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Article

On the Duality of Regular and Local Functions

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Abstract: In this paper, we relate Poisson's summation formula to Heisenberg's uncertainty principle. They both express Fourier dualities within the space of tempered distributions and these dualities are furthermore the inverses of one another. While Poisson's summation formula expresses a duality between discretization and periodization, Heisenberg's uncertainty principle expresses a duality between regularization and localization. We define regularization and localization on generalized functions and show that the Fourier transform of regular functions are local functions and, vice versa, the Fourier transform of local functions are regular functions.

Keywords: generalized functions; tempered distributions; regular functions; local functions; regularization-localization duality; regularity; Heisenberg's uncertainty principle

Classification: MSC 42B05, 46F10, 46F12

1. Introduction

Regularization is a popular trick in applied mathematics, see [1] for example. It is the technique "to approximate functions by more differentiable ones" [2]. Its terminology coincides moreover with the terminology used in generalized function spaces. They contain two kinds of functions, "regular functions" and "generalized functions". While regular functions are being functions in the ordinary functions sense which are infinitely differentiable in the ordinary functions sense, all other functions become "infinitely differentiable" in the "generalized functions sense" [3]. In this way, all functions are being infinitely differentiable. Localization, in contrast to that, is another popular technique. It allows for example to integrate functions which could not be integrated otherwise, if we think of "locally integrable" functions or if we think of the Short-Time Fourier Transform (STFT), capable to analyze infinitely extended signals. Although, regularization and localization appear to be quite different, a

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connection between these operations is no surprise. It is ubiquitous in the literature. The theorem below, however, appears in wider sense. It holds within the space of tempered distributions and is directly related to Heisenberg's uncertainty principle. It is moreover the inverse of an already known discretization-periodization duality.

Section 2 provides an introduction to the notations used and previous results. Section 3 presents a justification for Section 4 where regularization and localization within the space of tempered distributions are defined. Section 5 provides symbolic calculation rules based on these definitions, needed to prove the theorem in Section 6. Section 7 connects these results to results in a previous study and Section 8, finally, concludes this study and provides an outlook.

2. Preliminaries

Let δ_{kT} be the Dirac impulse shifted by $k \in \mathbb{Z}^n$ units of $T \in \mathbb{R}^n_+ = \{t \in \mathbb{R}^n : 0 < t_\nu < \infty,\ 1 \leqslant \nu \leqslant n\}$, kT being componentwise multiplication, within the space $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^n)$ of tempered distributions (generalized functions that do not grow faster than polynomials) and let

$$\text{III}_T := \sum_{k \in \mathbb{Z}^n} \delta_{kT}$$

be the Dirac comb. Then $\delta_{kT} \in \mathcal{S}'$ and $\text{III}_T \in \mathcal{S}'$ for any $T \in \mathbb{R}^n_+$ are tempered distributions [4–6]. We shortly write δ instead of δ_{kT} if $k = 0$. The Fourier transform $\mathcal{F}$ in $\mathcal{S}'$ is defined as usual and such that $\mathcal{F}1 = \delta$ and $\mathcal{F}\delta = 1$ where 1 is the function being constantly one [3,4,7–12]. The Dirac comb is moreover known for its excellent discretization (sampling) and periodization properties [7,11–13]. While multiplication $\text{III}_T \cdot f$ in $\mathcal{S}'$ samples a function $f \in \mathcal{S}'$, the corresponding convolution product $\text{III}_T * f$ periodizes f in $\mathcal{S}'$.

The following two lemmas summarize the demands that must be put on $f \in \mathcal{S}'$ such that f can be sampled or periodized in $\mathcal{S}'$. Recall that smoothness, i.e., infinite differentiability, is not a demand. It is a given fact for all functions in generalized function spaces. Also recall that $\mathcal{O}_M$ is the space of multiplication operators in $\mathcal{S}'$ and $\mathcal{O}_C{}'$ is the space of convolution operators in $\mathcal{S}'$ according to Laurent Schwartz' theory of distributions [4,5,14–20].

