Article

A Speculative Model of Thermally Forced Quasi-Geostrophic Flow in an Unbounded Ocean

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Abstract: Starting with a hypothetical geostrophic zonal current in an unbounded ocean, the investigation points out the response of this simple system to a thermal forcing, applied to the free surface and consistent with the maintenance of the geostrophic balance. The main result is the formation of a meridional component of the current, according to the Sverdrup relation, such that the full velocity vector rotates clockwise for heating and anticlockwise for cooling to adjust eventually in the initial zonal direction for large depths.

Keywords: Geophysical Fluid Dynamics, Geostrophic flows; Thermal forcing; Analytical model

1. Introduction

In the large-scale ocean circulation the effect of the thermal forcing is prevailingly indirect. It reveals itself essentially through the overlying wind field which is ultimately the response of the atmospere to the horizontal component of the air density gradient due to solar heating. This is the origin of the wind-driven circulation. On the other hand, a direct effect concerns with the heat flux at the air-sea interface, of seasonal or meteorological origin, the latter being able to induce a marked convective circulation on a local scale. Besides the “extreme” cases cited above, in which the forcing has a drastic influence, one can conceive a system in which a less intense heating/cooling at the sea surface makes rise to a weak disturbance superimposed to a hypothetical unperturbed circulation. In this context, most of the existing literature deals with numerically solved high-resolution nodels of circulation in rotating closed basins [1], [2] and papers quoted therein. However, remarkable analytical contributions are found in Barcilon, V., and J. Pedlosky, 1967 [3], Whitehead and Pedlosky ,2000 [4], and Pedlosky (2003) [5]. The aim of this investigation is to explore the interplay between the heating/cooling of a thin sub-superficial layer of an unbounded ocean and the consequent mechanical reaction of the quasi-geostrophic flow field induced by the redistribution of sea-water density inside the water column. The heat flux is weak enough to preserve the dominance of the geostrophic balance in the system. The method resorts (i) to a thermodynamic equation which links together the time rate of heat transfer into/out the ocean surface with the ageostrophic vertical current and (ii) to the Sverdrup relation which connects the vertical current to the meridional transport. In the framework of an analytic approach to above model, only a local solution seems to be accessible. In any case the latter solves *exactly* the vertical structure of the geostrophic flow and allows to

 doi:10.20944/preprints201705.0150.v1

elucidate the physics of the interplay. The main effect of the thermal forcing on a hypothetical unperturbed westward zonal current is the addition of a small meridional component. In particular, the velocity vector rotates clockwise for heating and anticlochwise for cooling. For increasing depth the current undergoes a further counter-rotation, far smaller than the first one, to adjust eventually in the initial zonal direction for large depths.

2. The link between thermal forcing and dynamics

In this section a relationship between the time rate of heat transferred into, or released by, a salinity conserving sea-water volume and the vertical current taking place in the same volume is inferred. Consider first the linearized equation of state $\rho=\rho(\theta)$, where ρ is the density and θ is the temperature of sea-water, which yields (see, for instance, eq. (2.17) of [6])

(1) $$\frac{\partial\rho}{\partial\theta}=-\rho_0\alpha_\theta$$

In (1) ρ_0 is the density around which the equation is linearized and α_θ is the coefficient of thermal expansion. Then assume the thermodynamic equation

(2) $$\frac{D\rho}{Dt}=\left(\frac{\partial\rho}{\partial\theta}\right)_{S=S_0}\frac{\dot{Q}}{c_p}$$

which comes from the first principle for an incompressible fluid (see, for instance, eq. (2.50) of [6]). In (2) S_0 is the constant value of salinity, $\dot{Q}$ is the time rate of transferred, or released, heat into sea-water and c_p is its specific heat at constant pressure. Equations (1) and (2) yield

(3) $$\frac{D\rho}{Dt}=-\frac{\rho_0\alpha_\theta}{c_p}\dot{Q}$$

The fundamental role of equation (3) lies in the fact that it includes implicitly the vertical velocity in the advective term of the Lagrangian derivative and, explicitly, the time rate of heat transfer; thus (3) links the dynamics of the flow with the thermal forcing which acts on it. Owing to the prevailing geostrophic and hydrostatic balances governing geophysical flows, the density field is suitably expressed as (see, for instance, eq. (2.554) of [6])

(4) $$\rho=\rho_s(z)\left[1+r\rho'(\mathbf{x}',t')\right]$$

In (4) ρ_s is the "standard" density, which retains the basic vertical structure of the overall field density while ρ' is the non dimensional contribute required by the geostrophic balance. Finally,

