INTEGRAL REPRESENTATIONS FOR BIVARIATE COMPLEX GEOMETRIC MEAN AND APPLICATIONS

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Abstract. In the paper, the authors survey integral representations (including the Lévy–Khintchine representations) and applications of some bivariate means (including the logarithmic mean, the identric mean, Stolarsky's mean, the harmonic mean, the (weighted) geometric means and their reciprocals, and the Toader–Qi mean) and the multivariate (weighted) geometric means and their reciprocals, derive integral representations of bivariate complex geometric mean and its reciprocal, and apply these newly-derived integral representations to establish integral representations of Heronian mean of power 2 and its reciprocal.

Contents

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1. Preliminaries

We recall some definitions and notion.

Recall from [66, Chapter IV] that an infinitely differentiable function f on an interval I is said to be completely monotonic on I if $(-1)^{n-1}f^{(n-1)}(x)\ge0$ for $x\in I$ and $n\in\mathbb{N}$, where $\mathbb{N}$ stands for the set of all positive integers. Theorem 12b in [66] reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is $f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\alpha(t)$, where $\alpha(t)$ is non-decreasing and the integral converges for $0<x<\infty$.

In [2, 31, 33, 44], it was defined implicitly and explicitly that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if inequalities $(-1)^k[\ln f(x)]^{(k)}\ge0$ on I for all $k\in\mathbb{N}$. In [4, Theorem 1.1], [13, Theorem 4], [31, Theorem 1], and [33, Theorem 4], it was found and verified once again that a logarithmically completely monotonic function must be completely monotonic, but not conversely.

In [64, Definition 2.1], it was defined that a Stieltjes function is a function $f:(0,\infty)\to[0,\infty)$ which can be written in the form

$$f(x)=\frac{a}{x}+b+\int_0^\infty\frac{1}{x+u}\,\mathrm{d}\,\mu(u), \tag{1.1}$$

where a,b are nonnegative constants and μ is a measure on $(0,\infty)$ such that $\int_0^\infty\frac{1}{1+s}\,\mathrm{d}\,\mu(s)<\infty$. In [4, Theorem 1.2], it was proved that a positive Stieltjes function must be a logarithmically completely monotonic function on $(0,\infty)$, but not conversely.

Recall from [64, Definition 3.1] that a function $f:(0,\infty)\to[0,\infty)$ is a Bernstein function if $(-1)^{k-1}f^{(k)}(x)\ge0$ for all $k\in\mathbb{N}$ and $x>0$. Generally, an infinitely differentiable function $f:I\to[0,\infty)$ is called a Bernstein function on an interval I if f' is completely monotonic on I. The Bernstein functions on $(0,\infty)$ can be characterized by [64, Theorem 3.2] which reads that a function $f:(0,\infty)\to[0,\infty)$ is a Bernstein function if and only if it admits the representation

$$f(x)=a+bx+\int_0^\infty\left(1-e^{-xt}\right)\mathrm{d}\,\mu(t), \tag{1.2}$$

where $a,b\ge0$ and μ is a measure on $(0,\infty)$ satisfying $\int_0^\infty\min\{1,t\}\,\mathrm{d}\,\mu(t)<\infty$. The integral representation (1.2) is called the Lévy–Khintchine representation of f. In [10, pp. 161–162, Theorem 3] and [64, Proposition 5.25], it was proved that the reciprocal of a Bernstein function is logarithmically completely monotonic.

Definition 6.1 in [64, p. 69] states that a Bernstein function f is said to be a complete Bernstein function if its Lévy measure μ in (1.2) has a completely monotonic density $m(t)$ with respect to

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the Lebesgue measure, that is,

$$f(x)=a+bx+\int_0^\infty\bigl(1-e^{-xt}\bigr)m(t)\,\mathrm{d}\,t,\quad a,b\ge0, \tag{1.3}$$

the function $m(t)$ is completely monotonic on $(0,\infty)$, and $\int_0^\infty\min\{1,t\}m(t)\,\mathrm{d}\,t<\infty$. Theorem 7.3 in [64] reads that a (non-trivial) function f is a complete Bernstein function if and only if $\frac1f$ is a (non-trivial) Stieltjes function.

Let $\mathbb{M}_n^+$ denote the space of $n\times n$ complex Hermitian positive semi-definite matrices with the usual ordering that $A\le B$ means that $B-A$ is a positive matrix. For a real function f on an interval I, if D is a diagonal matrix $\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)$, then define $f(D)=\operatorname{diag}(f(\lambda_1),f(\lambda_2),\dots,f(\lambda_n))$. If A is an Hermitian matrix with eigenvalues belonging to I, then define $f(A)=Uf(D)U^H$, where $A=UDU^H$ and the diagonal matrix D is constituted by the eigenvalues of A, with U being a unitary matrix and U^H being the conjugate transpose of U. Recall from [8, Definition 2] and [64, Definition 12.9] that a function $f:I\to(0,\infty)$ is said to be matrix monotone of order n if $A\le B$ implies $f(A)\le f(B)$, where $A,B\in\mathbb{M}_n^+$ and the eigenvalues of A and B are contained in the interval I. If for every $n\ge1$ a function f on an interval I is always matrix monotone of order n, then f is said to be operator monotone on I. Theorem 12.17 in [64] states that the families of complete Bernstein factions and positive operator monotone functions on $(0,\infty)$ coincide.

Recall from [14, 20, 21, 65] that, for $r,s\in\mathbb{R}$ and $a,b>0$ with $a\ne b$, the Stolarsky mean $E(r,s;a,b)$, or say, the extended mean value $E(r,s;a,b)$, can be defined by

$$\begin{aligned}
E(r,s;a,b)&=\left[\frac{r(b^s-a^s)}{s(b^r-a^r)}\right]^{1/(s-r)}, && rs(r-s)\ne0;\\
E(r,0;a,b)&=\left[\frac{b^r-a^r}{r(\ln b-\ln a)}\right]^{1/r}, && r\ne0;\\
E(r,r;a,b)&=\frac{1}{e^{1/r}}\left(\frac{a^{a^r}}{b^{b^r}}\right)^{1/(a^r-b^r)}, && r\ne0;\\
E(0,0;a,b)&=\sqrt{ab}\,.
\end{aligned}$$

Specially, the quantities

$$E(1,2;a,b)=A(a,b),\quad E(0,0;a,b)=G(a,b),\quad E(-2,-1;a,b)=H(a,b),$$
$$E(1,1;a,b)=I(a,b),\quad E(1,0;a,b)=L(a,b)$$

are respectively called the arithmetic, geometric, harmonic, identric, and logarithmic means of two positive numbers a,b with $a\ne b$.

For $a,b>0$ and $\lambda\in(0,1)$, the weight arithmetic and geometric means are respectively defined by

$$A(a,b;\lambda)=\lambda a+(1-\lambda)b\quad\text{and}\quad G(a,b;\lambda)=a^\lambda b^{1-\lambda}.$$

For $a_k>0$ and $\lambda_k>0$ satisfying $a_k<a_{k+1}$ and $\sum_{k=1}^n\lambda_k=1$ for $1\le k\le n$ and $n\ge2$, the arithmetic and geometric means and the weighted arithmetic and geometric means are respectively defined by

$$A_n(\boldsymbol{a})=\frac1n\sum_{k=1}^na_k,\qquad G_n(\boldsymbol{a})=\sqrt[n]{\prod_{k=1}^na_k}\,,$$
$$A_n(\boldsymbol{a},\boldsymbol{\lambda})=\sum_{k=1}^n\lambda_ka_k,\quad G_n(\boldsymbol{a},\boldsymbol{\lambda})=\prod_{k=1}^na_k^{\lambda_k},$$

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where $\boldsymbol{a}=(a_1,a_2,\dots,a_n)$ and $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_n)$. For $z\in\mathbb{C}$, we denote

$$\begin{aligned}
A_{a,b}(z)&=A(a+z,b+z), & G_{a,b}(z)&=G(a+z,b+z),\\
H_{a,b}(z)&=H(a+z,b+z), & I_{a,b}(z)&=I(a+z,b+z),\\
L_{a,b}(z)&=L(a+z,b+z), & A_{a,b;\lambda}(z)&=A(a+z,b+z;\lambda),\\
G_{a,b;\lambda}(z)&=G(a+z,b+z;\lambda), & A_{n,\boldsymbol{a}}(z)&=A_n(\boldsymbol{a}+z),\\
G_{n,\boldsymbol{a}}(z)&=G_n(\boldsymbol{a}+z), & G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(z)&=G_n(\boldsymbol{a}+z,\boldsymbol{\lambda}),
\end{aligned}$$

where $\boldsymbol{a}+z=(a_1+z,a_2+z,\dots,a_n+z)$. For more information on mathematical means, please refer to the monograph [9].

2. Origins and motivations

In mathematics, integral represenataions of functions provide us a lot of valuable information. By integral representations, one can obtain new properties of various functions. In recent decades, there have been much work related with establishing integral representations of special functions. By the Cauchy integral formula in the theory of complex functions, Berg and Pedersen [3, 4, 5, 6, 7] established integral representations of some functions involving the gamma and logarithmic functions. Later, by the Cauchy integral formula and other analytic techniques, Qi and his coauthors [8, 15, 24, 25, 32, 35, 37, 38, 40, 42, 43, 50, 54, 56, 57, 59, 60, 63] established integral representations and the Lévy-Khintchine representations of several generating functions of sequences in combinatorics and number theory, of some functions involving the identric and logarithmic functions, and of geometric means and their reciprocals.

In what follows, we will pay our main attention on the Bernstein function property, integral representations, and the Lévy-Khintchine representations of several mathematical means and their reciprocals.

2.1. Lévy–Khintchine representation of bivariate logarithmic mean.

In [22, Remark 3.7] and its preprint [23, Remark 3.7], the logarithmic mean $L_{a,b}(x)$ was proved to be increasing and concave in $x\in(-\min\{a,b\},\infty)$.

In [26, Theorem 1] and its formally published version [32, Thereom 1], starting from the integral representation

$$L(a,b)=\int_0^1 a^{1-u}b^u\,\mathrm{d}\,u, \tag{2.1}$$

the logarithmic mean $L_{a,b}(x)$ was further proved to be a Bernstein function of $x\in(-\min\{a,b\},\infty)$. Consequently, the function $\frac{1}{L_{a,b}(x)}$ is logarithmically completely monotonic in $x\in(-\min\{a,b\},\infty)$. See also [61, Section 1.6] and [63, Section 1.6].

