The Monty Hall Problem as a Bayesian Game

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Abstract

This paper formulates the classic Monty Hall problem as a Bayesian game. Allowing Monty a small amount of freedom in his decisions facilitates a variety of solutions. The solution concept used is the Bayes Nash Equilibrium (BNE), and the set of BNE relies on Monty's motives and incentives. We endow Monty and the contestant with common prior probabilities (p) about the motives of Monty, and show that under certain conditions on p, the unique equilibrium is one where the contestant is indifferent between switching and not switching. This coincides and agrees with the typical responses and explanations by experimental subjects. Finally, we show that our formulation can explain the experimental results in *Page (1998)* [12]; that more people gradually choose switch as the number of doors in the problem increases.

Keywords: Monty Hall; Equiprobability Bias; games of incomplete information; Bayes Nash Equilibrium

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1 Introduction

The "Monty Hall" problem is a famous scenario in decision theory. It is a simple problem, yet people confronted with this dilemma almost overwhelmingly seem to make the incorrect choice. This paper provides justification for why a rational agent may actually be reasoning correctly when he or she makes this ostensibly erroneous choice.

This notorious problem arrived in the public eye in the form of a September, 1990 column published by Marilyn vos Savant in *Parade Magazine*[1] (*vos Savant (1990)* [20]). vos Savant's solution drew a significant amount of ire, as people vehemently disagreed with her answer. The ensuing debate can be recounted in several subsequent articles by vos Savant [21] [22], as well as in a 1991 *New York Times* article by James Tierney [18]. The problem draws its name from the former host of the TV show "Let's Make a Deal", Monty Hall, and is formulated as follows:

There is a contestant (whom we shall call Amy (A)) on the show and its famous host Monty Hall (M). Amy and Monty are faced on stage by three curtains. Monty hides a car behind one of the curtains at random, and a goat behind the other two curtains. Amy selects a curtain, and then, before revealing to Amy the outcome of her choice, Monty opens one of the other unopened curtains to reveal a goat. Amy is then given the option of switching to the other unopened curtain, or of staying on her current choice. Should Amy switch?

There are a significant number of papers in both the economics and psychology literatures that look at the typical person's answers to this question. The overwhelming result presented in these bodies of work, including [2], [4], [13], is that that subjects confronted with the problem solve it incorrectly and fail to recognize that Amy should switch. The typical justification put forward by the subjects is that the likelihood of success if they were to switch curtains is the same as if they were to stay with their original choice of curtain. This mistake is often cited as an example of the equiprobability bias heuristic ([14], [15], [19], [9]) as described in *Lecoutre (1992)* [7].

[1] Note that this problem was published as early as 1975 by *Selvin (1975)* [16] [17]; and was also mentioned in *Nalebuff (1987)* [11], among others.

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It is important to recognize; however, that in the scenario presented above, as in vos Savant's columns, Selvin's articles, and many other formulations of the problem, it is unclear whether Monty must reveal a goat and allow Amy to switch. However, their subsequent analysis and solutions imply that this is an implicit condition on the problem.

One of the first properties of the model formulated by our paper is that the game show host is also a strategic player, in addition to the game's contestant. Indeed this agrees with the *New York Times* [18] article, in which the eponymous host Monty, himself, notes that he had a significant amount of freedom. To model this problem, we will formulate it as a non-cooperative game, as developed by Nash, von Neumann and others. There have been several other papers that model this scenario as a game: notably by *Fernandez and Piron (1999)* [3], *Mueser and Granberg (1999)* [10], *Bailey (2000)* [1], and *Gill (2011)* [6]. The most similar of these works to this paper is [3]. There, Fernandez and Piron also note Monty's freedom to choose, and in fact allow him to manipulate various behavioral parameters of Amy's. In addition to giving Monty this wide array of strategy choices, the authors also introduce a third player, "the audience". Monty's objective as a host is to make the situation difficult for the audience to predict, and this incentive, combined with Monty's menu of options, allow for a variety of equilibria.