(5) $$r=\frac{ULf_0}{gZ}\qquad O\left(\frac{ULf_0}{gZ}\right)\approx10^{-4}$$

 doi:10.20944/preprints201705.0150.v1

is a small quantity depending on the scales of the motion L, Z, U[1] and on the planetary constants f_0 and g. Substitution of (4) into (3) produces the desired relationship between the heat and the vertical current (see the Appendix for details) in the non dimensional form (primed variables)

(6)
$$w' = \frac{\varepsilon}{S}\left(\frac{D\rho'}{Dt'} + \mathrm{H}'\right)$$

In (6) w' is the vertical velocity of the fluid, ε is the Rossby number, S is the stratification parameter (unfortunately the notation is the same as salinity, but the latter will not be used in what follows) and $\mathrm{H}' = \dfrac{gZ\alpha_\theta \dot{Q}}{U^2 f_0 c_p}$ is a non dimensional heating function.

3. The quasi-geostrophic potential vorticity equation at the oceanic basin scale with thermal forcing

The quasi-geostrophic dynamics at the oceanic basin scale is widely described, for instance, in [6] and in [7]. Its basic feature is the dominance of planetary vorticity with respect to relative vorticity and this fact makes the formal aspect of the resulting governing equation of potential vorticity noticeably simple. Only the main points, not to be ignored, are recalled below. The non dimensional fields appearing in the primitive equations, that is velocity, density and pressure, are expanded in powers of the ordering parameter

(7a)
$$b = \varepsilon\beta$$

or, alternatively,

(7b)
$$b = \beta_0 L / f_0$$

where $\beta = \beta_0 L^2 / U$ and β_0 is the planetary vorticity gradient. At the basin scale

(8)
$$O(b) = 10^{-1}$$

and

(9)
$$O(\beta S) = 1$$

Thus (primes are dropped)

(10)
$$(u,\ v,\ \rho,\ p) = (u_0,\ v_0,\ \rho_0,\ p_0) + b(u_1,\ v_1,\ \rho_1,\ p_1) + \ldots$$
$$w = bw_1 + \ldots$$

where u_0 and v_0 are in geostrophic balance with the pressure gradient $\nabla_h p_0$ and

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[1] The depth of the motion is represented by Z while H means a heating function.

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(11) $$\rho_0 + \frac{\partial p_0}{\partial z} = 0$$

The first key equation involving two of the fields (10) is the fundamental Sverdrup relation (widely discussed, for instance, in [8])

(12) $$v_0 = \frac{\partial w_1}{\partial z}$$

which is one of the main ingredients of this investigation. Equation (12) is also the basis of the so called Sverdrup balance of the wind-driven circulation which, however, has nothing to do with this model. The second equation comes from (6) by using (7a) and (10). In fact, substitution of (10) into (6) gives

(13) $$bw_1 + \ldots = \frac{\varepsilon}{S}\left(\frac{D_0}{Dt}\rho_0 + \mathrm{H}\right) + \ldots$$

where $\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \nabla$ and $\mathbf{u}_0 = (u_0, v_0)$. The balance between the l.h.s. and the r.h.s. of (13), where $O(\mathrm{H}) = 1$ (see (A11) of the Appendix), is achieved noting that, according to (7a) and to (9), $\frac{\varepsilon}{S} = \frac{\beta\varepsilon}{\beta S} = \frac{b}{\beta S}$ and $O\left(\frac{b}{\beta S}\right) = O(b)$. Thus (13) gives

(14) $$w_1 = \frac{1}{\beta S}\left(\frac{D_0}{Dt}\rho_0 + \mathrm{H}\right)$$

Finally, recalling (11), equation (14) becomes

(15) $$w_1 = -\frac{1}{\beta S}\left(\frac{D_0}{Dt}\frac{\partial p_0}{\partial z} - \mathrm{H}\right)$$

At this point the vertical velocity w_1 can be eliminated from (12) and (15) in favour of the sole geostrophic fields to obtain

(16) $$\frac{\partial}{\partial z}\left[\frac{1}{\beta S}\left(\frac{D_0}{Dt}\frac{\partial p_0}{\partial z} - \mathrm{H}\right)\right] + v_0 = 0$$

The steady version of (16) written by means of the streamfunction $\psi = p_0$, in terms of which $\frac{D_0}{Dt}\frac{\partial p_0}{\partial z} = J\left(\psi, \frac{\partial \psi}{\partial z}\right)$ and $v_0 = \frac{\partial \psi}{\partial x}$, takes the form

(17) $$\frac{\partial}{\partial z}\left\{\frac{1}{\beta S}\left[J\left(\psi, \frac{\partial \psi}{\partial z}\right) - \mathrm{H}\right]\right\} + \frac{\partial \psi}{\partial x} = 0$$

By resorting to the same notation as in (17), equation (15) becomes

(18) $$w_1 = -\frac{1}{\beta S}\left[J\left(\psi, \frac{\partial \psi}{\partial z}\right) - \mathrm{H}\right]$$

Equation (18) will be used to express the vertical boundary condition of the model solution in $z=0$ by means of ψ under the assumption of the rigid lid approximation, in accordance with [7].