Lemma 1 (Discretization)**.** *Generalized functions $f \in \mathcal{S}'$ can be sampled in $\mathcal{S}'$ if and only if $f \in \mathcal{O}_M$.*

Proof. Any uniform discretization (sampling) in $\mathcal{S}'$ corresponds to forming the product

$$\text{III}_T \cdot f \quad \text{in} \quad \mathcal{S}'$$

where $\text{III}_T \in \mathcal{S}'$ is the Dirac comb. Furthermore, III_T is no regular function, i.e., $\text{III}_T \in \mathcal{S}' \setminus \mathcal{O}_M$. On the other hand, for any multiplication product in $\mathcal{S}'$, it is required that at least one of the two factors is in $\mathcal{O}_M$. Hence, $f \in \mathcal{O}_M \subset \mathcal{S}'$. Otherwise the product does not exist. Vice versa, if $f \in \mathcal{O}_M$ then $\text{III}_T \cdot f$ exists due to $\mathcal{S}' \cdot \mathcal{O}_M \subset \mathcal{S}'$. □

An equivalent statement is the following lemma.

Lemma 2 (Periodization)**.** *Generalized functions $f \in \mathcal{S}'$ can be periodized in $\mathcal{S}'$ if and only if $f \in \mathcal{O}_C{}'$.*

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Proof. Any periodization in $\mathcal{S}'$ corresponds to forming the convolution product

$$\mathrm{III}_T * f \quad \text{in} \quad \mathcal{S}'$$

where $\mathrm{III}_T \in \mathcal{S}'$ is the Dirac comb. Furthermore, III_T is of no rapid descent, i.e., $\mathrm{III}_T \in \mathcal{S}' \setminus \mathcal{O}_C{}'$. On the other hand, for any convolution product in $\mathcal{S}'$, it is required that at least one of the two factors is in $\mathcal{O}_C{}'$. Hence, $f \in \mathcal{O}_C{}' \subset \mathcal{S}'$. Otherwise the convolution product does not exist. Vice versa, if $f \in \mathcal{O}_C{}'$ then $\mathrm{III}_T * f$ exists due to $\mathcal{S}' * \mathcal{O}_C{}' \subset \mathcal{S}'$. $\square$

In a previous study [19], we used these insights in order to define operations of discretization $\sqcup\!\!\sqcup_T$ and periodization $\bigtriangleup\!\!\bigtriangleup_T$ in $\mathcal{S}'$. While discretization is an operation $\sqcup\!\!\sqcup_T : \mathcal{O}_M \to \mathcal{S}',\ f \mapsto \sqcup\!\!\sqcup_T f := \mathrm{III}_T \cdot f$, periodization is an operation $\bigtriangleup\!\!\bigtriangleup_T : \mathcal{O}_C{}' \to \mathcal{S}',\ g \mapsto \bigtriangleup\!\!\bigtriangleup_T g := \mathrm{III}_T * g$, respectively. Starting from these two definitions we proved that

$$\mathcal{F}(\sqcup\!\!\sqcup f) = \bigtriangleup\!\!\bigtriangleup(\mathcal{F}f) \qquad \text{and} \tag{1}$$

$$\mathcal{F}(\bigtriangleup\!\!\bigtriangleup g) = \sqcup\!\!\sqcup(\mathcal{F}g) \tag{2}$$

hold in $\mathcal{S}'$, both being expressions of Poisson's Summation Formula. We shortly write $\sqcup\!\!\sqcup$ and $\bigtriangleup\!\!\bigtriangleup$ instead of $\sqcup\!\!\sqcup_T$ and $\bigtriangleup\!\!\bigtriangleup_T$ if $T_\nu = 1$ for all $1 \leqslant \nu \leqslant n$.

Recall moreover that these rules are a consequence of the Fourier duality

$$\mathcal{F}(\alpha \cdot f) = \mathcal{F}\alpha * \mathcal{F}f \qquad \text{and} \tag{3}$$

$$\mathcal{F}(g * f) = \mathcal{F}g \cdot \mathcal{F}f \qquad \text{in} \quad \mathcal{S}' \tag{4}$$

for any $\alpha \in \mathcal{O}_M$, $g \in \mathcal{O}_C{}'$ and $f \in \mathcal{S}'$ which is, according to Laurent Schwartz' theory of generalized functions, the *widest* possible comprehension of both, multiplication and convolution within the space of tempered distributions [4,14,18]. It lies at the very heart of $\mathcal{S}'$. Many calculation rules in $\mathcal{S}'$, including Equations (1), (2), (7), (8), (11), (12) and Lemmas 1, 2, 3, 4 rely on it.

3. Feasibilities

The following two lemmas provide justifications for the way we will define regularizations and localizations in $\mathcal{S}'$ below. They will allow us to invert discretizations and periodizations in $\mathcal{S}'$.