In this paper, we are able to generate our results simply by giving Monty the option to allow or prevent Amy from switching choices. We note that Monty's incentives and personality matter, and show that his disposition towards Amy's success affects the equilibria of the game. We model this by considering a game of Incomplete Information [2]. Both Monty and Amy share a common prior about whether Monty is "sympathetic" or "antipathetic" towards Amy, but only Monty sees the realization of his "type" before the game is played. Using this, we are able to show that there may be an equilibrium where it is not optimal for Amy to always switch from her initial choice following a revelation by Monty. There is an equilibrium under which Amy's payoffs under switching and staying are the same (as people are wont to claim). We also generalize the three

[2] See, for example, *Fudenberg and Tirole (1991)* [5].

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door problem to one with n doors, and show that our formulation agrees with the experimental evidence in *Page (1998)* [12]. As the number of doors increases, the range of prior probabilities under which there is an equilibrium where Amy does not always switch, shrinks. Moreover, there is a cutoff number of doors, beyond which, the only equilibrium is the one where Amy always switches.

There are several contributions made by this paper. We construct a simple model of the Monty Hall problem where a rational player is indifferent between switching and not switching. We illustrate that, given only a slight amount of freedom, Monty's incentives matter significantly. Finally, we show how the experimental results in [12] can arise as a consequence of our model.

The structure of this paper is as follows. In section two, we formalize the problem as one where Monty may choose whether to reveal a goat and examine the cases where Monty derives utility from Amy's success or failure, respectively. In the next section, we further extend the model by allowing there to be uncertainty about whether Monty is sympathetic or antipathetic. We also then extend this analysis beyond the three door case to the more general n door case.

2 The Model

We consider the onstage scenario, where Monty (M) hides two goats and one car among three curtains and Amy (A) must then select a curtain. Suppose that following the initial selection of a curtain by Amy, Monty has two choices:

1. Reveal a goat behind one of the unselected curtains, what we shall call "reveal" (r).
2. Not reveal a goat (h).

Amy has different feasible strategies, depending on what Monty's choice is: If Monty reveals a goat, then Amy may,

1. Switch (s).

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2. Not switch (or keep) (k).

If Monty does not reveal (choice h), then Amy has no option to switch choices and must keep her original selection.

To simplify the analysis, we impose that Monty must hide the car completely at random. Consequently, we can model this situation as a game of **Incomplete Information**:

Amy's initial pick can be modeled as a random draw or a "move of nature". With probability 1/3, her first pick is correct and with probability 2/3, it is incorrect. Monty is able to observe the realization of this draw prior to making his decision and see whether or not Amy is correct. Thus, this game can be modeled as a sequential Bayesian game, where Amy and Monty have the common prior $\theta_1 = 1/3$ and $\theta_2 = 2/3$. Monty is then able to view the "draw" realization i.e. whether or not Amy is correct, before choosing his action. Amy, cannot observe the outcome and has only her prior to go by.

Before going further, we need to specify the two players' preferences in this strategic interaction. Naturally, Amy prefers to end up with the car than with the goat. It is reasonable to suppose that Monty, for a given choice of Amy's, would strictly prefer to reveal a goat[3], and thereby give Amy the option of switching, than to not reveal a goat, and prevent Amy from switching. Assigning Amy to be player 1 and Monty player 2, we write this formally as:

$$\begin{aligned} u_M(s,r) &\succ u_M(s,h) \\ u_M(k,r) &\succ u_M(k,h) \end{aligned} \tag{1}$$

We now split up the analysis into two cases:

1. Case 1: This is what we will call the **Sympathetic Case**: Monty and Amy's preferences are aligned in the sense that else equal Monty would prefer that Amy be correct rather than incorrect.

[3]Presumably, the tension engendered by Amy's decision on whether or not to switch is attractive to the audience of the show, and Monty recognizes this.

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2. Case 2: We call this the **Antipathetic Case**: Here, all else equal, Monty would prefer that Amy be incorrect. Their preferences are not aligned.