4. The Model

The model aims to focus the vertical structure of the geostrophic flow by means of suitable simplifications whose validity is essentially local. The system is steady and governed by a special version of (17) under the following hypotheses:

- The factor $\frac{1}{\beta S}$ is a constant of the order of unity according to (9);
- The fluid is vertically included in the interval $(-\infty < z \leq 0)$;
- $\mathrm{H} = \mathrm{H}(z)$, $O(\mathrm{H}) = 1$, $\mathrm{H}(-\infty) = 0$;
- The vertical boundary condition $w_1 = 0$ in $z=0$ is given by (18)

(19) $$J\left(\psi, \frac{\partial \psi}{\partial z}\right)_{z=0} = \mathrm{H}(0)$$

- At large depths the current is mainly zonal

(20) $$\psi(x, y, -\infty) = \psi_0(y)$$

Thus, equation (17) simplifies into

(21) $$\frac{\partial}{\partial z} J\left(\psi, \frac{\partial \psi}{\partial z}\right) + \beta S \frac{\partial \psi}{\partial x} = \frac{d\mathrm{H}}{dz}$$

and the model is summarized in equations (19), (20), (21). The thermal forcing can be conceived as a disturbance of the zonal current $u_0 = -\frac{d\psi_0}{dy}$, where $\psi = \psi_0(y)$, which identically satisfies the unperturbed motion (i.e., the motion in the absence of H) on the beta-plane. Hereafter, for simplicity, the constant westward current $u_0 = -1$ will be considered, so

(22) $$\psi_0(y) = y - \bar{y}$$

$\bar{y}$ being a constant. The effect of the forcing is expected to introduce a baroclinic term $\phi(x, y, z)$ superimposed to (22), that is to say

(23) $$\psi = y - \bar{y} + \phi(x, y, z)$$

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Substitution of (23) into (21) yields

(24) $$\beta S\frac{\partial \phi}{\partial x}-\frac{\partial^2}{\partial z^2}\frac{\partial \phi}{\partial x}+\frac{\partial}{\partial z}J\left(\phi,\ \frac{\partial \phi}{\partial z}\right)=\frac{d\mathrm{H}}{dz}$$

Because the r.h.s. of (24) is a function of z alone, it seems sensible to try a solution of the form

(25) $$\phi=\left(x-\overline{x}\right)\varphi\left(z\right)$$

where the constant $\overline{x}$ can be identified with the longitude of the eastern boundary of the fluid domain. Thus $x-\overline{x}\le 0$. Owing to (25), the Jacobian determinant of (24) is identically zero and the substitution of (25) into (24) produces the linear ODE

(26) $$\beta S\varphi-\frac{d^2\varphi}{dz^2}=\frac{d\mathrm{H}}{dz}$$

In turn, substitution of the full streamfunction (23) with (25) into (19) gives the boundary condition at the upper surface in the form

(27) $$\left(\frac{d\varphi}{dz}\right)_{z=0}=-\mathrm{H}\left(0\right)$$

while boundary condition (20) implies

(28) $$\varphi\left(-\infty\right)=0$$

Trivially $\nabla_h^2\psi=\nabla_h^2\left[y-\overline{y}+\left(x-\overline{x}\right)\varphi\left(z\right)\right]=0$, so the request that, at the basin scale, relative vorticity be far smaller than planetary vorticity is satisfied *a fortiori*. Problem (26), (27), (28) can be solved once $\mathrm{H}\left(z\right)$ is given. The hypothesis

(29) $$\mathrm{H}\left(z\right)=\mathrm{H}_0\exp\left(\lambda z\right)$$

is hereafter taken into account, where $O\left(\mathrm{H}_0\right)=1$ and $\lambda>>1$. A wide variety of functions could be considered in place of (29), however the latter seems to be preferable because of its simple form and because it is fit for describing the absorption process $\mathrm{H}\left(z-dz\right)=-\lambda\ \mathrm{H}\left(z\right)dz$ into sea water $\left(z\le 0\right)$. Heating implies $\mathrm{H}_0>0$ while cooling $\mathrm{H}_0<0$. Moreover, setting $z_*=Z\ z$, the quantity $\lambda z=\frac{z_*}{Z/\lambda}$ appearing in (29) introduces the typical heating/cooling depth $Z/\lambda<<Z$. Substitution of (29) into (26) yields

