); \end{array} |y \right]. \quad (3.3)$$

Corollary 3.2. *Let $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; with $\Re(\alpha) > \max\{0, \Re(c)-1\}; \Re(c) > 0$. $\Re(\gamma) > 0, \Re(2\rho+j) > 0$ and $x > 0$. Then the following formula holds true:*

$$\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} E^{\gamma}_{\alpha,\beta}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} {}_2\psi_2 \left[\begin{array}{c} (\rho,1),(\gamma,1); \\ \\ (\beta,\alpha),(2\rho+j,1); \end{array} ;|\tfrac{4}{9}y \right]. \quad (3.4)$$

Corollary 3.3. *Let $y, c, \rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; with $\Re(\alpha) > \max\{0, \Re(c)-1\}; \Re(c) > 0$. $\Re(\gamma) > 0, \Re(2\rho+j) > 0$ and $x > 0$. Then the following formula holds true:*

$$\int_0^1 x^{(\rho+j-1)}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{(2(\rho+j)-1)}\left(1-\frac{x}{4}\right)^{(\rho-1)} E^{\gamma,q}_{\alpha,\beta}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho+j)} \frac{\Gamma(\rho+j)}{\Gamma(\gamma)} {}_2\psi_2 \left[\begin{array}{c} (\rho,1),(\gamma,q); \\ \\ (\beta,\alpha),(2\rho+j,1); \end{array} ;|y \right] \quad (3.5)$$

Corollary 3.4. *Let $\rho, j, \alpha, \beta, \gamma \in \mathbb{C}$; with $\Re(\alpha) > 0, \Re(c) > \Re(\gamma) > 0, \Re(2\rho+j) > 0$ and $x > 0$. Then the following formula holds true:*

$$\int_0^1 x^{(\rho-1)}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{(2\rho-1)}\left(1-\frac{x}{4}\right)^{(\rho+j-1)} E^{\gamma,q}_{\alpha,\beta}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$
$$= \left(\frac{2}{3}\right)^{2(\rho)} \frac{\Gamma(\rho+j)}{\Gamma(\rho)} {}_2\psi_2 \left[\begin{array}{c} (\rho,1),(\gamma,q); \\ \\ (\beta,\alpha),(2\rho+j,1); \end{array} ;|\tfrac{4}{9}y \right]. \quad (3.6)$$

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Gauhar Rahman: Department of Mathematics, International Islamic University, Islamabad, Pakistan

E-mail address: gauhar55uom@gmail.com

A. Ghaffar: Department of Mathematical Science, BUITEMS Quetta, Pakistan

E-mail address: abdulghaffar.jaffar@gmail.com

Kottakkaran Sooppy Nisar: Department of Mathematics, College of Arts and Science-Wadi Al-dawaser, 11991, Prince Sattam bin Abdulaziz University, Alkharj, Kingdom of Saudi Arabia

E-mail address: ksnisar1@gmail.com, n.sooppy@psau.edu.sa

Shahid Mubeen: Department of Mathematics, University of Sargodha, Sargodha, Pakistan

E-mail address: smjhanda@gmail.com