Lemma 3 (Regularization)**.** *Let $\varphi \in \mathcal{S}$. Then for any $f \in \mathcal{S}'$, $\varphi * f$ can be sampled.*

Proof. This is a consequence of the fact that $\mathcal{S} * \mathcal{S}' \subset \mathcal{O}_M$ [4,10,14,18] and Lemma 1. $\square$

An equivalent statement is the following lemma.

Lemma 4 (Localization)**.** *Let $\varphi \in \mathcal{S}$. Then for any $f \in \mathcal{S}'$, $\varphi \cdot f$ can be periodized.*

Proof. It follows from the fact that $\mathcal{S} \cdot \mathcal{S}' \subset \mathcal{O}_C{}'$ [4,18], which is the Fourier dual $\mathcal{F}(\mathcal{S} * \mathcal{S}') = \mathcal{F}(\mathcal{O}_M)$ of $\mathcal{S} * \mathcal{S}' \subset \mathcal{O}_M$, and Lemma 2. $\square$

 doi:10.20944/preprints201705.0175.v1

It is interesting to observe that $\varphi *$ and $\varphi \cdot$ stretch and compress $f \in \mathcal{S}'$, respectively. This property is moreover independent of the actual choice of $\varphi \in \mathcal{S}$. It can therefore be attributed to the operations of convolution and multiplication themselves.

4. Definitions

"One of the main applications of convolution is the regularization of a distribution" [14] or the regularization of ordinary functions which are not being infinitely differentiable in the conventional functions sense. Its actual importance lies furthermore in the fact that it is the reversal of discretization.

Regularization is usually understood as the approximation of generalized functions via approximate identities [2,8–10,14,21–23]. In this paper, however, we extend this idea by allowing *any* $\varphi \in \mathcal{S}$ and by allowing even *ordinary* functions $f \in \mathcal{S}'$ to be used for regularizations in $\mathcal{S}'$. This approach naturally includes the special case of choosing approximate identities without unnecessarily restricting our theorem below. Lemmas 3 and 4 above justify the following two definitions.

Definition 5 (Regularization)**.** *Let $\varphi \in \mathcal{S}$. Then for any tempered distribution $f \in \mathcal{S}'$ we define another tempered distribution by*

$$\cap_\varphi f := \varphi * f \tag{5}$$

which is a regular, slowly growing function in $\mathcal{O}_M \subset \mathcal{S}'$. The operation $\cap_\varphi$ is called regularization, approximation, interpolation or smoothing of f by means of φ. It is a linear continuous operation $\cap_\varphi : \mathcal{S}' \to \mathcal{O}_M,\ f \mapsto \cap_\varphi f$. The result of $\cap_\varphi$ is called regular function of f in $\mathcal{S}'$.

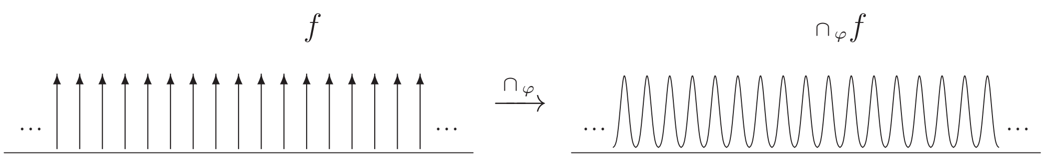

Figure 1. The regularization of generalized function f yields regular function $\cap_\varphi f$.

Regular functions are functions in the ordinary functions sense which are infinitely differentiable in the ordinary functions sense, a function property that is of immense value in many branches of mathematics. Regular functions belong to $\mathcal{O}_M$ because $\mathcal{S} * \mathcal{S}' \subset \mathcal{O}_M$, see e.g. [4,10,18]. They maintain the being 'tempered' property, i.e., they do not grow faster than polynomials, which is common to all tempered distributions but add the regularity of φ to f. It follows that regularized functions can always be sampled according to Lemma 1.

Regularizations are treated in many mathematical textbooks [2,5,12,14,15,21] and scientific papers [1,24–27]. They are also known in terms of "regularizers" [1,25–27], "smooth cutoff functions" [2,28] and "mollifiers" [16,29–33], a term that goes back (see [16], p.63) to K.O. Friedrichs [29]. Regularized $rect$ functions (characteristic functions of an interval) are known as "mesa function", "tapered box" [12] or "tapered characteristic function" and "taper function" [34] or as "C^∞ bell" or "smoothed top hat" function in [35]. Mostly, regularizations are required "to obtain regularized interpolating kernels" such as in [27].