(30) $$\beta S\varphi-\frac{d^2\varphi}{dz^2}=\lambda\mathrm{H}_0\exp\left(\lambda z\right)$$

The solutions of problem (27), (28), (30) is

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(31) $$\varphi(z)=\frac{\mathrm{H}_0}{\beta S-\lambda^2}\left[\lambda\exp(\lambda z)-\sqrt{\beta S}\exp\left(\sqrt{\beta S}\ z\right)\right]$$

where $\beta S-\lambda^2<<1$. Solution (31) allows to evaluate all the geostrophic fields, that is to say

$$u_0=-1,\ v_0=\varphi(z),\ p_0=y-\bar{y}+(x-\bar{x})\varphi(z),\ \rho_0=-(x-\bar{x})\frac{d\varphi}{dz}$$

Note that ρ_0 has the same sign as $\frac{d\varphi}{dz}$. In turn, substitution of (31) into (18) gives the ageostrophic vertical velocity

(32) $$w_1=\frac{\mathrm{H}_0}{\beta S-\lambda^2}\left[\exp(\lambda z)-\exp\left(\sqrt{\beta S}\ z\right)\right]$$

One can check that (31) and (32) identically satisfy equation (12). From (31), one finds $\int_{-\infty}^{0}\varphi(z)\ dz=0$, in accordance with the relationships $w_1(0)=w_1(-\infty)=0$ that follow from (32). In particular $\varphi(0)=-\frac{\mathrm{H}_0}{\sqrt{\beta S}+\lambda}$; moreover φ changes its sign, only once, at the depth

(33) $$z_0=\frac{1}{\sqrt{\beta S}-\lambda}\ln\frac{\lambda}{\sqrt{\beta S}}$$

Therefore, in the presence of heating, φ is negative (i.e., v_0 southward) from the free surface down to z_0 and positive (i.e., v_0 northward) below z_0. Because of the inequalities $\exp(\lambda z)\le\exp\left(\sqrt{\beta S}\ z\right)\ \forall z\in(0,\ -\infty)$ and $\beta S-\lambda^2<0$, the vertical velocity (32) has the same sign as H_0 in the full water column (upward for heating and downward for cooling) and reaches its extremum in (33), where the meridional velocity changes its sign. The density anomaly ρ_0 is negative for heating and positive for cooling in the layer from the free surface down to the depth $z=2z_0$. Although it changes its sign below $z=2z_0$, the total density anomaly $\int_{-\infty}^{0}\rho_0\,dx=(x-\bar{x})\frac{\mathrm{H}_0}{\sqrt{\beta S}+\lambda}$ retains a sign opposite to that of H_0 as in the upper layer $(2z_0<z\le0)$. All this summarizes the dynamics of the thermal forcing. In fact heating squizees the

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upper portion of the fluid column and stretches the remaining part in accordance with the upward ageostrophic vertical velocity. In doing so, heating decreases the density anomaly both below the free surface and, less markedly, on the whole fluid column. In turn, because of the Sverdrup relation, the meridional southward transport $M_y = \int_{z_0}^{0} v_0 \, dz$ with a negative density anomaly takes place in the upper part of the column where a direct computation yields

$$M_y = -w_1\left(z = z_0\right) \qquad (34)$$

while the meridional northward transport equal to $-M_y$ forms in the remaining part. The dynamics is symmetrically reverted for cooling. A further noticeable quantity describing the structure of the flow field is the angle, say δ, singled out, at each depth, by the vector $\mathbf{u}_0 = -\hat{\mathbf{i}} + \varphi(z)\hat{\mathbf{j}}$ with respect to the semi-axis $x > 0$, anticlockwise oriented. Openly

$$\tan\left[\delta(z)\right] = -\varphi(z) \qquad (35)$$

where the periodicity π of the tangent function must be taken into account in the inversion of (35). In fact, the geostrophic current at large depth $\mathbf{u}_0(z = -\infty) = -\hat{\mathbf{i}}$ implies $\delta(-\infty) = \pi$ and therefore, according to the identity $\tan(\delta) = \tan(\delta - \pi)$, the inversion of (35) gives