In [60, Theorem 1.2] and its preprint [58, Theorem 4.1], combining (2.1) with (2.11) below deduced that, for $b>a>0$ and $z\in\mathbb{C}\setminus[-b,-a]$, the principal branch of the logarithmic mean $L_{a,b}(z)$ has the Lévy–Khintchine representation

$$L_{a,b}(z)=L(a,b)+z+\frac{b-a}{\pi}\int_0^\infty\frac{P_{a,b}(s)}{s}e^{-as}\bigl(1-e^{-sz}\bigr)\,\mathrm{d}\,s, \tag{2.2}$$

where

$$P_{a,b}(s)=\int_0^1\sin(\lambda\pi)F(\lambda,(b-a)s)\,\mathrm{d}\,\lambda$$

and

$$F(\lambda,s)=\int_0^1\biggl(\frac{1}{u}-1\biggr)^\lambda\biggl(1-\frac{\lambda}{1-u}\biggr)e^{-us}\,\mathrm{d}\,u>0.$$

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Consequently, the logarithmic mean $L_{a,b}(t)$ is a complete Bernstein function and a positive operator monotone function of $t \in (0,\infty)$. See also [60, Remark 4.2].

In [58, Corollary 4.1] and [60, Remark 4.4], by taking $z \to \infty$ on both sides of (2.2), it was obtained that

$$A(a,b) - L(a,b) = \frac{b-a}{\pi} \int_0^\infty \frac{P_{a,b}(s)}{s} e^{-as} \,\mathrm{d}\, s, \quad b > a > 0.$$

This implies the inequality $A(a,b) > L(a,b)$ for $a,b > 0$ and $a \ne b$.

Since

$$\frac{1}{L_{a,b}(z)} = \frac{\ln(b+z) - \ln(a+z)}{b-a} = \frac{1}{b-a} \int_a^b \frac{1}{z+u} \,\mathrm{d}\, u,$$

comparing with (1.1) shows that the reciprocal $\frac{1}{L_{a,b}(z)}$ is trivially a Stieltjes function.

2.2. **Lévy–Khintchine representation of bivariate identric mean.** In [46, Remark 6] and its preprint [45, Remark 4], it was pointed out that by standard argument the reciprocal of the identric mean $I_{a,b}(x)$ can be proved to be a logarithmically completely monotonic function on $(-\min\{a,b\},\infty)$. Hereafter, by virtue of the integral representation

$$I(a,b) = \exp\left(\frac{1}{b-a} \int_a^b \ln u \,\mathrm{d}\, u\right),$$

it was claimed in [45, Remark 5] and [46, Remark 6] that the the identric mean $I_{a,b}(x)$ is a Bernstein function of $x \in (-\min\{a,b\},\infty)$. See also [63, Section 1.5] and its preprint [61, Section 1.5]. We do not doubt correctness of this claim, we even believe that $I_{a,b}(x)$ is a complete Bernstein function. But we need a complete proof for it. We also believe that such a proof should not be straightforward, direct, and trivial.

2.3. **Monotonicity and convexity concerning Stolasky's mean.** In [36, Theorems 3 to 5] and their formally published versions [18, Theorems 4.1 to 4.3], it was obtained that

(1) the function $F_{r,s;a,b}(t) = E(r+t, s+t; a,b)$ is logarithmically convex on $\left(-\infty, -\frac{r+s}{2}\right)$ and logarithmically concave on $\left(-\frac{r+s}{2}, \infty\right)$;
(2) the product $F_{r,s;a,b}(t)F_{r,s;a,b}(-t)$ is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$;
(3) if $r+s>0$, the function $t \ln F_{r,s;a,b}(t)$ is convex on $\left(-\frac{r+s}{2}, 0\right)$; if $r+s<0$, the function $t \ln F_{r,s;a,b}(t)$ is convex on $\left(0, -\frac{r+s}{2}\right)$.

2.4. **Stolasky's mean is a Bernstein function.** Motivated by the papers [26, 32], among other things, Besenyei obtained in [8, Theorem 2] that, for $q > 0$, if $|r| \le 1$ and $|s| \le 2$, or $|s| \le 1$ and $|r| \le 2$, then the function $x \mapsto E_{r,s;q,0}(x) = E(r,s;x+q,x)$ is a Bernstein function on $(0,\infty)$.

2.5. **Lévy–Khintchine representation of bivariate harmonic mean.** In the preprint [61, Theorem 3.1] and its formally published version [63, Theorem 3.1], the harmonic mean $H_{a,b}(t)$ was proved to be a Bernstein function of $t \in (-\min\{a,b\},\infty)$ and to have the Lévy–Khintchine representation

$$H_{a,b}(t) = H(a,b) + t + \frac{(b-a)^2}{4} \int_0^\infty \left(1 - e^{-tu}\right) e^{-(a+b)u/2} \,\mathrm{d}\, u \tag{2.3}$$

and

$$H(a,b) = A(a,b) - \frac{(b-a)^2}{2} \int_0^\infty e^{-(a+b)u} \,\mathrm{d}\, u. \tag{2.4}$$

From the representation (2.4), the well-known inequality $H(a,b) < A(a,b)$ for $a \ne b$ follows immediately.

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Comparing (2.3) with (1.3) reveals that the harmonic mean $H_{a,b}(t)$ is a complete Bernstein function and a positive operator monotone function in $t\in(0,\infty)$. Consequently, the reciprocal $\frac{1}{H_{a,b}(t)}=\frac{a+b+2t}{2(a+t)(b+t)}$ is a Stieltjes function on $(0,\infty)$.

2.6. **Lévy–Khintchine representation of bivariate geometric mean.** In [63, Theorem 4.1] and its preprint [61, Theorem 4.1], by several approaches, the geometric mean $G_{a,b}(t)$ was proved to be a Bernstein function of $t\in(-\min\{a,b\},\infty)$.

In [61, Theorem 4.2] and its formally published version [63, Theorem 4.2], with the help of the Cauchy integral formula and other techniques, it was proved that the principal branch of $G_{a,b}(z)$ for $b>a>0$ and $z\in\mathbb{C}\setminus[-b,-a]$ has the Lévy–Khintchine representation

$$G_{a,b}(z)=G(a,b)+z+\frac{b-a}{2\pi}\int_0^\infty\frac{\rho((b-a)s)}{s}e^{-as}\bigl(1-e^{-sz}\bigr)\,\mathrm{d}\,s, \tag{2.5}$$

where

$$\rho(s)=\int_0^{1/2}q(u)\bigl[1-e^{-(1-2u)s}\bigr]e^{-us}\,\mathrm{d}\,u$$

is nonnegative on $(0,\infty)$ and

$$q(u)=\sqrt{\frac{1}{u}-1}-\frac{1}{\sqrt{1/u-1}},\quad u\in(0,1)$$

is positive on $\bigl(0,\frac12\bigr)$ and symmetric with respect to $u=\frac12$. In [61, Remark 4.2] and its formally published version [63, Remark 4.2], the integral representation

$$A(a,b)=G(a,b)+\frac{b-a}{2\pi}\int_0^\infty\frac{\rho((b-a)s)}{s}e^{-as}\,\mathrm{d}\,s,\quad b>a>0 \tag{2.6}$$

was derived from (2.5). From the representation (2.6), the well-known mean inequality $G(a,b)<A(a,b)$ for $a\ne b$ follows immediately.

Since e^{-as}, $e^{-u(b-a)s}$, and

$$\frac{1-e^{-(1-2u)(b-a)s}}{s}=\int_{e^{-(1-2u)(b-a)}}^1 v^{s-1}\,\mathrm{d}\,v$$

for fixed $u\in\bigl(0,\frac12\bigr)$ are completely monotonic functions of $s\in(0,\infty)$, and since the product of finitely many completely monotonic functions is still completely monotonic, then the density

$$\frac{b-a}{2\pi}\frac{\rho((b-a)s)}{s}e^{-as}=\frac{b-a}{2\pi}e^{-as}\int_0^{1/2}\frac{1-e^{-(1-2u)(b-a)s}}{s}e^{-u(b-a)s}q(u)\,\mathrm{d}\,u$$

in the Lévy–Khintchine representation (2.5) is completely monotonic on $(0,\infty)$. Therefore, the geometric mean $G_{a,b}(t)$ is a complete Bernstein function and a positive operator monotone function on $(0,\infty)$. This coincides with [15, Theorem 2.3].

Letting $n=2$ and $\lambda=\frac12$ in [15, Theorem 2.2] and [57, Theorems 3.1 and 4.6] leads to

$$G_{a,b}(z)=\sqrt{ab}+z+\frac{z}{\pi}\int_a^b\frac{\sqrt{(t-a)(b-t)}}{t}\frac{1}{t+z}\,\mathrm{d}\,t,\quad b>a>0. \tag{2.7}$$

Taking $n=2$ in [59, Theorem 1.1], that is, the integral representations (2.17) and (2.18) below, results in (2.7) and, for $b>a>0$,

$$G_{a,b}(z)=\sqrt{ab}+z+\frac{1}{\pi}\int_0^\infty\biggl[\int_a^b\sqrt{(t-a)(b-t)}\,e^{-tu}\,\mathrm{d}\,t\biggr]\bigl(1-e^{-zu}\bigr)\,\mathrm{d}\,u \tag{2.8}$$

which is more significant than (2.5) and is equivalent to (2.7).

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2.7. **Integral representation for reciprocal of bivariate geometric mean and applications.** In [42, Lemma 2.4], by two methods, including the Cauchy integral formula, it was proved that the principal branch of the reciprocal $\frac{1}{G_{a,b}(z)}$ for $b > a$ and $z \in \mathbb{C} \setminus [-b, -a]$ with $a, b \in \mathbb{R}$ can be represented by

$$\frac{1}{\sqrt{(z+a)(z+b)}} = \frac{1}{\pi} \int_a^b \frac{1}{\sqrt{(t-a)(b-t)}} \frac{1}{t+z} \,\mathrm{d}\, t. \tag{2.9}$$

Comparing (2.9) with (1.1) shows that the reciprocal $\frac{1}{G_{a,b}(x)}$ is a Stieltjes transform, a logarithmically completely monotonic function on $(-a, \infty)$, and a completely monotonic function on $(-a, \infty)$.

In [42, Theorem 1.3], employing (2.9) derives an integral representation

$$D(k) = \frac{1}{\pi} \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{1}{\sqrt{\bigl(t-3+2\sqrt{2}\,\bigr)\bigl(3+2\sqrt{2}\,-t\bigr)}} \frac{1}{t^{k+1}} \,\mathrm{d}\, t, \quad k \ge 0, \tag{2.10}$$

where $D(k)$ denotes central Delannoy number which is the number of "king walks" from the $(0,0)$ corner of an $n \times n$ square to the upper right corner (n,n) and can be generated by

$$G(x) = \frac{1}{\sqrt{1-6x+x^2}} = \sum_{k=0}^{\infty} D(k) x^k = 1 + 3x + 13x^2 + 63x^3 + \cdots.$$

As consequences of the integral representations (2.9) and (2.10), the (complete) monotonicity, (logarithmic) convexity, product inequalities, and positivity of determinants concerning the central Delannoy numbers $D(k)$ were deduced in [42, Theorems 1.4 to 1.7] and [42, Corollaries 1.1 and 1.2].