2.1 Case 1, Sympathetic

We stipulate that all parameters of the game are common knowledge, including Monty's preferences; i.e. it is common knowledge that Monty is a sympathetic player. As above, Amy's and Monty's sets of strategies S_A[4] and S_M are,

$$\begin{aligned} S_A &= \{s, k\} \\ S_M &= \{r, h\} \end{aligned} \tag{2}$$

The payoff matrix is:

θ_1 :

A \ M	r	h
s	$-1, 0$	$1, 1$
k	$1, 2$	$1, 1$

θ_2 :

A \ M	r	h
s	$1, 2$	$-1, -1$
k	$-1, 0$	$-1, -1$

Lemma 2.1. *The unique Bayes-Nash Equilibrium is in pure strategies*[5]*; it is:*

$$(s, (h, r)) \tag{3}$$

Amy always switches, and type θ_1 of Monty chooses not to reveal, and type θ_2 of Monty chooses to reveal. It is clear that following Monty's choice, Amy knows exactly whether her initial choice was correct. Monty's choice completely reveals to Amy the state of the world.

[4]Since a strategy for a player must specify an action at **every** information set encountered by a player, Amy's strategy should, strictly speaking, be: $S_A = \{s, k\} \times \{k'\}$. Our simplification of this; however, is clearer and does not affect the analysis.

[5]For derivation of this, see Appendix A.1.

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2.2 Case 2, Antipathetic

As in Case 1 (section 2.1), Monty's preferences are common knowledge; i.e. that he is antipathetic towards Amy. Defining Amy and Monty's strategies as before, the payoff matrix is:

θ_1 :

M

A	r	h
s	$-1, 2$	$1, -1$
k	$1, 0$	$1, -1$

θ_2 :

M

A	r	h
s	$1, 0$	$-1, 1$
k	$-1, 2$	$-1, 1$

Interestingly, there are no pure strategy Bayes-Nash Equilibria.

Lemma 2.2. *The unique BNE is in mixed strategies*[6]*; it is:*

$$\left(\Pr_A(s) = \frac{1}{2}, (r, \Pr_M(r) = \frac{1}{2}) \right) \tag{4}$$

Amy switches half the time, while type θ_1 of Monty chooses to reveal, and type θ_2 of Monty chooses to reveal half of the time. Using Bayes' law, we write:

$$\Pr(Correct|Reveal) = \frac{\Pr(Reveal|Correct)\Pr(Correct)}{\Pr(Reveal)} = \frac{1(1/3)}{1(1/3) + (1/2)(2/3)} = \frac{1}{2}$$
$$\Pr(Correct|NotReveal) = \frac{\Pr(NotReveal|Correct)\Pr(Correct)}{\Pr(NotReveal)} = \frac{0(1/3)}{1/3} = 0 \tag{5}$$

Thus, a completely rational Amy knows that if Monty chooses to reveal a goat, there is a 50 percent chance that her initial choice was correct. Simply by allowing Monty the option to not reveal, we have shown that if Monty and Amy's preferences are opposed, the unique equilibrium is one where Amy does not always switch. Moreover, this equilibrium is one under which Amy's beliefs are exactly those proposed so often by subjects in the many experiments: that her chances of success are equal under her decision to switch or not.

[6] For derivation of this, see Appendix A.2.

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3 Uncertainty about Monty's Motives

We now take the framework from the previous section and extend it as follows. Suppose now that Amy is unsure about whether Monty is sympathetic or antipathetic. We model this by saying that Amy and Monty have a common prior p, defined as,

$$p = \Pr(M = M_S)$$

p is the probability that Monty is sympathetic (type M_S); and naturally, $1-p$ is the probability that Monty is antipathetic (type M_A). Consequently, there are 4 states of the world: $\{\theta_1, \theta_2\} \times \{M_S, M_A\}$ Thus, we may present the game matrices:

$M_S\theta_1$:

A \ **M**	r	h
s	$-1, 0$	$1, 1$
k	$1, 2$	$1, 1$

$M_S\theta_2$:

A \ **M**	r	h
s	$1, 2$	$-1, -1$
k	$-1, 0$	$-1, -1$

$M_A\theta_1$:

A \ **M**	r	h
s	$-1, 2$	$1, -1$
k	$1, 0$	$1, -1$

$M_A\theta_2$:

A \ **M**	r	h
s	$1, 0$	$-1, 1$
k	$-1, 2$	$-1, 1$

We write the following lemma:

Lemma 3.1. *Given uncertainty about Monty's motives as formulated above, there is a pure strategy BNE,*

$$(s, h, r, r, h)$$

for $p \geq \frac{1}{3}$.