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Away from the generalized functions literature, we furthermore encounter regularizations in terms of "smoothings", "interpolations", "zero-paddings" or "approximations" because they are not only applied to generalized functions, they are also applied to ordinary functions, usually to obtain better "regularity" properties for functions, i.e. better differentiability. Regularity is also a topic discussed in [36], for example. It is closely related to localization.

Definition 6 (Localization)**.** *Let $\varphi \in \mathcal{S}$. Then for any tempered distribution $f \in \mathcal{S}'$ we define another tempered distribution by*

$$\sqcap_\varphi f := \varphi \cdot f \tag{6}$$

which is a generalized function of rapid descent in ${\mathcal{O}_C}' \subset \mathcal{S}'$. The operation $\sqcap_\varphi$ is called localization or restriction of f by means of φ. It is a linear continuous operation $\sqcap_\varphi : \mathcal{S}' \to {\mathcal{O}_C}'$, $f \mapsto \sqcap_\varphi f$. If $\varphi \in \mathcal{D} \subset \mathcal{S}$, it is also called finitization. The result of $\sqcap_\varphi$ is called local function of f in $\mathcal{S}'$.

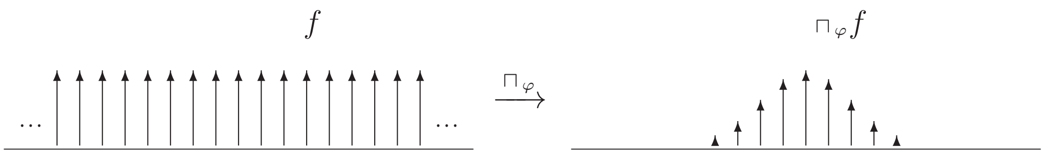

Figure 2. The localization of generalized function f yields local function $\sqcap_\varphi f$.

Local functions belong to ${\mathcal{O}_C}'$ because $\mathcal{S}' \cdot \mathcal{S} \subset {\mathcal{O}_C}'$ [4,14]. They add the 'rapid descent' property of Schwartz functions $\varphi \in \mathcal{S}$ to $f \in \mathcal{S}'$. It follows that localized functions can always be periodized according to Lemma 2.

The term "local" and the treatment of localizations have a long history in mathematics. It culminated, however, in the term "localization operator". It appears 1988 for the first time (see [37], p.133 in [38]) in Daubechies' article [39] and later in Daubechies' 1992 standard textbook [36]. Meanwhile, "localizations" occur in many textbooks [2,36,38–47], amongst others as "localized trigonometric functions" or "localized sine basis" [36,41,48], as "localized frames" [49], "local trigonometric bases", as "local representations" [9] or simply in terms of "locally integrable" functions.

5. Calculation Rules

"One of the basic principles in classical Fourier analysis is the impossibility to find a function f being arbitrarily well localized together with its Fourier transform $\mathcal{F}f$" [50]. This, in particular, can easily be seen if one tries to localize the function that is constantly 1.

Lemma 7 (Localization Balance)**.** *Let $\varphi \in \mathcal{S}$ and let $\hat{\varphi} := \mathcal{F}\varphi$. Then*

$$\mathcal{F}(\cap_\varphi \delta) = \sqcap_{\hat{\varphi}} 1 \qquad \in {\mathcal{O}_C}' \quad \textit{and} \tag{7}$$

$$\mathcal{F}(\sqcap_{\hat{\varphi}} 1) = \cap_\varphi \delta \qquad \in \mathcal{O}_M \quad \textit{in } \mathcal{S}'. \tag{8}$$

In (8) we see that by localizing 1, we delocalize δ, i.e., 1 and its Fourier transform δ cannot be both arbitrarily well localized. This phenomenon is known as Heisenberg's uncertainty principle [9,10,12, 50–52]. Vice versa, in (7) we see that by regularizing δ we increasingly deregularize 1. The entity $\cap_\varphi \delta$

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is also known as an "approximate identity" of δ, usually denoted as δ_ϵ where ϵ is a parameter describing the proximity to δ (see e.g. [14] p.316, p.401 or [21] p.5). Convolving any $f \in \mathcal{S}'$ with δ_ϵ, it creates an approximate identity f_ϵ of f which is a function in the ordinary sense being infinitely differentiable.