2.8. **Lévy–Khintchine representation of bivariate weighted geometric mean.** In [58, Theorem 3.1] and its formally published version [56, Theorem 1.1], it was elementarily proved that the weighted geometric mean $G_{a,b;\lambda}(t)$ is a Bernstein function of $t > -\min\{a, b\}$.

In [51, Theorem 1], with the aid of [56, Lemma 2.1] and its preprint [58, Lemma 2.1], several identities and recurrence relations involving the falling and rising factorials, the Cauchy and Lah numbers, and the Stirling numbers of the first kind were found. For detailed information on the falling and rising factorials, the Cauchy and Lah numbers, and the Stirling numbers of the first kind, please refer to [11, 16, 24, 27, 28, 29] and the closely related references therein.

In [60, Theorem 1.1] and its preprint [58, Theorem 3.2], with the help of the Cauchy integral formula, the principal branch of the weighted geometric mean $G_{a,b;\lambda}(z)$ for $b > a > 0$ and $z \in \mathbb{C} \setminus [-b, -a]$ was proved to have the integral representation

$$G_{a,b;\lambda}(z) = a^\lambda b^{1-\lambda} + z + \frac{\sin(\lambda\pi)}{\pi}(b-a) \int_0^\infty \frac{F(1-\lambda, (b-a)s)}{s} e^{-as}\bigl(1-e^{-sz}\bigr) \,\mathrm{d}\, s. \tag{2.11}$$

Consequently, the geometric mean $G_{a,b;\lambda}(t)$ is a complete Bernstein function and a positive operator monotone function of $t \in (0, \infty)$.

In [60, Remark 4.3] and its preprint [58, Corollary 3.1], it was derived that the difference between the weighted arithmetic mean $A(a, b; \lambda)$ and the weighted geometric mean $G(a, b; \lambda)$ for $b > a > 0$ can be expressed by

$$\bigl[\lambda a + (1-\lambda) b\bigr] - a^\lambda b^{1-\lambda} = \frac{\sin(\lambda\pi)}{\pi}(b-a) \int_0^\infty \frac{F(1-\lambda, (b-a)s)}{s} e^{-sa} \,\mathrm{d}\, s. \tag{2.12}$$

The integral representation (2.12) gives an alternative proof of the famous inequality $A(a, b; \lambda) > G(a, b; \lambda)$ for $a, b > 0$ with $a \ne b$ and $\lambda \in (0, 1)$.

Letting $n=2$ in [15, Theorem 2.2] and [57, Theorems 3.1 and 4.6] reduces to

$$G_{a,b;\lambda}(z)=a^\lambda b^{1-\lambda}+z+\frac{\sin(\lambda\pi)}{\pi}z\int_a^b\frac{(t-a)^\lambda(b-t)^{1-\lambda}}{t}\frac{1}{t+z}\,\mathrm{d}\,t, \tag{2.13}$$

where $b>a>0$, $\lambda\in(0,1)$, and $z\in\mathbb{C}\setminus[-b,-a]$. Consequently, the weighted geometric mean $G_{a,b;\lambda}(t-a)$ for $\lambda\in(0,1)$ and $b>a>0$ is a complete Bernstein function and a positive operator monotone function on $(0,\infty)$ and $\frac{1}{G_{a,b;\lambda}(t-a)}$ is a Stieltjes function.

2.9. **Lévy–Khintchine representation of Toader–Qi mean.** Due to the paper [52] and its preprint [53], the mean

$$\frac{1}{2\pi}\int_0^{2\pi}a^{\cos^2\theta}b^{\sin^2\theta}\,\mathrm{d}\,\theta=\frac{\pi}{2}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}\,\mathrm{d}\,\theta,\quad b>a>0$$

was named the Toader–Qi mean and denoted by $TQ(a,b)$ in the papers [67, 68, 69]. Similar to the deduction of the Lévy–Khintchine representation (2.2), by virtue of the integral representations (2.11) and (2.13), it was acquired in [35, Theorem 1.1] that the Toader–Qi mean $TQ(x+a,x+b)$ for $x>-a$ has the integral representation

$$TQ(x+a,x+b)=TQ(a,b)+x\left[1+\frac{1}{\pi}\int_a^b\frac{h(a,b;t)}{t}\frac{1}{t+x}\,\mathrm{d}\,t\right]$$

and the Lévy–Khintchine representation

$$TQ(x+a,x+b)=TQ(a,b)+x+\frac{b-a}{\pi}\int_0^\infty\frac{H(a,b;s)}{s}e^{-as}(1-e^{-xs})\,\mathrm{d}\,s,$$

where

$$h(a,b;t)=\frac{2}{\pi}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)(t-a)^{\cos^2\theta}(b-t)^{\sin^2\theta}\,\mathrm{d}\,\theta,\quad t\in[a,b]$$

and

$$H(a,b;s)=\frac{2}{\pi}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)F\bigl(\sin^2\theta,(b-a)s\bigr)\,\mathrm{d}\,\theta,\quad s\in(0,\infty).$$

Consequently,

(1) the Toader–Qi mean $TQ(x,x+b-a)$ is a complete Bernstein function and a positive operator monotone function of x on $(0,\infty)$;
(2) the divided difference

$$\frac{TQ(x,x+b-a)-TQ(a,b)}{x-a}$$

is a Stieltjes function of x on $(0,\infty)$.

2.10. **Integral representation for reciprocal of bivariate weighted geometric mean and applications.** In [38] and its preprint [39], with the help of the Cauchy integral formula, the following conclusions were obtained.

(1) For $a,b\in\mathbb{R}$ with $b>a$ and $z\in\mathbb{C}\setminus[-b,-a]$, the principal branch of the reciprocal $\frac{1}{G_{a,b;\lambda}(z)}$ can be represented as

$$\frac{1}{(z+a)^\lambda(z+b)^{1-\lambda}}=\frac{\sin(\lambda\pi)}{\pi}\int_a^b\frac{1}{(t-a)^\lambda(b-t)^{1-\lambda}}\frac{1}{t+z}\,\mathrm{d}\,t. \tag{2.14}$$

Consequently, it follows that the reciprocal $\frac{1}{G_{a,b;\lambda}(x-a)}$ is a Stieltjes function and, consequently, a (logarithmically) completely monotonic function on $(0,\infty)$.

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(2) For $b>a>0$ and $n\ge0$, we have the integral formulas

$$\int_a^b \frac{1}{(t-a)^\lambda(b-t)^{1-\lambda}}\frac{1}{t^{n+1}}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)}\frac{1}{n!}\frac{1}{a^\lambda b^{1-\lambda}}\frac{1}{b^n}\sum_{\ell=0}^{n}\binom{n}{\ell}(\lambda)_\ell(1-\lambda)_{n-\ell}\left(\frac{b}{a}\right)^\ell$$

and

$$\int_a^b \frac{1}{(t-a)^\lambda(b-t)^{1-\lambda}}\frac{1}{t^{n+1}}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)}\frac{1}{a^\lambda b^{1-\lambda}}\frac{1}{b^n}\sum_{k=0}^{n}\left(\frac{b}{a}\right)^k\sum_{\ell=0}^{k}\langle\lambda\rangle_\ell\binom{k-1}{\ell-1}\frac{1}{\ell!}\left(1-\frac{a}{b}\right)^\ell,$$

where the quantities

$$(x)_n=\begin{cases}\prod\limits_{k=0}^{n-1}(x+k), & n\ge1\\ 1, & n=0\end{cases} \quad\text{and}\quad \langle x\rangle_n=\begin{cases}\prod\limits_{k=0}^{n-1}(x-k), & n\ge1\\ 1, & n=0\end{cases}$$

are called the rising and falling factorial respectively.

(3) Let $a,b\in\mathbb{R}$ with $b>a$ and $z\in\mathbb{C}\setminus[-b,-a]$. Then the integral formulas and representations

$$\int_a^b \frac{1}{(t-a)^\lambda(b-t)^{1-\lambda}}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)},$$

$$\left(\frac{b+z}{a+z}\right)^\lambda=1+\frac{\sin(\lambda\pi)}{\pi}\int_a^b\left(\frac{b-t}{t-a}\right)^\lambda\frac{1}{t+z}\,\mathrm{d}\,t,$$

$$\left(\frac{a+z}{b+z}\right)^\lambda=1-\frac{\sin(\lambda\pi)}{\pi}\int_a^b\left(\frac{t-a}{b-t}\right)^\lambda\frac{1}{t+z}\,\mathrm{d}\,t,$$

and

$$\frac{z}{(a+z)^\lambda(b+z)^{1-\lambda}}=1-\frac{\sin(\lambda\pi)}{\pi}\int_a^b\frac{t}{(t-a)^\lambda(b-t)^{1-\lambda}}\frac{1}{t+z}\,\mathrm{d}\,t$$

are valid. Consequently, the functions

$$\left(1+\frac{b-a}{t}\right)^\lambda,\quad 1-\frac{1}{\left(1+\frac{b-a}{t}\right)^\lambda},\quad 1-\frac{t-a}{t^\lambda(b-a+t)^{1-\lambda}}$$

are Stieltjes functions and the integral formulas

$$\int_a^b\left(\frac{b-t}{t-a}\right)^\lambda\frac{1}{t}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)}\left[\left(\frac{b}{a}\right)^\lambda-1\right],$$

$$\int_a^b\left(\frac{t-a}{b-t}\right)^\lambda\frac{1}{t}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)}\left[1-\left(\frac{a}{b}\right)^\lambda\right],$$

$$\int_a^b\left(\frac{b-t}{t-a}\right)^\lambda\frac{1}{t^{k+1}}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)}\left(\frac{b}{a}\right)^\lambda\frac{1}{a^k}\sum_{\ell=0}^{k}\frac{\langle\lambda\rangle_\ell}{\ell!}\binom{k-1}{\ell-1}\left(1-\frac{a}{b}\right)^\ell,$$

$$\int_a^b\left(\frac{t-a}{b-t}\right)^\lambda\frac{1}{t^{k+1}}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)}\left(\frac{a}{b}\right)^\lambda\frac{1}{b^k}\sum_{\ell=0}^{k}\frac{\langle\lambda\rangle_\ell}{\ell!}\binom{k-1}{\ell-1}\left(1-\frac{b}{a}\right)^\ell,$$

$$\frac{1}{b-a}\int_a^b\left(\frac{t-a}{b-t}\right)^\lambda\mathrm{d}\,t=\frac{\lambda\pi}{\sin(\lambda\pi)},$$

$$[\lambda a+(1-\lambda)b]-a^\lambda b^{1-\lambda}=\frac{\sin(\lambda\pi)}{\pi}\int_a^b\frac{(t-a)^\lambda(b-t)^{1-\lambda}}{t}\,\mathrm{d}\,t$$

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hold for $b>a>0$, $k\in\mathbb{N}$, and $\lambda\in(0,1)$.