$$\delta(z) = \pi - \tan^{-1}\left[\varphi(z)\right] \qquad (36)$$

To visualize a definite solution, the following values

$$\beta S = 1, \;\; \mathrm{H}_0 = 1, \;\; \lambda = 10 \qquad (37)$$

are hereafter fixed. Therefore

$$v_0(z) = \frac{1}{99}\left[\exp(z) - 10 \; \exp(10 \; z)\right] \qquad (38)$$

$$w_1(z) = \frac{1}{99}\left[\exp(z) - \exp(10 \; z)\right] \qquad (39)$$

$$\frac{\rho_0}{\overline{x} - x} = \frac{1}{99}\left[\exp(z) - 100 \; \exp(10 \; z)\right] \qquad (40)$$

$$\delta(z) = \pi - \tan^{-1}\left\{\left[\exp(z) - 10 \; \exp(10 \; z)\right]/99\right\} \qquad (41)$$

The plots of (38), (39), (40) and (41) are shown in panels (a), (b), (c) and (d) of Fig. 1, respectively.

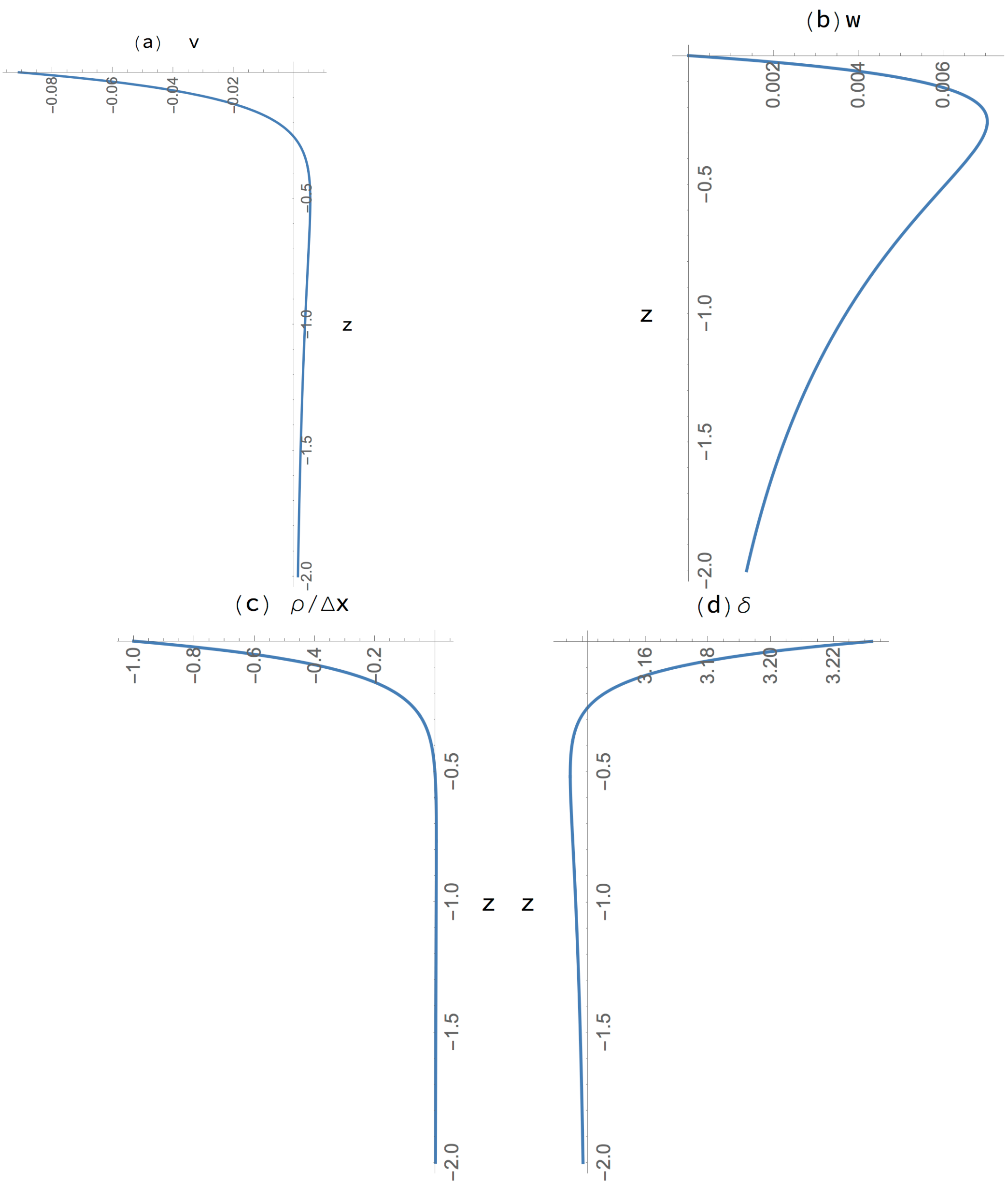

Figure 1. *Panel (a)* Vertical profile of the meridional geostrophic current in the presence of heating; *Panel (b)* Vertical profile of the a-geostrophic vertical velocity; *Panel (c)* Vertical profile of the ratio $\rho_0/(\bar{x}-x)$; *Panel (d)* Vertical profile of the angle δ between the velocity vector and the negative x-axis, evaluated counter-clockwise. Note the small counter-clockwise rotation below z_0.