Proof. According to (4), $\delta \in \mathcal{O}_C{}'$ can be convolved with $\varphi \in \mathcal{S} \subset \mathcal{S}'$ and, equivalently, $1 \in \mathcal{O}_M$ can be multiplied with $\psi \in \mathcal{S} \subset \mathcal{S}'$, hence

$$\mathcal{F}(\cap_\varphi \delta) = \mathcal{F}(\varphi * \delta) = \mathcal{F}\varphi \cdot \mathcal{F}\delta = \hat{\varphi} \cdot 1 = \sqcap_{\hat{\varphi}} 1$$

holds in $\mathcal{S}'$. The second formula is shown in an analogous manner. □

It is moreover interesting to observe that in analogy to the Dirac comb identity [19]

$$\triangle\!\triangle\!\triangle\,\delta \equiv \text{III} \equiv \perp\!\!\perp\!\!\perp 1$$

the following identity, let's say a "localization balance"

$$\cap_\varphi \delta \equiv \Omega \equiv \sqcap_\varphi 1$$

holds a balance in $\mathcal{S}'$ if $\varphi \equiv \hat{\varphi}$ is satisfied for $\varphi \in \mathcal{S}$, which obviously is the best achievable compromise in localizing 1 and thereby delocalizing δ. It is true for the Gaussian $\Omega(t) \equiv e^{-\pi t^2}$ and herewith expresses Hardy's uncertainty principle [53]. But it is also true for the Hyperbolic secant $\Omega(t) \equiv 2/(e^t + e^{-t})$, see e.g. [12], and for every fourth Hermite function H, i.e., all H satisfying $\mathcal{F}H \equiv H$. A connection between Gaussians and Hyperbolic Secants is that both belong to a class of "Pólya frequency functions" [54,55]. Gaussians, Hyperbolic Secants and Hermite functions are treated in [56,57] for example. Hyperbolic Secants may also replace Gaussians in Gabor systems, see e.g. Janssen and Strohmer [58]. A link between Gaussians and Hyperbolic Secants is furthermore known in soliton physics where the "initial Gaussian beam reshapes to a squared hyperbolic secant profile" [59]. Studying fixpoints Ω of the Fourier transform in $\mathcal{S}$ is therefore worthwhile goal.

Another calculation rule we need to prove the theorem below is the following. It holds in analogy to already shown properties of discretizations and periodizations [19].

Lemma 8. *Let $\varphi \in \mathcal{S}$, $\alpha \in \mathcal{O}_M$, $g \in \mathcal{O}_C{}'$ and $f \in \mathcal{S}'$. Then αf and $g * f$ exist in $\mathcal{S}'$ and*

$$\alpha \cdot (\sqcap_\varphi f) = \sqcap_\varphi(\alpha f) = (\sqcap_\varphi \alpha) \cdot f \qquad \in \mathcal{O}_C{}' \quad \textit{and} \tag{9}$$

$$g * (\cap_\varphi f) = \cap_\varphi(g * f) = (\cap_\varphi g) * f \qquad \in \mathcal{O}_M \quad \textit{in } \mathcal{S}'. \tag{10}$$

Proof. We may allow that at most one of the operands in $\varphi * g * f$ is no element in $\mathcal{O}_C{}'$. This is indeed true as $\varphi \in \mathcal{S} \subset \mathcal{O}_C{}'$, $g \in \mathcal{O}_C{}'$ and f is an arbitrary element in $\mathcal{S}'$. It follows that $\varphi * g * f$ exists in $\mathcal{S}'$ and, hence, operands may be interchanged arbitrarily. Using $\mathcal{O}_C{}' * \mathcal{S}' \subset \mathcal{S}'$ twice and (5), we obtain

$$g * (\cap_\varphi f) = g * (\varphi * f) = \varphi * g * f = \cap_\varphi(g * f)$$

in $\mathcal{S}'$. The other half of this equation results from the fact that the roles of f and g can be exchanged due to commutativity. The second formula is then shown in an analogous manner. □

doi:10.20944/preprints201705.0175.v1

6. A Regularization-Localization Duality

The interaction between regularizations and localizations is ubiquitous in the literature today, for example as ”regularization” and multiplication with smooth ”cutoff functions” in Hörmander [2], as ”two components of the approximation procedure” in $\mathcal{S}'$, see Strichartz [6], or as ”approximation by cutting and regularizing” in Trèves [15], p.302, or in terms of ”cutting out” one period of f and applying ”(quasi-)interpolation” [47]. Detailed studies of the interaction of both, regularizations and localizations, can be found for example in [34,37,38,60] and in engineering literature, we encounter these interactions in terms of the interplay between ”windowing” on one hand and ”interpolation” on the other. Another equivalent is the so-called ”zero-padding” technique found in engineering textbooks as a way to implement interpolations. It corresponds to the regularization of a discrete function by embedding it into a higher-dimensional space where it is smooth.