The last equation above gives an integral representation for the difference between the weighted arithmetic and geometric means of two positive numbers and gives an alternative proof of the well-known inequality $A(a,b;\lambda)>G(a,b;\lambda)$ for $a,b>0$ with $a\ne b$ and $\lambda\in(0,1)$. As did in [25, Theorem 1.1], this integral representation and (2.12) can be applied to estimate the difference between the weighted arithmetic and geometric means of two positive numbers.

(4) When $b>a>0$ and $\lambda\in(0,1)$, the integral representations (2.13) and (2.14) are equivalent to each other, that is, any one of the integral representations (2.13) and (2.14) can be directly derived from another one.

(5) For $n\ge0$, we have

$$\int_a^b\frac{1}{\sqrt{(t-a)(b-t)}\,}\frac{1}{t^{n+1}}\,\mathrm{d}\,t=\frac{\pi}{\sqrt{ab}\,}\frac{1}{b^n}\sum_{\ell=0}^n\frac{(2\ell-1)!!}{(2\ell)!!}\frac{[2(n-\ell)-1]!!}{[2(n-\ell)]!!}\biggl(\frac{b}{a}\biggr)^\ell,$$

$$\int_a^b\frac{1}{\sqrt{(t-a)(b-t)}\,}\frac{1}{t^{n+1}}\,\mathrm{d}\,t\\ =\frac{(-1)^n\pi}{(a+b)^n\sqrt{ab}\,}\sum_{\ell=0}^n(-1)^\ell2^{2\ell}\frac{(2\ell-1)!!}{(2\ell)!!}\binom{\ell}{n-\ell}\biggl(\frac{a+b}{2}\biggr)^\ell\biggl(\frac{1/a+1/b}{2}\biggr)^\ell,$$

and

$$\int_a^b\frac{1}{\sqrt{(t-a)(b-t)}\,}\frac{1}{t^{n+1}}\,\mathrm{d}\,t=\frac{\pi}{\sqrt{ab}\,}\frac{1}{b^n}\sum_{k=0}^n\biggl(\frac{b}{a}\biggr)^k\sum_{\ell=0}^k(-1)^{\ell-1}\frac{(2\ell-3)!!}{(2\ell)!!}\binom{k-1}{\ell-1}\biggl(1-\frac{a}{b}\biggr)^\ell,$$

where the double factorial of negative odd integers $-(2n+1)$ is defined by

$$(-2n-1)!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^nn!}{(2n)!},\quad n\ge0.$$

See also [30, pp. 105–106].

(6) For $b>a>0$ and $\lambda\in(-1,1)$,

$$\int_a^b\biggl(\frac{b-t}{t-a}\biggr)^\lambda\frac{1}{t}\ln\frac{b-t}{t-a}\,\mathrm{d}\,t=\begin{cases}\dfrac{\pi}{\sin(\lambda\pi)}\biggl\{\biggl(\dfrac{b}{a}\biggr)^\lambda\ln\dfrac{b}{a}-\pi\cot(\lambda\pi)\biggl[\biggl(\dfrac{b}{a}\biggr)^\lambda-1\biggr]\biggr\}, & \lambda\ne0;\\ \dfrac12\biggl(\ln\dfrac{b}{a}\biggr)^2, & \lambda=0\end{cases}\tag{2.15}$$

and

$$\int_0^\infty\frac{s^\lambda\ln s}{(1+s)(as+b)}\,\mathrm{d}\,s=\begin{cases}\dfrac{\pi}{\sin(\lambda\pi)}\dfrac{\bigl(\frac{b}{a}\bigr)^\lambda\ln\frac{b}{a}-\pi\cot(\lambda\pi)\bigl[\bigl(\frac{b}{a}\bigr)^\lambda-1\bigr]}{b-a}, & \lambda\ne0;\\ \dfrac{1}{2(b-a)}\biggl(\ln\dfrac{b}{a}\biggr)^2, & \lambda=0.\end{cases}\tag{2.16}$$

When $0>b>a$ and $\lambda\in(-1,1)$, the formulas (2.15) and (2.16) are also valid.

(7) For $n\ge0$, the central Delannoy numbers $D(n)$ can be computed by

$$\begin{aligned}D(n)&=\frac1\pi\int_{3-2\sqrt2}^{3+2\sqrt2}\frac{1}{\sqrt{\bigl(t-3+2\sqrt2\,\bigr)\bigl(3+2\sqrt2\,-t\bigr)}\,}\frac{1}{t^{n+1}}\,\mathrm{d}\,t\\ &=\bigl(3-2\sqrt2\,\bigr)^n\sum_{\ell=0}^n\frac{(2\ell-1)!!}{(2\ell)!!}\frac{[2(n-\ell)-1]!!}{[2(n-\ell)]!!}\bigl(3+2\sqrt2\,\bigr)^{2\ell},\end{aligned}$$

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$$D(n)=\frac{(-1)^n}{6^n}\sum_{\ell=0}^{n}(-1)^\ell 6^{2\ell}\frac{(2\ell-1)!!}{(2\ell)!!}\binom{\ell}{n-\ell},$$

and

$$D(n)=\bigl(3-2\sqrt{2}\bigr)^n\sum_{k=0}^{n}\bigl(3+2\sqrt{2}\bigr)^{2k}\sum_{\ell=0}^{k}(-1)^{\ell-1}\frac{(2\ell-3)!!}{(2\ell)!!}\binom{k-1}{\ell-1}4^\ell\bigl(3\sqrt{2}-4\bigr)^\ell,$$

where $\binom{p}{q}=0$ for $q>p\ge 0$.

2.11. **Lévy–Khintchine representation for multivariate geometric mean and applications.** Theorem 1.1 in [59, Theorem 1.1] (see also its preprints [55, Theorem 3.1] and [62, Theorem 3.1]) can be reformulated as that the principal branch of the geometric mean $G_{n,\boldsymbol{a}}(z)$ can be represented by the integral representation

$$G_{n,\boldsymbol{a}}(z)=A_n(\boldsymbol{a})+z-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_\ell}^{a_{\ell+1}}\sqrt[n]{\prod_{k=1}^{n}|a_k-t|}\,\frac{\mathrm{d}\,t}{t+z}\tag{2.17}$$

and the Lévy–Khintchine representation

$$G_{n,\boldsymbol{a}}(z)=G_n(\boldsymbol{a})+z+\int_0^\infty Q_n(u)\bigl(1-e^{-zu}\bigr)\,\mathrm{d}\,u,\tag{2.18}$$

where

$$Q_n(u)=\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_\ell}^{a_{\ell+1}}\sqrt[n]{\prod_{k=1}^{n}|a_k-t|}\,e^{-tu}\,\mathrm{d}\,t.$$

Consequently, the geometric mean $\sqrt[n]{\prod_{\ell=1}^{n}(t+a_\ell)}$ is a Bernstein function and a positive operator monotone function of $t\in(-a_1,\infty)$.

In [59, Section 4] and its preprints [55, Section 4] and [62, Section 4], the integral representation (2.17) was utilized to alternatively prove the arithmetic-geometric mean inequality $A_n(\boldsymbol{a})>G_n(\boldsymbol{a})$ and the validity of its equality.

In [29], the integral representation (2.17) was applied to establish three integral representations and then to discover many properties of the Stirling numbers of the first kind. For more information on the Stirling numbers of the first kind, please refer to [27, 28] and the closely related references therein.

2.12. **Integral representation for reciprocal of multivariate geometric mean.** In [37, Theorem 2.1], it was obtained that the principal branch of the reciprocal $\frac{1}{G_{n,\boldsymbol{a}}(z)}$ for $z\in\mathbb{C}\setminus[-a_n,-a_1]$ can be represented by

$$\frac{1}{G_{n,\boldsymbol{a}}(z)}=\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\Bigl(\frac{\ell\pi}{n}\Bigr)\int_{a_\ell}^{a_{\ell+1}}\frac{1}{\sqrt[n]{\prod_{k=1}^{n}|t-a_k|}}\,\frac{1}{t+z}\,\mathrm{d}\,t.\tag{2.19}$$

As a result, the function $\frac{1}{G_{n,\boldsymbol{a}}(z)}$ is a Stieltjes function and, consequently, a (logarithmically) completely monotonic function on $(0,\infty)$.

When $n=2$, the integral representation (2.19) becomes (2.9).

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2.13. **Lévy–Khintchine representation for multivariate weighted geometric mean and applications.** In [15, Theorem 2.2] and [57, Theorems 3.1 and 4.6], it was proved and reformulated that the principal branch of the weighted geometric mean $G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(z)$ for $z\in\mathbb{C}\setminus[-a_n,-a_1]$ has the Lévy–Khintchine representation

$$G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(z)=G_n(\boldsymbol{a},\boldsymbol{\lambda})+z+\int_0^\infty m_{n,\boldsymbol{a},\boldsymbol{\lambda}}(u)(1-e^{-zu})\,\mathrm{d}\,u, \tag{2.20}$$

where the density

$$m_{n,\boldsymbol{a},\boldsymbol{\lambda}}(u)=\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left(\pi\sum_{j=1}^{\ell}\lambda_j\right)\int_{a_\ell}^{a_{\ell+1}}\prod_{k=1}^{n}|a_k-t|^{\lambda_k}e^{-ut}\,\mathrm{d}\,t.$$

Consequently, the weighted geometric mean $G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(t-a_1)$ is a complete Bernstein function and a positive operator monotone function on $(0,\infty)$ and $\frac{1}{G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(t-a_1)}$ is a Stieltjes function.

In [40, Remark 4.1] and its preprint [41, Remark 4.1], it was proved that letting $n=2$ in (2.20) leads to (2.13). Differentiating with respect to z on both sides of (2.13) and letting $z\to0$ arrive at

$$\frac{A(a,b;\lambda)}{G(a,b;\lambda)}=1+\frac{\sin(\lambda\pi)}{\pi}\int_a^b\frac{(t-a)^{\lambda}(b-t)^{1-\lambda}}{t^2}\,\mathrm{d}\,t \tag{2.21}$$

which, when $\lambda=\frac12$, becomes [30, Remark 4.3, Eq. (4.2)]. The equation (2.21) has been applied in [25, Theorem 1.2] to bound the difference and ratio between the weighted arithmetic and geometric means.

From the expression (2.21), one can derive readily the famous and fundamental inequality

$$A(a,b;\lambda)\ge G(a,b;\lambda),\quad \lambda\in(0,1),\quad a,b>0 \tag{2.22}$$

and the equality in the above inequality is valid if and only if $a=b$ for $\lambda\in(0,1)$. This can be regarded as an alternative proof of the inequality (2.22) whose proofs are presumably over one hundred, as said in [9, Section 2.4], in the literature.

