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Article

Lempel-Ziv Complexity of Photonic Quasicrystals

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Abstract: The properties of photonic quasicrystals ultimate rely on their inherent long-range order, a hallmark that can be quantified in many ways depending on the specific aspects to be studied. We use the Lempel-Ziv measure, a basic tool for information theoretic problems, to characterize the complexity of the specific structure under consideration. Using the generalized Fibonacci quasicrystals as our thread, we adress the relation between the optical response and the associated complexity.

Keywords: quasicrystals; photonic crystals; photonic bandgap materials

1. Introduction

The spatial modulation of the properties of a medium enforces drastic alterations in light propagation. There are two extreme instances of this: when the spatial profile is periodic and when it is fully random. In the first case, the structure is called a photonic crystal [1,2], underlining the strong similarities between the distinctive features of light in these structures and those of electrons in solids. Bragg mirrors, consisting of alternating low- and high-index layers, constitute the simplest example of one-dimensional photonic crystals [3,4]: the name stems from the presence of photonic band gaps; i.e., ranges of frequency in which strong reflection occurs for all angles of incidence and all polarizations [5–9].

The converse case of random media has found a limited number of applications thus far. Nonetheless, it also deserves attention because it lies in the realm of some intriguing fundamental effects, such as Anderson localization [10,11], coherent backscattering [12], and optical Hall effect [13].

Deterministic aperiodic spatial patterns, dubbed photonic quasicrystals [14–21], display stunning optical responses that cannot be found in either periodic or random systems. Among other issues, they include a self-similar energy spectrum [22,23], a pseudo-bandgap of forbidden frequencies [24–26], and critically localized states [27,28] whose wave functions are distinguished by power law asymptotes and self-similarity [29–31]. All these features are of utmost significance, because they can provide remarkable functionalities [32,33].

The first illustration of an aperiodic lattice possessing long-range order was a one-dimensional structure assembled according to the Fibonacci sequence [34–36]. Subsequently, a wealth of other photonic quasicrystals have been conceived, the most outstanding ones being Thue-Morse [37–39] and Cantor [40,41].

In this work, we examine the so-called generalized Fibonacci quasicrystals. This has developed into a cool topic [42–56], which besides is underpinned by remarkable mathematical properties [57]. This confirms that the deterministic aperiodic nanostructures is a highly interdisciplinary and fascinating research field, conceptually rooted in several branches of mathematics.

To characterize the system, we use here the Lempel-Ziv measure [58]. This is one amongst many measures of complexity available [59]: it is related to the number of distinct patterns and the rate of their occurrence along a given structure. In recent years this metric has been extensively used in several areas, because it is relatively easy to compute and gives results of very direct interpretation [60].

The Lempel-Ziv complexity allows us to provide a comprehensive understanding of the relation between structural and spectral properties that governs the optical behavior of one-dimensional quasicrystals with controllable degree of aperiodic order.

2. Generalized Fibonacci quasicrystals

Let us first review some basic notions that are important in the discussion that follows. A word (also called a sequence) is an ordered list of letters, which are elements of a finite alphabet. We shall be concerned with a two-letter alphabet, denoted by $\{L,H\}$, but alphabets can be of any size. In physical realizations, each letter corresponds to a different type of building block (e.g., dielectric layers, atoms, etc).

A straightforward way to generate deterministic aperiodic words is using symbolic substitutions [61,62]. A specific substitution rule replaces each letter in the alphabet by a finite word:

$$L \mapsto \varphi_1(L,H)\,, \qquad H \mapsto \varphi_2(L,H)\,, \tag{1}$$

φ_1 and φ_2 being any string of L and H. One must start from a given letter, which is called a seed.

More specifically, we are interested in the generalized Fibonacci sequences $\mathrm{FS}(h,\ell)$, which are generated by the inflation rule

$$L \mapsto H\,, \qquad H \mapsto H^h L^\ell\,, \tag{2}$$

where h and ℓ are arbitrary positive integers and we adopt the convention that the seed is L.

The words $\{W_\alpha\}$ of $\mathrm{FS}(h,\ell)$ can be alternatively defined by the recursive scheme

$$W_{\alpha+1} = W_\alpha^h \, W_{\alpha-1}^\ell\,, \tag{3}$$

with $W_0 = L$ and $W_1 = H$. Here, the integer α labels the corresponding iteration, which is also known as the generation.

The length (i.e., the total number of letters L and H) of the word W_α is denoted by w_α and satisfies the relation

$$w_{\alpha+1} = h w_\alpha + \ell w_{\alpha-1}\,. \tag{4}$$

In the limit of an infinite sequence, we have that

$$\lim_{\alpha\to\infty} \frac{w_\alpha}{w_{\alpha-1}} \equiv \sigma(h,\ell) = \frac{1}{2}\left(h + \sqrt{h^2+4\ell}\right). \tag{5}$$

For $\ell = 1$, the resulting sequences fulfill

$$\sigma(h,1) = h + \cfrac{h}{h + \cfrac{h}{h + \ddots}} \tag{6}$$

where we denote the continued-fraction in this equation as $[\bar{h}] = [h,h,h,\ldots]$. In this way, we get

$$\sigma(1,1) = \frac{1}{2}(1+\sqrt{5}) \equiv \Phi = [\bar{1}]\,, \qquad \sigma(2,1) = 1+\sqrt{2} = [\bar{2}]\,, \qquad \sigma(3,1) = \frac{1}{2}(3+\sqrt{13}) = [\bar{3}]\,. \tag{7}$$

For $\sigma(1,1)$ (i.e., the standard Fibonacci sequence), we get the golden mean, whereas $\sigma(1,2)$ gives the silver mean and $\sigma(3,1)$ the bronze mean. This family generalizes in quite a natural way the golden ratio [63] and will be designated here, by obvious reasons, the Olympic-metal family. Among all these metallic means, the slowest converging one is the golden mean, since its denominators are the smallest possible. This corresponds to the traditional statement that the golden mean Φ is the most irrational of all irrational numbers [57].