We see that if Amy is sufficiently optimistic about the nature of Monty, there is a pure strategy equilibrium where she switches.

There may also be mixed strategy BNE, and we write:

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Lemma 3.2. *Given uncertainty about Monty's motives as formulated above, there is a mixed strategy BNE,*

$$\left(\Pr_A(s) = \frac{1}{2}, (\alpha, r, r, \beta)\right)$$

where type $M_S\theta_1$ plays $\Pr(r) = \alpha$ and type $M_S\theta_2$ plays $\Pr(r) = \beta$, where α, β satisfy

$$-p\alpha + 3p + 2\beta - 2p\beta - 1 = 0 \tag{6}$$

Equation 6 has a solution[7] *in acceptable (i.e. $\alpha, \beta \in [0, 1]$) α and β for all $p \leq \frac{1}{2}$.*

How should we interpret these equilibria? We see that as long as Amy believes that there is at least a 50 percent chance that Monty is antipathetic, there is an equilibrium where she switches half of the time. Interestingly, as long as the common belief as to Monty's type falls in that range, Amy's mixing strategy is independent of the actual value of p. Amy mixes equally regardless of whether p is 0 or $\frac{1}{2}$. On the other hand, Monty's strategies are not independent of p. We may rearrange equation (5) and take derivatives to see how Monty's optimal choice of α and β vary depending on p. Rearranging (5), we obtain,

$$\beta(p, \alpha) = \frac{1 + p\alpha - 3p}{2 - 2p} \tag{7}$$

Then,

$$\frac{\partial \beta}{\partial p} = \frac{2\alpha - 4}{2 - 2p^2} \tag{8}$$

This is strictly less than 0 for all $p \neq \frac{1}{2}$ and for all permissible α. Similarly,

$$\alpha(p, \beta) = \frac{3p + 2\beta - 2p\beta - 1}{p} \tag{9}$$

Then,

$$\frac{\partial \alpha}{\partial p} = \frac{1 - 2\beta}{p^2} \tag{10}$$

This is greater than or equal to 0 for all p and strictly greater than 0 for $\beta \neq \frac{1}{2}$.

[7] For derivation of this, see Appendix A.3.

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As the common belief that Monty is sympathetic increases, type $M_S\theta_1$ plays reveal more often. Likewise, as p increases, type $M_A\theta_2$: plays reveal more often. Both relationships are somewhat counter-intuitive: as p increases, the sympathetic Monty reveals more often in the state where Amy was correct, and the unsympathetic Monty reveals less in the state where Amy is incorrect.

3.1 Generalizing to n Doors

We now generalize the previous scenario to one with $n-1$ doors ($n \geq 3$) concealing goats and one door concealing a car.

Thus, it is easy to obtain the following lemma:

Lemma 3.3. *The unique pure strategy BNE is*

$$(s, (h, r, r, h))$$

for $p \geq \frac{1}{n}$.

As before, there may also be a mixed strategy BNE, presented in the lemma,

Lemma 3.4. *There is a set of mixed strategy BNE given by:*

$$\left(\Pr_A(s) = \frac{1}{2}, (\alpha, r, r, \beta) \right)$$

where type $M_S\theta_1$ *plays* $\Pr(r) = \alpha$ *and type* $M_S\theta_2$ *plays* $\Pr(r) = \beta$*. This equation defines a BNE for all acceptable* α*,* β *(i.e.* $\alpha, \beta \in [0, 1]$*) satisfying*

$$-p\alpha + np + (n-1)\beta - (n-1)p\beta - 1 = 0 \tag{11}$$

As a corollary,

Corollary 3.1. *Equation 11,*

$$-p\alpha + np + (n-1)\beta - (n-1)p\beta - 1 = 0$$

has a solution in acceptable α *and* β *for all* p *that satisfy:*

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$$\begin{gathered}\frac{1}{n-\alpha} \geq p \\ \frac{1-(n-1)\beta}{(n-1)-(n-1)\beta} \geq p \geq \frac{1-(n-1)\beta}{n-(n-1)\beta}\end{gathered} \tag{12}$$

Note that the second inequality in Corollary 3.1 requires that $\beta \leq \frac{1}{n-1}$. The proof of this corollary is simple and may be found in Appendix A.3.