In [40, Remark 4.2] and its preprint [41, Remark 4.2], it was derived that

$$\int_a^b\frac{(t-a)^{\lambda}(b-t)^{1-\lambda}}{t^{k+2}}\,\mathrm{d}\,t=\frac{\pi}{\sin(\lambda\pi)}\left(\frac{a}{b}\right)^{\lambda}\frac{(-1)^k}{b^k}\frac{1}{(k+1)!}\sum_{\ell=0}^{k+1}\binom{k+1}{\ell}\langle\lambda\rangle_\ell\langle1-\lambda\rangle_{k-\ell+1}\left(\frac{b}{a}\right)^{\ell},$$

where $k\in\mathbb{N}$, $b>a>0$, and $\lambda\in(0,1)$. When $\lambda=\frac12$,

$$\int_a^b\frac{\sqrt{(t-a)(b-t)}}{t^{k+2}}\,\mathrm{d}\,t=-\pi\frac{\sqrt{ab}}{b^{k+1}}\sum_{\ell=0}^{k+1}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(k-\ell+1)-3]!!}{[2(k-\ell+1)]!!}\left(\frac{b}{a}\right)^{\ell}, \tag{2.23}$$

where $k\in\mathbb{N}$, $b>a>0$, and the double factorial of negative odd integers $-(2n+1)$ is defined by

$$(-2n-1)!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^n n!}{(2n)!},\quad n\ge0.$$

In [30, Remark 4.3], it was also established that

$$\int_a^b\frac{\sqrt{(t-a)(b-t)}}{t^{k+2}}\,\mathrm{d}\,t=\pi\frac{(-1)^k}{(a+b)^k}\sum_{\ell=0}^{k}(-1)^{\ell}\frac{(2\ell-1)!!}{[2(\ell+1)]!!}\binom{\ell+1}{k-\ell}\left(\frac{a+b}{\sqrt{ab}}\right)^{2\ell+1} \tag{2.24}$$

for $b>a>0$ and $k\in\mathbb{N}$.

In [47, Theorem 1.1], by virtue of the Lévy–Khintchine representations (2.5), (2.11), (2.18), and (2.20), the large Schröder number S_n, who describes the number of paths from the southwest corner $(0,0)$ of an $n\times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the southwest-northeast diagonal, and its generating functions were represented by several integral representations, one of them is

$$S_n=\frac{1}{2\pi}\int_{3-2\sqrt2}^{3+2\sqrt2}\frac{\sqrt{\bigl(u-3+2\sqrt2\,\bigr)\bigl(3+2\sqrt2\,-u\bigr)}}{u^{n+2}}\,\mathrm{d}\,u,\quad n\ge0.$$

Therefore, when letting $a=3-2\sqrt2$ and $b=3+2\sqrt2$ in (2.23) and (2.24) respectively, we obtain two explicit formulas

$$S_n=-\frac12\bigl(\sqrt2\,-1\bigr)^{2(n+1)}\sum_{\ell=0}^{n+1}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+1)-3]!!}{[2(n-\ell+1)]!!}\bigl(\sqrt2\,+1\bigr)^{4\ell}$$

and

$$S_n=\frac{1}{12}\frac{(-1)^n}{6^n}\sum_{\ell=0}^{n}(-1)^\ell\frac{(2\ell-1)!!}{[2(\ell+1)]!!}\binom{\ell+1}{n-\ell}6^{2\ell}$$

for the large Schröder numbers S_n. For more information on recent results about the large Schröder numbers S_n, please refer to the recently published papers [17, 34, 47, 48, 49] and the closely related references therein.

2.14. **Integral representation for reciprocal of multivariate weighted geometric mean and applications.** In [40, Theorem 2.1] and its preprint [41, Theorem 2.1], by virtue of the Cauchy integral formula, it was established that the principal branch of the reciprocal $\frac{1}{G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(z)}$ for $z\in\mathbb{C}\setminus[-a_n,-a_1]$ can be represented by

$$\frac{1}{G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(z)}=\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left(\pi\sum_{k=1}^{\ell}w_k\right)\int_{a_\ell}^{a_{\ell+1}}\frac{1}{\prod_{k=1}^{n}|t-a_k|^{w_k}}\frac{1}{t+z}\,\mathrm{d}\,t.$$

Consequently, the reciprocal $\frac{1}{G_{n,\boldsymbol{a},\boldsymbol{\lambda}}(t-a_1)}$ is a Stieltjes function and a (logarithmically) completely monotonic function on $(0,\infty)$.

2.15. **Integral representations for bivariate Heronian mean of power type.** For $a,b>0$ with $a\ne b$, taking $r=1$ and $s=3$ gives

$$E(1,3;a,b)=\sqrt{\frac{a^2+ab+b^2}{3}}.$$

This is also a special case $p=2$ of the mean

$$h_p(a,b)=\left[\frac{a^p+(ab)^{p/2}+b^p}{3}\right]^{1/p},\quad p\ne0$$

which is called [19, p. 58] the Heronian mean of power type. It is not difficult to see that

$$E(1,3;a,b)=h_2(a,b)=\sqrt{\left(\frac{2a+b}{2\sqrt3}-\frac{b}{2}i\right)\left(\frac{2a+b}{2\sqrt3}+\frac{b}{2}i\right)},$$

where $i=\sqrt{-1}$ is the imaginary unit.

It is clear that one can not use conclusions in Sections 2.1 to 2.14 to establish integral representations for

$$E(1,3;a+z,b+z)=h_2(a+z,b+z)=\sqrt{\left(z+\frac{a+b}{2}-\frac{b-a}{2\sqrt3}i\right)\left(z+\frac{a+b}{2}+\frac{b-a}{2\sqrt3}i\right)}\quad(2.25)$$

and
$$\sqrt{\biggl(z+\frac{2a+b}{2\sqrt{3}}-\frac{b}{2}i\biggr)\biggl(z+\frac{2a+b}{2\sqrt{3}}+\frac{b}{2}i\biggr)} \tag{2.26}$$
or to discuss whether they are Bernstein functions or not. Therefore, we naturally pose a problem: can one find an integral representation for the principal branch of the bivariate complex geometric mean
$$G_{\alpha,\beta}(z)=\sqrt{(z+\alpha)(z+\beta)}\,?$$
where $\alpha,\beta\in\mathbb{C}$ with $\alpha,\beta\ne0$ and $\alpha\ne\beta$, the notation $[-\beta,-\alpha]$ denotes the closed line segment between $-\beta$ and $-\alpha$, and $z\in\mathbb{C}\setminus[-\beta,-\alpha]$.

3. Integral representations for bivariate complex geometric mean and its reciprocal

Now we start out to establish integral representations for the principal branches of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ and its reciprocal $\frac{1}{G_{\alpha,\beta}(z)}$ step by step.

Theorem 3.1. *Let $\alpha,\beta\in\mathbb{C}$, $\alpha\ne\beta$, and $\theta=\arg(\beta-\alpha)\in(-\pi,\pi]$. If $\Re\bigl(\alpha e^{-\theta i}\bigr)>0$, then the principal branch of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z\in\mathbb{C}\setminus[-\beta,-\alpha]$ can be represented by*
$$\begin{aligned}G_{\alpha,\beta}(z)&=\sqrt{\Re\bigl(\alpha e^{-\theta i}\bigr)\Re\bigl(\beta e^{-\theta i}\bigr)}\,e^{\theta i}+z+\Im\bigl(\alpha e^{-\theta i}\bigr)ie^{\theta i}\\&\quad+\frac{z+\Im\bigl(\alpha e^{-\theta i}\bigr)ie^{\theta i}}{\pi}\int_{\Re(\alpha e^{-\theta i})}^{\Re(\beta e^{-\theta i})}\frac{\sqrt{\bigl[t-\Re\bigl(\alpha e^{-\theta i}\bigr)\bigr]\bigl[\Re\bigl(\beta e^{-\theta i}\bigr)-t\bigr]}}{t}\frac{1}{t+ze^{-\theta i}+i\Im\bigl(\alpha e^{-\theta i}\bigr)}\,\mathrm{d}\,t\end{aligned} \tag{3.1}$$
and
$$\begin{aligned}G_{\alpha,\beta}(z)&=\sqrt{\Re\bigl(\alpha e^{-\theta i}\bigr)\Re\bigl(\beta e^{-\theta i}\bigr)}\,e^{\theta i}+z+\Im\bigl(\alpha e^{-\theta i}\bigr)ie^{\theta i}\\&+\frac{e^{\theta i}}{\pi}\int_0^\infty\biggl(\int_{\Re(\alpha e^{-\theta i})}^{\Re(\beta e^{-\theta i})}\sqrt{\bigl[t-\Re\bigl(\alpha e^{-\theta i}\bigr)\bigr]\bigl[\Re\bigl(\beta e^{-\theta i}\bigr)-t\bigr]}e^{-tu}\,\mathrm{d}\,t\biggr)\Bigl(1-e^{-[ze^{-\theta i}+i\Im(\alpha e^{-\theta i})]u}\Bigr)\,\mathrm{d}\,u.\end{aligned} \tag{3.2}$$

Proof. Since $\theta=\arg(\beta-\alpha)$, it follows that
$$\beta-\alpha=|\beta-\alpha|e^{\theta i},\quad(\beta-\alpha)e^{-\theta i}=|\beta-\alpha|,\quad\beta e^{-\theta i}=\alpha e^{-\theta i}+|\beta-\alpha|,$$
$$\Im\bigl(\alpha e^{-\theta i}\bigr)=\Im\bigl(\beta e^{-\theta i}\bigr),\quad\Re\bigl(\beta e^{-\theta i}\bigr)=\Re\bigl(\alpha e^{-\theta i}\bigr)+|\beta-\alpha|.$$
Then, write the numbers $\alpha e^{-\theta i}$ and $\beta e^{-\theta i}$ as
$$\alpha e^{-\theta i}=L+Ni,\quad\beta e^{-\theta i}=M+Ni,$$
where $L=\Re\bigl(\alpha e^{-\theta i}\bigr)>0$, $M=\Re\bigl(\beta e^{-\theta i}\bigr)=L+|\beta-\alpha|>L$, and $N=\Im\bigl(\alpha e^{-\theta i}\bigr)=\Im\bigl(\beta e^{-\theta i}\bigr)$. Therefore, it follows that
$$\begin{aligned}\sqrt{(z+\alpha)(z+\beta)}&=\sqrt{(z+\alpha)(z+\beta)e^{-2\theta i}}\sqrt{e^{2\theta i}}=\sqrt{(ze^{-\theta i}+\alpha e^{-\theta i})(ze^{-\theta i}+\beta e^{-\theta i})}\,e^{\theta i}\\&=\sqrt{(ze^{-\theta i}+L+Ni)(ze^{-\theta i}+M+Ni)}\,e^{\theta i}=\sqrt{[(ze^{-\theta i}+Ni)+L][(ze^{-\theta i}+Ni)+M]}\,e^{\theta i}.\end{aligned}$$
As a result, by (2.7) and (2.8), we can acquire
$$G_{\alpha,\beta}(z)=e^{\theta i}\biggl[\sqrt{LM}+ze^{-\theta i}+Ni+\frac{ze^{-\theta i}+Ni}{\pi}\int_L^M\frac{\sqrt{(t-L)(M-t)}}{t}\frac{1}{t+ze^{-\theta i}+Ni}\,\mathrm{d}\,t\biggr]$$

and

$$G_{\alpha,\beta}(z)=e^{\theta i}\biggl(\sqrt{LM}+ze^{-\theta i}+Ni+\frac{1}{\pi}\int_0^\infty\biggl[\int_L^M\sqrt{(t-L)(M-t)}\,e^{-tu}\,\mathrm{d}\,t\biggr]\Bigl[1-e^{-(ze^{-\theta i}+Ni)u}\Bigr]\,\mathrm{d}\,u\biggr),$$

which can be rearranged as (3.1) and (3.2), for

$$ze^{-\theta i}+Ni\in\mathbb{C}\setminus[-M,-L]\Longleftrightarrow z\in\mathbb{C}\setminus[-\beta,-\alpha].$$