However, we may summarize this regularization-localization duality in the following way.

Theorem 9 (Regularization *vs.* Localization)**.** *Let $\varphi \in \mathcal{S}$, $f \in \mathcal{S}'$ and let $\hat{\varphi} := \mathcal{F}\varphi$. Then*

$$\mathcal{F}(\cap_\varphi f) = \sqcap_{\hat{\varphi}}(\mathcal{F}f) \qquad \in {\mathcal{O}_C}' \quad \textit{and} \tag{11}$$

$$\mathcal{F}(\sqcap_{\hat{\varphi}} f) = \cap_\varphi(\mathcal{F}f) \qquad \in \mathcal{O}_M \quad \textit{in } \mathcal{S}'. \tag{12}$$

So, this duality asserts that regularizing a function means to localize its Fourier transform and, vice versa, localizing a function means to regularize its Fourier transform. It is the one-to-one counterpart of a discretization-periodization duality in $\mathcal{S}'$, given in (1) and (2).

Proof. Formally, according to the calculation rules shown above the following equalities hold

$$\begin{aligned} \mathcal{F}(\cap_\varphi f) &= \mathcal{F}\cap_\varphi(\delta * f) = \mathcal{F}(\cap_\varphi \delta \, * \, f) = \mathcal{F}(\cap_\varphi \delta) \cdot \mathcal{F}f \\ &= \sqcap_{\hat{\varphi}} 1 \cdot \mathcal{F}f \, = \sqcap_{\hat{\varphi}}(1 \cdot \mathcal{F}f) = \sqcap_{\hat{\varphi}}(\mathcal{F}f) \end{aligned}$$

in $\mathcal{S}'$. We start using $f = \delta * f$ where $\delta \in {\mathcal{O}_C}' \subset \mathcal{S}'$ is the identity element with respect to convolution in $\mathcal{S}'$. Then we apply Equations (10), (4), (7) and (9), in this order. Finally, with $\mathcal{F}f = g \in \mathcal{S}'$ we use $g = 1 \cdot g$ where $1 \in \mathcal{O}_M \subset \mathcal{S}'$ is the identity element with respect to multiplication in $\mathcal{S}'$. The second formula is now shown in an analogous manner. □

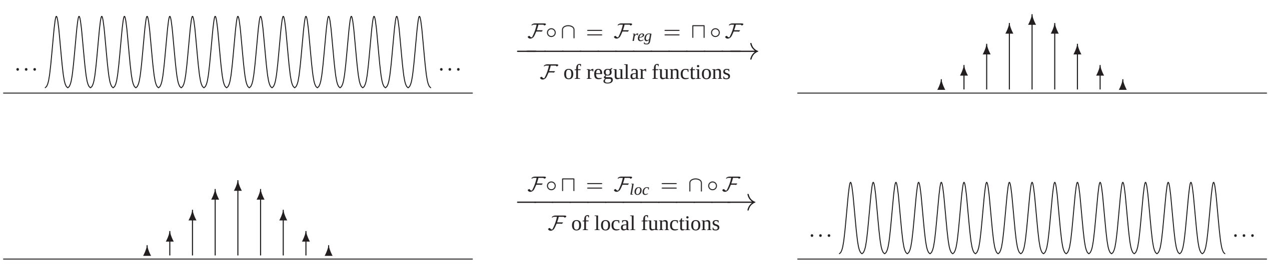

Figure 3. The Regularization-Localization Theorem.