Define set $\mathcal{A}_n$ as

$$\mathcal{A}_n = \left[\frac{1}{n}, 1\right] \tag{13}$$

That is, for a given situation with n doors, $\mathcal{A}_n$ is the set of priors, p, under which there is a pure strategy BNE where Amy switches. We can then obtain the following theorem:

Theorem 3.1. *As the number of doors increases, the set of priors that support a BNE where Amy switches is monotonic in the following sense:*

$$\mathcal{A}_n \subset \mathcal{A}_{n+1} \tag{14}$$

Moreover, in the limit, as the number of doors goes to infinity, we see that any interior probability supports a BNE where Amy switches.

Now, take the derivatives of the three constraints on the set of priors, p, given by equation 12. They are:

$$\begin{gathered}-\frac{1}{(n-\alpha)^2} \\ \frac{1}{(\beta-1)(n-1)^2} \\ -\frac{1}{((\beta-1)n-\beta)^2}\end{gathered} \tag{15}$$

Evidently, each of these is negative for all acceptable values of the parameters and therefore, we see that the constraints are all strictly decreasing in n. We shall also find it useful to look at the limit of the two possible upper bounds for p:

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$$\lim_{n\to\infty} \frac{1}{n-\alpha} = 0 \tag{16}$$

$$\lim_{n\to\infty} \frac{1-(n-1)\beta}{(n-1)-(n-1)\beta} = -\frac{\beta}{1-\beta} \tag{17}$$

The term in equation 16 is strictly positive for all values of n, whereas the term in equation 17 is less than 0 for all n greater than some cutoff value. Thus, there must be some cutoff value $\hat{n}$, beyond which the first constraint, that $\frac{1}{n-\alpha} \geq p$, does not bind. Indeed, it is simple to derive the following lemma:

Lemma 3.5. *The constraint* $\frac{1}{n-\alpha} \geq p$ *does not bind for all* $n > \hat{n}$, *where* $\hat{n}$ *is given by*

$$\hat{n} = \frac{1}{2}\left(\alpha + 2 + \sqrt{\frac{\beta\alpha^2 - 4\alpha + 4}{\beta}}\right) \tag{18}$$

Define the set $\mathcal{B}_n$ as the interval of prior probabilities p, that satisfy the conditions 27 in Corollary 3.1 for a given number of doors n, $n > \hat{n}$. Because $n > \hat{n}$, the constraint

$$\frac{1-(n-1)\beta}{(n-1)-(n-1)\beta} \geq p \geq \frac{1-(n-1)\beta}{n-(n-1)\beta}$$

binds. Let b denote the left hand side of this expression and a the right hand side:

$$b = \frac{1-(n-1)\beta}{(n-1)-(n-1)\beta} \qquad a = \frac{1-(n-1)\beta}{n-(n-1)\beta} \tag{19}$$

Thus, $\mathcal{B}_n$ is defined as

$$\mathcal{B}_n = [a, b] \tag{20}$$

Having defined the set $\mathcal{B}_n$, we can now state a natural corollary to Lemma 3.5:

Corollary 3.2. *There is a number* $N \in \mathbb{N}$ *such that if* $n > N$, $\mathcal{B}_n = \emptyset$

Proof. This corollary follows immediately from Lemma 3.5 and equations 28 and 30 (monotonicity of the constraints, and behavior of left-hand side constraint in the limit). □

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Since each set $\mathcal{B}_n$ is a closed interval, it is natural to define the size of the set, ν_n, as the Lebesgue measure of the interval. That is, if a, b are the two endpoints of the interval $\mathcal{B}_n$,

$$\nu_n = |b - a| \tag{21}$$

We can obtain the following theorem:

Theorem 3.2. $\nu_n > \nu_{n+1}$. *That is, the size of the set of priors under which there is a mixed strategy BNE is strictly decreasing in* n.