Panel (a) points out the inversion of the meridional velocity around the depth $z_0 \approx -0.256$, from southward to northward. At the same depth the a-geostrophic vertical velocity reaches its maximum, according to panel (b). For every fixed longitude x, the density anomaly induced by heating is almost exactly negative as one can ascertain from panel (c). Finally, the anticlockwise rotation of the velocity vector $\mathbf{u}_0$ is made evident in panel (d): in particular, the right zonal direction takes place both at $z = z_0$ and at great depths, in accordance with (28). Note the small counter-rotation of $\mathbf{u}_0$ below z_0. The vertical motion of a parcel, say $z = z(t)$, can be singled out by solving the problem

$$\begin{cases} \dot{z} = b w_1(z) \\ z(t_i) = z_i \end{cases}$$

where the subscript *i* refers to the "initial" position of the parcel. A set of trajectories, each for a given z_i, is reported in Fig. 2.

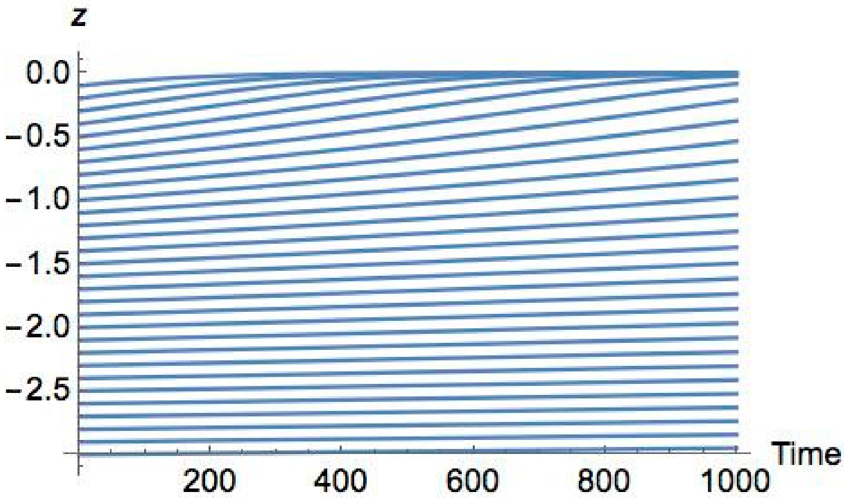

Figure 2. Vertical displacements of a set of parcels located at different depths at a certain initial time t_i. Note the intensification of the displacements in the course of time only for parcels initially close enough to the free surface.

The increasing slope of the trajectories for increasing z_i is due to the squeezing of the fluid column in the proximity of the free surface, whose effect is governed by the Sverdrup relation (12).

To carry out an estimate of the dimensional meridional transport M_{y*} in the layer $(z_0 \le z \le 0)$ due to the thermal forcing, consider the dimensional version of (34) in SI units, that is

$$M_{y*} = -(f_0 / \beta_0) w_* \qquad (42)$$

By using the typical heating/cooling depth Z/λ and (7b) to evaluate w_*, one obtains

$$O(w_*) = \frac{UZ/\lambda}{L} \frac{\beta_0 L}{f_0} = \frac{UZ\beta_0}{\lambda f_0} \qquad (43)$$

Thus, from (42) and (43) the estimate

$$O(M_{y*}) = \frac{UZ}{\lambda} \qquad (44)$$

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follows. For instance, if $U=10^{-2}ms^{-1},\ Z=10^{3}m,\ \lambda=10$, equation (44) yields $O\left(M_{y\,*}\right)=1m^{2}s^{-1}$. The latter estimate is hundred times smaller than the transport of the Gulf Stream.

Remark.

If the quasi-geostrophic dynamics is disregarded, the relationship between the vertical velocity and the thermal forcing, consistent with above results, can be obtained on the basis of a simple two-layer model. Consider two configurations, the former before the action of the thermal forcing and the latter in the presence of it, according to Fig. 3 , left and right panel, respectively.