The proof of Theorem 3.1 is complete. □

Corollary 3.1. *Let $\alpha,\beta\in\mathbb{C}$, $\alpha\ne\beta$, and $\arg(\beta-\alpha)=\pm\frac{\pi}{2}$. If $\pm\Im(\alpha)>0$, then the principal branch of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z\in\mathbb{C}\setminus[-\beta,-\alpha]$ can be represented by*

$$G_{\alpha,\beta}(z)=\pm i\sqrt{\Im(\alpha)\Im(\beta)}+z+\Re(\alpha)+\frac{z+\Re(\alpha)}{\pi}\int_{\pm\Im(\alpha)}^{\pm\Im(\beta)}\frac{\sqrt{[t\mp\Im(\alpha)][\Im(\beta)\mp t]}}{t}\frac{1}{t\mp[z+\Re(\alpha)]i}\,\mathrm{d}\,t \tag{3.3}$$

and

$$G_{\alpha,\beta}(z)=\pm i\sqrt{\Im(\alpha)\Im(\beta)}+z+\Re(\alpha)\pm\frac{i}{\pi}\int_0^\infty\biggl(\int_{\pm\Im(\alpha)}^{\pm\Im(\beta)}\sqrt{[t\mp\Im(\alpha)][\mp\Im(\beta)-t]}\,e^{-tu}\,\mathrm{d}\,t\biggr)\Bigl(1-e^{\pm i[z+\Re(\alpha)]u}\Bigr)\,\mathrm{d}\,u. \tag{3.4}$$

Proof. Taking $\theta=\pm\frac{\pi}{2}$ in (3.1) and (3.2) respectively leads to the integral representations (3.3) and (3.4). □

Theorem 3.2. *Let $\alpha,\beta\in\mathbb{C}$, $\alpha\ne\beta$, and $\theta=\arg(\beta-\alpha)\in(-\pi,\pi]$. If $\Re\bigl(\beta e^{-\theta i}\bigr)>0>\Re\bigl(\alpha e^{-\theta i}\bigr)$, then the principal branch of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z\in\mathbb{C}\setminus[-\beta,-\alpha]$ can be represented by*

$$G_{\alpha,\beta}(z)=\sqrt{\Re\bigl(\alpha e^{-\theta i}\bigr)\Re\bigl[(2\alpha-\beta)e^{-\theta i}\bigr]}\,e^{\theta i}+z+\alpha+\Re\bigl(\alpha e^{-\theta i}\bigr)e^{\theta i}+\frac{z+\alpha+\Re\bigl(\alpha e^{-\theta i}\bigr)e^{\theta i}}{\pi}\times\int_{-\Re(\alpha e^{-\theta i})}^{-\Re[(2\alpha-\beta)e^{-\theta i}]}\frac{\sqrt{\bigl[t+\Re\bigl(\alpha e^{-\theta i}\bigr)\bigr]\bigl(-\Re\bigl[(2\alpha-\beta)e^{-\theta i}\bigr]-t\bigr)}}{t}\frac{1}{t+(z+\alpha)e^{-\theta i}+\Re\bigl(\alpha e^{-\theta i}\bigr)}\,\mathrm{d}\,t \tag{3.5}$$

and

$$G_{\alpha,\beta}(z)=\sqrt{\Re\bigl(\alpha e^{-\theta i}\bigr)\Re\bigl[(2\alpha-\beta)e^{-\theta i}\bigr]}\,e^{\theta i}+z+\alpha+\Re\bigl(\alpha e^{-\theta i}\bigr)e^{\theta i}+\frac{e^{\theta i}}{\pi}\int_0^\infty\biggl[\int_{-\Re(\alpha e^{-\theta i})}^{-\Re[(2\alpha-\beta)e^{-\theta i}]}\sqrt{\bigl[t+\Re\bigl(\alpha e^{-\theta i}\bigr)\bigr]\bigl(-\Re\bigl[(2\alpha-\beta)e^{-\theta i}\bigr]-t\bigr)}\,e^{-tu}\,\mathrm{d}\,t\biggr]\times\Bigl(1-e^{-[(z+\alpha)e^{-\theta i}+\Re(\alpha e^{-\theta i})]u}\Bigr)\,\mathrm{d}\,u. \tag{3.6}$$

Proof. As did in the proof of Theorem 3.1, we can write the numbers $\alpha e^{-\theta i}$ and $\beta e^{-\theta i}$ as

$$\alpha e^{-\theta i}=L+Ni,\quad \beta e^{-\theta i}=M+Ni,$$

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where $L=\Re(\alpha e^{-\theta i})<0$, $M=\Re(\beta e^{-\theta i})=L+|\beta-\alpha|>0$, and $N=\Im(\alpha e^{-\theta i})=\Im(\beta e^{-\theta i})$. Therefore, it follows that

$$\begin{aligned}\sqrt{(z+\alpha)(z+\beta)}&=\sqrt{(z+\alpha)(z+\beta)e^{-2\theta i}}\,\sqrt{e^{2\theta i}}\\&=\sqrt{(ze^{-\theta i}+\alpha e^{-\theta i})(ze^{-\theta i}+\beta e^{-\theta i})}\,e^{\theta i}\\&=\sqrt{(ze^{-\theta i}+Ni+L)(ze^{-\theta i}+Ni+M)}\,e^{\theta i}\\&=\sqrt{[(ze^{-\theta i}+Ni+2L)-L][(ze^{-\theta i}+Ni+2L)+(M-2L)]}\,e^{\theta i},\end{aligned}$$

where $-L>0$ and $M-2L>0$. Accordingly, by (2.7) and (2.8), we can obtain

$$\begin{aligned}G_{\alpha,\beta}(z)&=\sqrt{L(2L-M)}\,e^{\theta i}+z+(Ni+2L)e^{\theta i}\\&\quad+\frac{z+(Ni+2L)e^{\theta i}}{\pi}\int_{-L}^{M-2L}\frac{\sqrt{(t+L)(M-2L-t)}}{t}\frac{1}{t+ze^{-\theta i}+Ni+2L}\,\mathrm{d}\,t\end{aligned}$$

and

$$\begin{aligned}G_{\alpha,\beta}(z)&=\sqrt{L(2L-M)}\,e^{\theta i}+z+(Ni+2L)e^{\theta i}\\&\quad+\frac{e^{\theta i}}{\pi}\int_0^\infty\biggl[\int_{-L}^{M-2L}\sqrt{(t+L)(M-2L-t)}\,e^{-tu}\,\mathrm{d}\,t\biggr]\Bigl[1-e^{-(ze^{-\theta i}+Ni+2L)u}\Bigr]\,\mathrm{d}\,u\end{aligned}$$

for

$$ze^{-\theta i}+Ni+2L\in\mathbb{C}\setminus[2L-M,L]\iff z\in\mathbb{C}\setminus[-\beta,-\alpha].$$

These can be rearranged as integral representations (3.5) and (3.6). The proof of Theorem 3.2 is complete. $\square$

Corollary 3.2. *Let $\alpha,\beta\in\mathbb{C}$, $\alpha\ne\beta$, and $\arg(\beta-\alpha)=\pm\frac{\pi}{2}$. If $\pm\Im(\alpha)<0<\pm\Im(\beta)$, then the principal branch of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z\in\mathbb{C}\setminus[-\beta,-\alpha]$ can be represented by*

$$\begin{aligned}G_{\alpha,\beta}(z)&=\pm i\sqrt{\Im(\alpha)\Im(2\alpha-\beta)}+z+\alpha+\Im(\alpha)i\\&\quad+\frac{z+\alpha+\Im(\alpha)i}{\pi}\int_{\mp\Im(\alpha)}^{\mp\Im(2\alpha-\beta)}\frac{\sqrt{[t\pm\Im(\alpha)][\mp\Im(2\alpha-\beta)-t]}}{t}\frac{1}{t\pm[\Im(\alpha)-(z+\alpha)i]}\,\mathrm{d}\,t\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}G_{\alpha,\beta}(z)&=\pm i\sqrt{\Im(\alpha)\Im(2\alpha-\beta)}+z+\alpha+\Im(\alpha)i\\&\quad\pm\frac{i}{\pi}\int_0^\infty\biggl[\int_{\mp\Im(\alpha)}^{\mp\Im(2\alpha-\beta)}\sqrt{[t\pm\Im(\alpha)][\mp\Im(2\alpha-\beta)-t]}\,e^{-tu}\,\mathrm{d}\,t\biggr]\Bigl(1-e^{\mp[\Im(\alpha)-(z+\alpha)i]u}\Bigr)\,\mathrm{d}\,u.\end{aligned}\tag{3.8}$$

Proof. Taking $\theta=\pm\frac{\pi}{2}$ in (3.5) and (3.6) respectively deduces the integral representations (3.7) and (3.8). $\square$

Theorem 3.3. *Let $\alpha,\beta\in\mathbb{C}$, $\alpha\ne\beta$, and $\theta=\arg(\beta-\alpha)\in(-\pi,\pi]$. If $\Re(\beta e^{-\theta i})<0$, then the principal branch of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z\in\mathbb{C}\setminus[-\beta,-\alpha]$ can be represented by*

$$G_{\alpha,\beta}(z)=\sqrt{\Re(\alpha e^{-\theta i})\Re(\beta e^{-\theta i})}\,e^{\theta i}-z-\Im(\alpha e^{-\theta i})ie^{\theta i}-\frac{z+\Im(\alpha e^{-\theta i})ie^{\theta i}}{\pi}$$