Proof may be found in Appendix A.4.

This theorem, in conjunction with theorem 3.1, above, support the experimental evidence from *Page (1998)* [12]. There, the author finds evidence that supports the hypotheses that people's performance on the Monty Hall problem increases as the number of doors increases, and that, moreover, their improvement in performance is gradual in nature. Both of these follow from theorems 3.1 and 3.2 in our paper. If we view the prior probability p as itself being drawn from some population distribution of prior probabilities, we see that as the number of doors increases, the likelihood that there is a BNE where Amy always switches is strictly increasing. The probability that the random draw of p falls in the required interval is strictly increasing for a fixed distribution, since the interval grows monotonically as n increases.

4 Conclusion

In this paper we have developed and pursued several ideas. The first is that even a small amount of freedom on the part of Monty begets equilibria that differ from the canonical "always switch" solution. Moreover, given this freedom, Monty's incentives and preferences matter, and affect optimal play.

One of the characteristics of a mixed strategy equilibrium in games in which agents have two pure strategies is that each player's mixing strategy leaves the other player (or player's) indifferent between their two pure strategies. Thus, in a sense, the equiprobability bias may not as irrational as it seems. When the agent is confronted with a strategic adversary, and the decision problem becomes a game, any mixed strategy equilibrium yields for the agent the exact same expected reward. Indeed, the frequent remarks and comments in the literature by subjects confronted with this problem, coincide exactly with this equality of reward in expectation.

Of course, that is not to say that the participants in these experiments must be correct in doing so. In the variations posed there is almost always at least an implicit restriction requiring Monty to reveal a door (and often an explicit restriction). In such phrasings of the problem, regardless of Monty's motivation, the situation is merely a decision problem on the part of Amy, and the optimal solution is the one where she should always switch. However, many decision-making situations encountered by people in social settings, perhaps especially atypical ones–think of a sales situation–are strategic in nature and so it could be that mistakes such as the equiprobability heuristic are indicative of this. That is, so many situations are strategic that decision makers treat every situation as if it were strategic. Perhaps people do not solve probability problems correctly because they do not need to.

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A Appendix

A.1 Lemma 2.1 Proof

Proof. First we look for pure strategy BNE. The possible candidates for pure strategy BNE are $(s,(h,r))$ and $(k,(r,r))$. To establish that these are equilibria we check that Amy does not deviate profitably. That is, we need to check that $Eu_A(s,(h,r)) \geq Eu_A(k,(h,r))$ and $Eu_A(k,(r,r)) \geq Eu_A(s,(r,r))$.

$$\begin{aligned} & Eu_A(s,(h,r)) \geq Eu_A(k,(h,r)) \\ \Leftrightarrow \quad & 1 \geq -\frac{1}{3} \end{aligned} \tag{22}$$

$$\begin{aligned} & Eu_A(k,(r,r)) \geq Eu_A(s,(r,r)) \\ \Leftrightarrow \quad & -\frac{1}{3} \geq \frac{1}{3} \end{aligned} \tag{23}$$

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We see that $(s,(h,r))$ is an equilibrium, but $(k,(r,r))$ is not.

From Lemma 3.2, using $p=1$, it is clear that there is no mixed strategy BNE. □

A.2 Lemma 2.2 Proof

Proof. First we look for pure strategy BNE. The possible candidates for pure strategy BNE are $(s,(r,h))$ and $(k,(r,r))$. To establish that these are equilibria we check that Amy does not deviate profitably. That is, we need to check that $Eu_A(s,(r,h)) \geq Eu_A(k,(r,h))$ and $Eu_A(k,(r,r)) \geq Eu_A(s,(r,r))$.

$$\begin{aligned} & Eu_A(s,(r,h)) \geq Eu_A(k,(r,h)) \\ \Leftrightarrow \quad & -1 \geq -\frac{1}{3} \end{aligned} \tag{24}$$

$$\begin{aligned} & Eu_A(k,(r,r)) \geq Eu_A(s,(r,r)) \\ \Leftrightarrow \quad & -\frac{1}{3} \geq \frac{1}{3} \end{aligned} \tag{25}$$

We see that neither $(s,(r,h))$ nor $(k,(r,r))$ are equilibria.