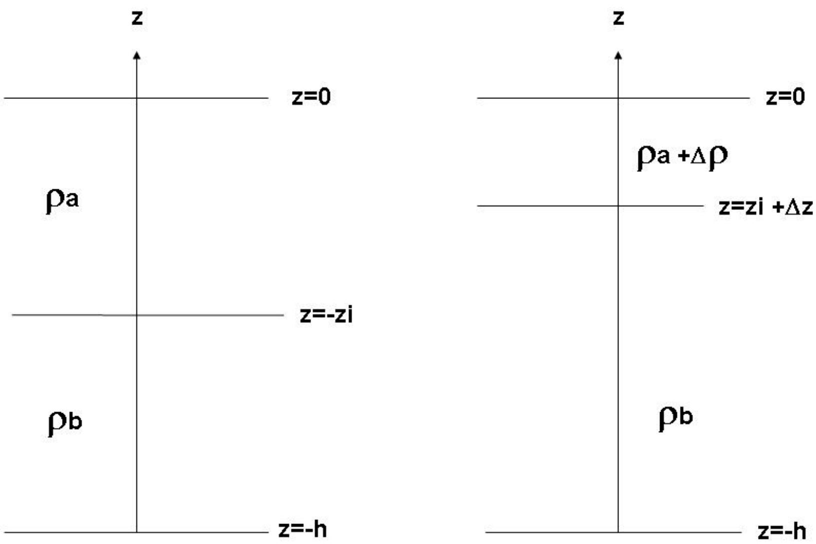

Figure 3. Scheme of the two-layer system to evaluate the vertical velocity of the interface for a varying density of the upper layer.

Mass conservation in a fluid column [2] implies

$$z_i\rho_a+\left(h-z_i\right)\rho_b=\left(z_i-\Delta z\right)\left(\rho_a+\Delta\rho\right)+\left(h-z_i+\Delta z\right)\rho_b$$

and, hence,

$$z_i\Delta\rho+\Delta z\left(\rho_b-\rho_a-\Delta\rho\right)=0 \qquad (45)$$

Moreover, conservation of the hydrostatic equilibrium during the process demands

$$\rho_b-\rho_a>\Delta\rho \qquad (46)$$

Putting $\delta_\rho=\rho_b-\rho_a$, equation (45) gives

[2] Mass conservation is explicitly invoked because of the 1D nature of this model.

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(47)
$$\Delta z = -\frac{z_i \Delta\rho}{\delta_\rho - \Delta\rho}$$

Division of (47) by Δt and the subsequent application of $\lim_{\Delta t \to 0}$ yields

(48)
$$w_i = \frac{z_i}{\delta_\rho - \Delta\rho} \frac{\alpha_\theta \rho_0}{c_p} \dot{Q}$$

where $w_i = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$ and equation (3) has been used. Recalling (46), equation (48) shows that the vertical velocity of the interface is upward for heating and downward for cooling. This is in accordance with the behaviour of w_1 in the continuum case.

5. Conclusions

In the framework of the quasi-geostrophic dynamics at the basin scale, the model of section 4 is probably the simplest and exactly solvable that is able to show how heating affects locally the streamfunction and how the resulting current field arises from the interplay between the thermal forcing and the Sverdrup relation. If the analytic approach is released in favour of a numerical one, the model equations can be made more realistic, although mathematically more complex, on the basis of several generalizations, some of them being listed below:

- The stratification parameter takes explicitly into account its depth dependence;
- The heating is a function of all the coordinates;
- The unperturbed streamfunction is a prescribed function of latitude (not necessarily linear) and of depth.

In doing so one could expect to derive a 3D current field whose validity covers a large portion of the beta-plane, with the obvious exception of the westernmost area.

The transition from an almost conceptual model, like that of section 4, to a numerically solved one introduces another question, that is to say the possibility to detect the phenomenology predicted by the numerical model solution. Because the model refers mainly to the upper ocean, where wind-waves and mixing are in permanent activity, the comparison of predictions with field observations might be problematic. This is the case, for instance, of the Ekman's spiral which, on the other hand, can be successfully simulated in a rotating tank. Reasoning in analogy, one could try to resort to a rotating tank experiment to check whether the predicted phenomenology does take place or not. This should be the first step to do.