An immediate consequence of the theorem is that f and its Fourier transform $\mathcal{F}f$ cannot be both arbitrarily well localized, a fact that is known as Heisenberg’s uncertainty principle. Also note that $\mathcal{F}_{loc}$,

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see figure above, is the Short-Time Fourier Transform (STFT) with window function $\varphi \in \mathcal{S}$ and it is the Gabor transform if φ is a Gaussian. Consequently, the result of Gabor transforms will be smooth, i.e., they cannot be discrete for example. Its Fourier dual, the Fourier transform of regular functions $\mathcal{F}_{reg}$ in contrast to that, see figure above, corresponds to first regularizing functions before Fourier transforming them. Consequently, the result of such transforms will be local, i.e., they cannot be periodic for example.

Obviously, by looking at these interactions, one may think of discrete functions as the 'opposite' of regular functions and, equivalently, one may think of periodic functions as the 'opposite' of local functions. This is examined more closely in the next section.

7. Four Subspaces

Let $\complement\mathcal{O}_M$ be the complement of regular functions $\mathcal{O}_M$ in $\mathcal{S}'$. It is the space of all ordinary or generalized functions in $\mathcal{S}'$ which are not infinitely differentiable in the ordinary functions sense. Let, furthermore, $\complement\mathcal{O}_C{}'$ be the complement of local functions $\mathcal{O}_C{}'$ in $\mathcal{S}'$. It consists of all ordinary or generalized functions in $\mathcal{S}'$ which do either not fade to zero as $|t|$ increases (periodic functions for example) or they fall to zero but too slowly (with polynomial decay rather than with exponential decay).

Then, the following diagram holds in $\mathcal{S}'$.

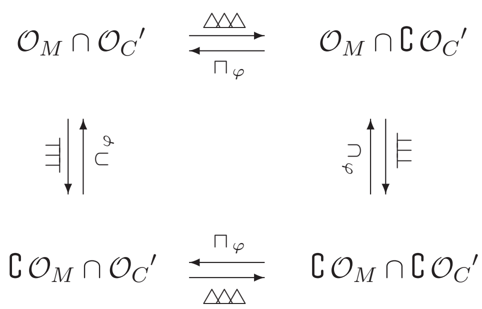

Figure 4. Four subspaces in $\mathcal{S}'$, linked via operations Ш, $\triangle\triangle\triangle$, $\cap_\varphi$, $\sqcap_{\hat{\varphi}}$.

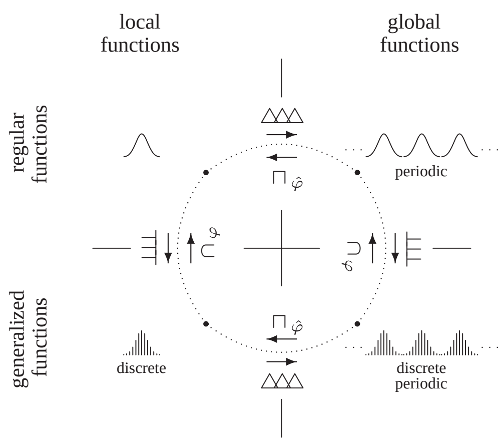

Figure 5. Same as above, drawn in another fashion.

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Apparently, the Schwartz space $\mathcal{S} \equiv \mathcal{O}_M \cap \mathcal{O}_C{}'$, the "smooth world", in some sense, is the 'opposite' of $\complement\,\mathcal{O}_M \cap \complement\,\mathcal{O}_C{}'$, the "discrete world". One may also note that no additional information is used yet beside pure operator definitions. There is also no statement yet on the reversibility of our operations $\sqcup\!\sqcup\!\sqcup$ and $\triangle\!\triangle\!\triangle$ and $\cap_\varphi$ and $\sqcap_\varphi$ in $\mathcal{S}'$. Such inversions will be treated in a follow-on study.

8. Conclusions and Outlook

It is shown that in analogy to a discretization-periodization duality in $\mathcal{S}'$ there is also a regularization-localization duality in $\mathcal{S}'$. Proving these dualities even follows the same pattern. In addition, the two dualities are inverses of each other in the sense that the first one maps towards discreteness and the latter one maps towards smoothness. A more detailed statement on the reversibility of discreteness in $\mathcal{S}'$ will be given in a next study.

Acknowledgements

This paper is primarily based on studies conducted in the years 1995-1997 at the Institute of Mathematics, Ludwig Maximilians University (LMU), Munich. The author is in particular very grateful to Professor Otto Forster. The author would also like to thank his colleagues at the Microwaves and Radar Institute, German Aerospace Center (DLR), for deep insights into signal processing and a great cooperation for many years.

Conflicts of Interest

The author declares no conflicts of interest.

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