From Lemma 3.2, using $p=0$, it is clear that the unique mixed strategy BNE is

$$\left(\Pr_A(s)=\frac{1}{2},(r,\Pr_M(r)=\frac{1}{2})\right)$$

□

A.3 Corollary 3.1 Proof

Proof. First, we rewrite equation 11:

$$\begin{aligned} \alpha &= \frac{np+(n-1)\beta-(n-1)p\beta-1}{p} \\ \beta &= \frac{1+p\alpha-np}{(n-1)(1-p)} \end{aligned} \tag{26}$$

Since α and β are probability distributions, we know that $\alpha \in [0,1]$ and $\beta \in [0,1]$. Thus, we may examine each of them in turn:

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$$\begin{aligned}\beta \geq 0 \quad &\Leftrightarrow \quad \frac{1+p\alpha-np}{(n-1)(1-p)} \geq 0 \\ &\Leftrightarrow \quad 1+p\alpha-np \geq 0 \\ &\Leftrightarrow \quad \frac{1}{n-\alpha} \geq p\end{aligned} \tag{27}$$

$$\begin{aligned}\beta \leq 1 \quad &\Leftrightarrow \quad \frac{1+p\alpha-np}{(n-1)(1-p)} \leq 1 \\ &\Leftrightarrow \quad 1+p\alpha-np \leq (n-1)(1-p) \\ &\Leftrightarrow \quad p(\alpha-1) \leq n-2\end{aligned} \tag{28}$$

$$\begin{aligned}\alpha \geq 0 \quad &\Leftrightarrow \quad \frac{np+(n-1)\beta-(n-1)p\beta-1}{p} \geq 0 \\ &\Leftrightarrow \quad np+(n-1)\beta-(n-1)p\beta-1 \geq 0 \\ &\Leftrightarrow \quad p \geq \frac{1-(n-1)\beta}{n-(n-1)\beta}\end{aligned} \tag{29}$$

$$\begin{aligned}\alpha \leq 1 \quad &\Leftrightarrow \quad \frac{np+(n-1)\beta-(n-1)p\beta-1}{p} \leq 1 \\ &\Leftrightarrow \quad np+(n-1)\beta-(n-1)p\beta-1 \leq p \\ &\Leftrightarrow \quad p \leq \frac{1-(n-1)\beta}{(n-1)-(n-1)\beta}\end{aligned} \tag{30}$$

Evidently, 12,

$$\frac{1-(n-1)\beta}{n-(n-1)\beta} \leq p \leq \frac{1-(n-1)\beta}{(n-1)-(n-1)\beta}$$

holds if and only if $\beta \leq \frac{1}{n-1}$. □

A.4 Theorem 3.2 Proof

Proof. Evidently, it is sufficient to show that

$$\frac{\partial b}{\partial n} < \frac{\partial a}{\partial n} \tag{31}$$

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Suppose for the sake of contradiction that

$$\frac{\partial b}{\partial n} > \frac{\partial a}{\partial n} \tag{32}$$

Which holds,

$$\begin{aligned} \Leftrightarrow \quad & \frac{1}{(\beta-1)(n-1)^2} > -\frac{1}{((\beta-1)n-\beta)^2} \\ \Leftrightarrow \quad & \beta^2(n-1)^2 - \beta(n-1)(n+1) + 2n - 1 < 0 \end{aligned} \tag{33}$$

Denote the left-hand side of this expression by $\phi = \phi(\beta, n)$. Then, $\frac{\partial \phi}{\partial \beta} < 0$ and so we see that ϕ is strictly decreasing in β. Since by Corollary 3.1 β is bounded above by $\frac{1}{n-1}$, $\phi > \phi(\beta = \frac{1}{n-1})$ for all permissible β.

We evaluate ϕ at $\beta = \frac{1}{n-1}$ and obtain

$$\begin{aligned} & 1 - (n+1) + 2n - 1 < 0 \\ \Leftrightarrow \quad & n - 1 < 0 \end{aligned} \tag{34}$$

We have obtained a contradiction and have proved our result.

□