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$$\times\int_{-\Re(\beta e^{-\theta i})}^{-\Re(\alpha e^{-\theta i})}\frac{\sqrt{\left[t+\Re\left(\beta e^{-\theta i}\right)\right]\left[-\Re\left(\alpha e^{-\theta i}\right)-t\right]}}{t}\frac{1}{t-\left[ze^{-\theta i}+\Im\left(\alpha e^{-\theta i}\right)i\right]}\,\mathrm{d}\,t \tag{3.9}$$

and

$$G_{\alpha,\beta}(z)=\sqrt{\Re\left(\alpha e^{-\theta i}\right)\Re\left(\beta e^{-\theta i}\right)}\,e^{\theta i}-z-\Im\left(\alpha e^{-\theta i}\right)ie^{\theta i}$$
$$+\frac{e^{\theta i}}{\pi}\int_0^\infty\left(\int_{-\Re(\beta e^{-\theta i})}^{-\Re(\alpha e^{-\theta i})}\sqrt{\left[t+\Re\left(\beta e^{-\theta i}\right)\right]\left[-\Re\left(\alpha e^{-\theta i}\right)-t\right]}\,e^{-tu}\,\mathrm{d}\,t\right)\left(1-e^{[ze^{-\theta i}+\Im(\alpha e^{-\theta i})i]u}\right)\mathrm{d}\,u. \tag{3.10}$$

Proof. As did in proofs of Theorems 3.1 and 3.2, we can write the numbers $\alpha e^{-\theta i}$ and $\beta e^{-\theta i}$ as

$$\alpha e^{-\theta i}=L+Ni,\quad \beta e^{-\theta i}=M+Ni,$$

where $L=\Re\left(\alpha e^{-\theta i}\right)<M=\Re\left(\beta e^{-\theta i}\right)=L+|\beta-\alpha|<0$ and $N=\Im\left(\alpha e^{-\theta i}\right)=\Im\left(\beta e^{-\theta i}\right)$. Therefore, it follows that

$$\begin{aligned}\sqrt{(z+\alpha)(z+\beta)}&=\sqrt{(z+\alpha)(z+\beta)e^{-2\theta i}}\,\sqrt{e^{2\theta i}}\\&=\sqrt{(ze^{-\theta i}+\alpha e^{-\theta i})(ze^{-\theta i}+\beta e^{-\theta i})}\,e^{\theta i}\\&=\sqrt{(ze^{-\theta i}+Ni+L)(ze^{-\theta i}+Ni+M)}\,e^{\theta i}\\&=\sqrt{[-(ze^{-\theta i}+Ni)-L][-(ze^{-\theta i}+Ni)-M]}\,e^{\theta i},\end{aligned}$$

where $-L>-M>0$. Accordingly, by (2.7) and (2.8), we can obtain

$$G_{\alpha,\beta}(z)=\sqrt{LM}\,e^{\theta i}-z-Nie^{\theta i}-\frac{z+Nie^{\theta i}}{\pi}\int_{-M}^{-L}\frac{\sqrt{(t+M)(-L-t)}}{t}\frac{1}{t-(ze^{-\theta i}+Ni)}\,\mathrm{d}\,t$$

and

$$G_{\alpha,\beta}(z)=\sqrt{LM}\,e^{\theta i}-z-Nie^{\theta i}+\frac{e^{\theta i}}{\pi}\int_0^\infty\left[\int_{-M}^{-L}\sqrt{(t+M)(-L-t)}\,e^{-tu}\,\mathrm{d}\,t\right]\left[1-e^{(ze^{-\theta i}+Ni)u}\right]\mathrm{d}\,u$$

for

$$-(ze^{-\theta i}+Ni)\in\mathbb{C}\setminus[L,M]\iff z\in\mathbb{C}\setminus[-\beta,-\alpha].$$

These can be rearranged as integral representations (3.9) and (3.10). The proof of Theorem 3.3 is complete. ☐

Corollary 3.3. *Let $\alpha,\beta\in\mathbb{C}$, $\alpha\ne\beta$, and $\arg(\beta-\alpha)=\pm\frac{\pi}{2}$. If $\pm\Im(\beta)<0$, then the principal branch of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z\in\mathbb{C}\setminus[-\beta,-\alpha]$ can be represented by*

$$\begin{aligned}G_{\alpha,\beta}(z)&=\pm i\sqrt{\Im(\alpha)\Im(\beta)}-z-\Re(\alpha)\\&\quad-\frac{z+\Re(\alpha)}{\pi}\int_{\mp\Im(\beta)}^{\mp\Im(\alpha)}\frac{\sqrt{[t\pm\Im(\beta)][\mp\Im(\alpha)-t]}}{t}\frac{1}{t+[z+\Re(\alpha)]i}\,\mathrm{d}\,t\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}G_{\alpha,\beta}(z)&=\sqrt{\Im(\alpha)\Im(\beta)}\,i-z-\Re(\alpha)\\&\quad+\frac{i}{\pi}\int_0^\infty\left(\int_{\mp\Im(\beta)}^{\mp\Im(\alpha)}\sqrt{[t\pm\Im(\beta)][\mp\Im(\alpha)-t]}\,e^{-tu}\,\mathrm{d}\,t\right)\left(1-e^{-i[z+\Re(\alpha)]u}\right)\mathrm{d}\,u.\end{aligned} \tag{3.12}$$

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Proof. Taking $\theta = \pm\frac{\pi}{2}$ in (3.9) and (3.10) respectively derives the integral representations (3.11) and (3.12). $\square$

Theorem 3.4. *Let $\alpha, \beta \in \mathbb{C}$, $\alpha \ne \beta$, and $\theta = \arg(\beta-\alpha) \in (-\pi, \pi]$. Then the principal branch of the reciprocal $\frac{1}{G_{\alpha,\beta}(z)}$ of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z \in \mathbb{C} \setminus [-\beta, -\alpha]$ has the following integral representations:*

(1) *if* $\Re(\alpha e^{-\theta i}) > 0$,

$$\frac{1}{G_{\alpha,\beta}(z)} = \frac{1}{\pi}\int_{\Re(\alpha e^{-\theta i})}^{\Re(\beta e^{-\theta i})} \frac{1}{\sqrt{[t-\Re(\alpha e^{-\theta i})][\Re(\beta e^{-\theta i})-t]}} \frac{1}{t+ze^{-\theta i}+i\Im(\alpha e^{-\theta i})} \,\mathrm{d}\,t;$$

(2) *if* $\Re(\beta e^{-\theta i}) > 0 > \Re(\alpha e^{-\theta i})$,

$$\frac{1}{G_{\alpha,\beta}(z)} = \\ \frac{1}{\pi}\int_{-\Re(\alpha e^{-\theta i})}^{-\Re[(2\alpha-\beta)e^{-\theta i}]} \frac{1}{\sqrt{[t+\Re(\alpha e^{-\theta i})](-\Re[(2\alpha-\beta)e^{-\theta i}]-t)}} \frac{1}{t+(z+\alpha)e^{-\theta i}+\Re(\alpha e^{-\theta i})} \,\mathrm{d}\,t;$$

(3) *if* $0 > \Re(\beta e^{-\theta i})$,

$$\frac{1}{G_{\alpha,\beta}(z)} = -\frac{1}{\pi}\int_{-\Re(\beta e^{-\theta i})}^{-\Re(\alpha e^{-\theta i})} \frac{1}{\sqrt{[t+\Re(\beta e^{-\theta i})][-\Re(\alpha e^{-\theta i})-t]}} \frac{1}{t-[ze^{-\theta i}+\Im(\alpha e^{-\theta i})i]} \,\mathrm{d}\,t.$$

Proof. This follows from combining the integral representation (2.9) and with proofs of Theorems 3.1 to 3.3. $\square$

Corollary 3.4. *Let $\alpha, \beta \in \mathbb{C}$, $\alpha \ne \beta$, and $\arg(\beta-\alpha) = \pm\frac{\pi}{2}$. Then the principal branch of the reciprocal $\frac{1}{G_{\alpha,\beta}(z)}$ of the bivariate complex geometric mean $G_{\alpha,\beta}(z)$ for $z \in \mathbb{C} \setminus [-\beta, -\alpha]$ has the following integral representations:*

(1) *if* $\pm\Im(\alpha) > 0$,

$$\frac{1}{G_{\alpha,\beta}(z)} = \frac{1}{\pi}\int_{\pm\Im(\alpha)}^{\pm\Im(\beta)} \frac{1}{\sqrt{[t\mp\Im(\alpha)][\Im(\beta)\mp t]}} \frac{1}{t\mp[z+\Re(\alpha)]i} \,\mathrm{d}\,t;$$

(2) *if* $\pm\Im(\alpha) < 0 < \pm\Im(\beta)$,

$$\frac{1}{G_{\alpha,\beta}(z)} = \frac{1}{\pi}\int_{\mp\Im(\alpha)}^{\mp\Im(2\alpha-\beta)} \frac{1}{\sqrt{[t\pm\Im(\alpha)][\mp\Im(2\alpha-\beta)-t]}} \frac{1}{t\pm[\Im(\alpha)-(z+\alpha)i]} \,\mathrm{d}\,t; \tag{3.13}$$

(3) *if* $\pm\Im(\beta) < 0$,

$$\frac{1}{G_{\alpha,\beta}(z)} = -\frac{1}{\pi}\int_{\mp\Im(\beta)}^{\mp\Im(\alpha)} \frac{1}{\sqrt{[t\pm\Im(\beta)][\mp\Im(\alpha)-t]}} \frac{1}{t+[z+\Re(\alpha)]i} \,\mathrm{d}\,t.$$

Proof. This follows from taking $\theta = \pm\frac{\pi}{2}$ in Theorem 3.4. $\square$

4. Applications to Heronian mean of power 2 and its reciprocal

In this section, we would like to apply Corollaries 3.2 and 3.4 to establish integral representations for Heronian mean $E(1,3;a,b) = h_2(a,b)$ of power 2 and its reciprocal.