Appendix A*:* proof of equation (6) and estimate of H

Substitution of (4) into (3) yields

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(A1)
$$\frac{D}{Dt}\left[\rho_s\left(1+r\rho'\right)\right]=\left(1+r\rho'\right)\frac{D\rho_s}{Dt}+r\rho_s\frac{D\rho'}{Dt}\approx w\frac{d\rho_s}{dz}+r\rho_s\frac{U}{L}\frac{D\rho'}{Dt'}$$
$$=\frac{UZ}{L}w'\frac{d\rho_s}{dz}+\frac{U^2 f_0}{gZ}\rho_s\frac{D\rho'}{Dt'}=\left(\frac{UZ}{L}\frac{d\rho_s}{dz}\right)\left[w'+\frac{ULf_0}{Z^2}\frac{\rho_s}{g}\left(\frac{d\rho_s}{dz}\right)^{-1}\frac{D\rho'}{Dt'}\right]$$

In terms of the Brunt-Vaisala frequency square $N_s^2=-\frac{g}{\rho_s}\frac{d\rho_s}{dz}$ one can write

(A2)
$$\frac{ULf_0}{Z^2}\frac{\rho_s}{g}\left(\frac{d\rho_s}{dz}\right)^{-1}=-\frac{U}{Lf_0}\left(\frac{Lf_0}{ZN_s}\right)^2=-\frac{\varepsilon}{S}$$

where ε is the Rossby number and S is the stratification parameter. Thus, because of (A2), equation (A1) is equivalent to

(A3)
$$\frac{D}{Dt}\left[\rho_s\left(1+r\rho'\right)\right]=\left(\frac{UZ}{L}\frac{d\rho_s}{dz}\right)\left(w'-\frac{\varepsilon}{S}\frac{D\rho'}{Dt'}\right)$$

and, by using (A3), the full equation (3) takes the form

$$w'-\frac{\varepsilon}{S}\frac{D\rho'}{Dt'}=-\frac{L}{UZ}\left(\frac{d\rho_s}{dz}\right)^{-1}\frac{\rho_0\alpha_\theta}{c_p}\dot{Q}$$

which is conveniently restated as follows

(A4)
$$w'=\frac{\varepsilon}{S}\left[\frac{D\rho'}{Dt'}-\frac{S}{\varepsilon}\frac{L}{UZ}\left(\frac{d\rho_s}{dz}\right)^{-1}\frac{\rho_0\alpha_\theta}{c_p}\dot{Q}\right]$$

Noting that

(A5)
$$\frac{S}{\varepsilon}\frac{L}{UZ}\left(\frac{d\rho_s}{dz}\right)^{-1}\frac{\rho_0\alpha_\theta}{c_p}\dot{Q}=\frac{Lf_0}{U}\frac{N_s^2Z^2}{L^2f_0^2}\frac{L}{UZ}\left(\frac{d\rho_s}{dz}\right)^{-1}\frac{\rho_0\alpha_\theta}{c_p}\dot{Q}$$
$$=\frac{N_s^2Z}{U^2f_0}\left(\frac{d\rho_s}{dz}\right)^{-1}\frac{\rho_0\alpha_\theta}{c_p}\dot{Q}=-\frac{gZ}{U^2f_0}\frac{\alpha_\theta}{c_p}\dot{Q}$$

substitution of (A5) into (A4) gives

(A6)
$$w'=\frac{\varepsilon}{S}\left(\frac{D\rho'}{Dt'}+\frac{gZ}{U^2f_0}\frac{\alpha_\theta}{c_p}\dot{Q}\right)$$

Finally, by using position

(A7)
$$\frac{gZ\alpha_\theta\dot{Q}}{U^2f_0c_p}=\mathrm{H}'$$

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one ascertains that equation (6) is nothing but (A.6). By using the l.h.s. of (A7) the order of magnitude of H' can be estimated as follows. Recalling (3), equation (A7) can be restated as $\mathrm{H}' = -\frac{gZ}{U^2 f_0 \rho_0}\frac{D\rho}{Dt}$ and hence

(A8) $$O(\mathrm{H}') = \frac{gZ}{U^2 f_0 \rho_0}\frac{\delta\rho}{T}$$

where

(A9) $$\frac{\delta\rho}{T} = O\left(\frac{D\rho}{Dt}\right)$$

Reasonable values of the depth of the motion and of the horizontal velocity substituted into (A8) lead to

(A10) $$O(\mathrm{H}') = 4\times 10^{7} \times \frac{\delta\rho}{T}$$

Keeping, for instance, $\delta\rho = 0.2\ kg\ m^{-3}$ and $T = 2.6\times 10^{6}\, s$, that is to say $T = 1\ month$, equation (A10) gives

(A11) $$\mathrm{O}(\mathrm{H}') = 1$$

Conflicts of Interest: The authors declare no conflict of interest."

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