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Theorem 4.1. *For $b>a>0$, the Heronian mean $\sqrt{\frac{a^2+ab+b^2}{3}}$ can be represented by*

$$\begin{aligned}\sqrt{\frac{a^2+ab+b^2}{3}}&=\frac{a+b}{2}+\left(\frac{1}{2}-\frac{b-a}{\sqrt{3}}\right)i\\&\quad+\frac{1}{\pi}\left(\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i\right)\int_{\frac{b-a}{2\sqrt{3}}}^{\frac{\sqrt{3}\,(b-a)}{2}}\frac{\sqrt{\left(t-\frac{b-a}{2\sqrt{3}}\right)\left[\frac{\sqrt{3}\,(b-a)}{2}-t\right]}}{t}\frac{1}{t-\left(\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i\right)i}\,\mathrm{d}\,t,\end{aligned}\tag{4.1}$$

$$\begin{aligned}\sqrt{\frac{a^2+ab+b^2}{3}}&=\frac{2a+b}{2\sqrt{3}}+\left(\frac{\sqrt{3}}{2}-1\right)bi\\&\quad+\frac{1}{\pi}\left(\frac{2a+b}{2\sqrt{3}}-bi\right)\int_{b/2}^{3b/2}\frac{\sqrt{\left(t-\frac{b}{2}\right)\left(\frac{3}{2}b-t\right)}}{t}\frac{1}{t-\left(\frac{2a+b}{2\sqrt{3}}-bi\right)i}\,\mathrm{d}\,t,\end{aligned}\tag{4.2}$$

$$\begin{aligned}\sqrt{\frac{a^2+ab+b^2}{3}}&=\frac{a+b}{2}+\left(\frac{1}{2}-\frac{b-a}{\sqrt{3}}\right)i\\&\quad+\frac{i}{\pi}\int_0^\infty\left[\int_{\frac{b-a}{2\sqrt{3}}}^{\frac{\sqrt{3}\,(b-a)}{2}}\sqrt{\left(t-\frac{b-a}{2\sqrt{3}}\right)\left[\frac{\sqrt{3}\,(b-a)}{2}-t\right]}\,e^{-tu}\,\mathrm{d}\,t\right]\left[1-e^{i\left(\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i\right)u}\right]\mathrm{d}\,u,\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\sqrt{\frac{a^2+ab+b^2}{3}}&=\frac{2a+b}{2\sqrt{3}}+\left(\frac{\sqrt{3}}{2}-1\right)bi\\&\quad+\frac{i}{\pi}\int_0^\infty\left[\int_{b/2}^{3b/2}\sqrt{\left(t-\frac{b}{2}\right)\left(\frac{3}{2}b-t\right)}\,e^{-tu}\,\mathrm{d}\,t\right]\left[1-e^{i\left(\frac{2a+b}{2\sqrt{3}}-bi\right)u}\right]\mathrm{d}\,u.\end{aligned}\tag{4.4}$$

Proof. Applying the integral representations (3.7) and (3.8) to the mean in (2.25) yields

$$\begin{aligned}\sqrt{\left(z+\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}i\right)\left(z+\frac{a+b}{2}+\frac{b-a}{2\sqrt{3}}i\right)}&=z+\frac{a+b}{2}+\left(\frac{1}{2}-\frac{b-a}{\sqrt{3}}\right)i\\+\frac{z+\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i}{\pi}\int_{\frac{b-a}{2\sqrt{3}}}^{\frac{\sqrt{3}\,(b-a)}{2}}&\frac{\sqrt{\left(t-\frac{b-a}{2\sqrt{3}}\right)\left[\frac{\sqrt{3}\,(b-a)}{2}-t\right]}}{t}\frac{1}{t-\left(z+\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i\right)i}\,\mathrm{d}\,t\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}\sqrt{\left(z+\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}i\right)\left(z+\frac{a+b}{2}+\frac{b-a}{2\sqrt{3}}i\right)}&=z+\frac{a+b}{2}+\left(\frac{1}{2}-\frac{b-a}{\sqrt{3}}\right)i\\+\frac{i}{\pi}\int_0^\infty\left[\int_{\frac{b-a}{2\sqrt{3}}}^{\frac{\sqrt{3}\,(b-a)}{2}}\sqrt{\left(t-\frac{b-a}{2\sqrt{3}}\right)\left[\frac{\sqrt{3}\,(b-a)}{2}-t\right]}\,e^{-tu}\,\mathrm{d}\,t\right]&\left[1-e^{i\left(z+\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i\right)u}\right]\mathrm{d}\,u.\end{aligned}\tag{4.6}$$

Further taking $z\to0$ arrives at (4.1) and (4.3).

Similarly, applying the integral representations (3.7) and (3.8) to the mean in (2.26) yields

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$$\begin{aligned}\sqrt{\biggl(z+\frac{2a+b}{2\sqrt{3}}-\frac{b}{2}i\biggr)\biggl(z+\frac{2a+b}{2\sqrt{3}}+\frac{b}{2}i\biggr)}&=z+\frac{2a+b}{2\sqrt{3}}+\biggl(\frac{\sqrt{3}}{2}-1\biggr)bi\\&\quad+\frac{z+\frac{2a+b}{2\sqrt{3}}-bi}{\pi}\int_{b/2}^{3b/2}\frac{\sqrt{\bigl(t-\frac{b}{2}\bigr)\bigl(\frac{3}{2}b-t\bigr)}}{t}\frac{1}{t-\Bigl(z+\frac{2a+b}{2\sqrt{3}}-bi\Bigr)i}\,\mathrm{d}\,t \end{aligned}\tag{4.7}$$

and

$$\begin{aligned}\sqrt{\biggl(z+\frac{2a+b}{2\sqrt{3}}-\frac{b}{2}i\biggr)\biggl(z+\frac{2a+b}{2\sqrt{3}}+\frac{b}{2}i\biggr)}&=z+\frac{2a+b}{2\sqrt{3}}+\biggl(\frac{\sqrt{3}}{2}-1\biggr)bi\\&\quad+\frac{i}{\pi}\int_0^\infty\biggl[\int_{b/2}^{3b/2}\sqrt{\biggl(t-\frac{b}{2}\biggr)\biggl(\frac{3}{2}b-t\biggr)}\,e^{-tu}\,\mathrm{d}\,t\biggr]\Bigl[1-e^{i\bigl(z+\frac{2a+b}{2\sqrt{3}}-bi\bigr)u}\Bigr]\,\mathrm{d}\,u. \end{aligned}\tag{4.8}$$

Further taking $z\to0$ arrives at (4.2) and (4.4). The proof of Theorem 4.1 is complete. $\square$

Theorem 4.2. *For $b>a>0$, the reciprocal of Heronian mean $\sqrt{\frac{a^2+ab+b^2}{3}}$ can be represented by*

$$\frac{1}{\sqrt{a^2+ab+b^2}}=\frac{1}{\sqrt{3}\,\pi}\int_{\frac{b-a}{2\sqrt{3}}}^{\frac{\sqrt{3}\,(b-a)}{2}}\frac{1}{\sqrt{\Bigl(t-\frac{b-a}{2\sqrt{3}}\Bigr)\Bigl[\frac{\sqrt{3}\,(b-a)}{2}-t\Bigr]}}\frac{1}{t-\Bigl(\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i\Bigr)i}\,\mathrm{d}\,t \tag{4.9}$$

and

$$\frac{1}{\sqrt{a^2+ab+b^2}}=\frac{1}{\sqrt{3}\,\pi}\int_{b/2}^{3b/2}\frac{1}{\sqrt{\bigl(t-\frac{b}{2}\bigr)\bigl(\frac{3}{2}b-t\bigr)}}\frac{1}{t-\Bigl(\frac{2a+b}{2\sqrt{3}}-bi\Bigr)i}\,\mathrm{d}\,t. \tag{4.10}$$

Proof. Applying the integral representation (3.13) to the reciprocal of the mean in (2.25) yields

$$\begin{aligned}&\frac{1}{\sqrt{\Bigl(z+\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}i\Bigr)\Bigl(z+\frac{a+b}{2}+\frac{b-a}{2\sqrt{3}}i\Bigr)}}\\&\quad=\frac{1}{\pi}\int_{\frac{b-a}{2\sqrt{3}}}^{\frac{\sqrt{3}\,(b-a)}{2}}\frac{1}{\sqrt{\Bigl(t-\frac{b-a}{2\sqrt{3}}\Bigr)\Bigl[\frac{\sqrt{3}\,(b-a)}{2}-t\Bigr]}}\frac{1}{t-\Bigl(z+\frac{a+b}{2}-\frac{b-a}{\sqrt{3}}i\Bigr)i}\,\mathrm{d}\,t. \end{aligned}\tag{4.11}$$

Further taking $z\to0$ arrives at (4.9).

Similarly, applying the integral representation (3.13) to the reciprocal of the mean in (2.26) yields

$$\begin{aligned}&\frac{1}{\sqrt{\Bigl(z+\frac{2a+b}{2\sqrt{3}}-\frac{b}{2}i\Bigr)\Bigl(z+\frac{2a+b}{2\sqrt{3}}+\frac{b}{2}i\Bigr)}}\\&\quad=\frac{1}{\pi}\int_{b/2}^{3b/2}\frac{1}{\sqrt{\bigl(t-\frac{b}{2}\bigr)\bigl(\frac{3}{2}b-t\bigr)}}\frac{1}{t-\Bigl(z+\frac{2a+b}{2\sqrt{3}}-bi\Bigr)i}\,\mathrm{d}\,t. \end{aligned}\tag{4.12}$$

Further taking $z\to0$ arrives at (4.10). The proof of Theorem 4.2 is complete. $\square$

5. Remarks

Finally we would like to list several remarks on related results.

Remark 5.1. For complex numbers z_j with $|z_j-1|<r<1$. In [12], the complex arithmetic mean $A_n(\boldsymbol{z})=\frac{1}{n}\sum_{j=1}^n z_j$ and the bivariate complex geometric mean $G_n(\boldsymbol{z})=\sqrt[n]{\prod_{j=1}^n z_j}$ were considered and they were proved to satisfy the inequalities

$$\frac{1-r^2}{|2-A_n(\boldsymbol{z})|}\le|G_n(\boldsymbol{z})|\le\frac{|A_n(\boldsymbol{z})|}{\sqrt{1-r^2}},$$

$$(1-r)^{(r+|1-A_n(\boldsymbol{z})|)/2r}(1+r)^{(r-|1-A_n(\boldsymbol{z})|)/2r}\le|G_n(\boldsymbol{z})|\le\sqrt{1+2|1-A_n(\boldsymbol{z})|+r^2}\,,$$

and

$$|A_n(\boldsymbol{z})|^{1/n}(1-r)^{1-1/n}\le|G_n(\boldsymbol{z})|\le|A_n(\boldsymbol{z})|^{1/n}(1+r)^{1-1/n}.$$

Let $\phi\in\left(0,\frac{\pi}{2}\right)$ and $W_\phi=\{z\in\mathbb{C}:|\arg(z)|\le\phi\}$. In [1, Theorem 2.2], it was obtained that, for all $z_1,z_2,\dots,z_n\in W_\phi$, the inequality

$$\prod_{k=1}^n|z_k|^{1/n}\le\sigma\left(\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor,\cos(2\phi)\right)\left|\frac{1}{n}\sum_{k=1}^n z_k\right|$$

is valid and the constant factor is best possible for every n and ϕ, where $\lfloor x\rfloor$ denotes the floor function whose value is the largest integer less than or equal to x and

$$\sigma(\mu,a)=\frac{\sqrt{2}\left[a(2\mu-1)+\sqrt{4\mu(1-\mu)+a^2(1-2\mu)^2}\,\right]^\mu}{(2\mu)^\mu\sqrt{2(1-\mu)+a^2(2\mu-1)+a\sqrt{4\mu(1-\mu)+a^2(1-2\mu)^2}}}.$$

For more information, please refer to the papers [1, 12] and the closely related references therein.

Remark 5.2. As did in [38], differentiating with respect to z on both sides of integral representations in Theorems 3.1 to 3.4, Corollaries 3.1 to 3.4, and integral representations (4.5) to (4.8), (4.11), and (4.12) can find more integral formulas or integral representations related to bivariate means.

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