On the other hand, when we fix $h = 1$, the sequences satisfy

$$\sigma(1,2) = [2,\bar{0}]\,, \qquad \sigma(1,3) = [2,\bar{3}]\,, \tag{8}$$

and the like. These two examples are known as the copper and nickel means and the complete FS(1, h) series will be termed as the non-Olympic-metal family.

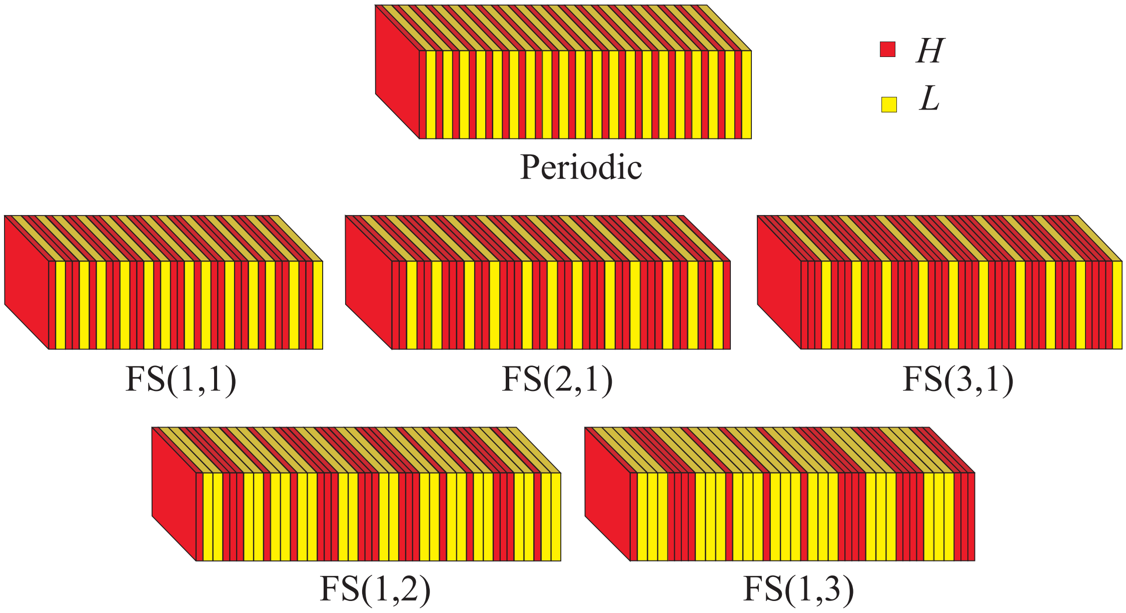

Figure 1. Illustrating different quasiperiodic chains considered in this work. In the top panel, we have the periodic case (40 letters). In the mid panel, the Olympic-metal family FS (h,1), with (from left to right) $h = 1$ (golden mean, 34 letters), $h = 2$ (silver mean, 41 letters), and $h = 3$ (bronze mean, 43 letters). In the bottom panel, the non-Olympic-metal family FS (1, ℓ), with $\ell = 2$ (copper mean, 43 letters) and $\ell = 3$ (nickel mean, 40 letters).

In an optical implementation of the generalized Fibonacci sequences FS (h,ℓ), the letters in the alphabet $\{L,H\}$ are realized as layers made of materials with refractive indices (n_L,n_H) and thicknesses (d_L,d_H), respectively. The material L has a low refractive index, while H is of a high refractive index, which justifies the notation employed here. An illustration of the resulting systems is presented in Fig. 1.

To compute the optical response of these structures we will use the transfer-matrix technique [64]. The αth word of FS(h,ℓ) has the associated transfer matrix

$$\mathsf{M}_{\alpha+1} = \mathsf{M}_{\alpha}^{h}\,\mathsf{M}_{\alpha-1}^{\ell}\,, \tag{9}$$

starting from $\mathsf{M}_0 = \mathsf{M}_L$ and $\mathsf{M}_1 = \mathsf{M}_H$, which are the basic matrices for each layer. Once M_α is known, the transmittance

$$\mathscr{T}_\alpha = \frac{4}{|\mathsf{M}_\alpha|^2+2}\,, \tag{10}$$

where $|\mathsf{M}_\alpha|^2 = \sum_{ij}|m_{ij}|^2$ (the sum of the absolute squares of the matrix elements) is the (Frobenius) norm of M_α. Note that $\mathscr{T}_\alpha$ will depend on the wavelength, incidence angle and polarization of the incident radiation.

3. Spectral measures of generalized Fibonacci quasicrystals

To each rule (1) we associate a substitution matrix T, defined as

$$\mathsf{T} = \begin{pmatrix} |\varphi_1(L,H)|_L & |\varphi_2(L,H)|_L \\ |\varphi_1(L,H)|_H & |\varphi_2(L,H)|_H \end{pmatrix}, \tag{11}$$

where $|\cdot|_{L,H}$ is the number of letters L (resp. H). This matrix does not depend on the precise form of the substitutions, only on the number of letters L or H.

The eigenvalues of T contain a lot of information. Actually, as discovered by Bombieri and Taylor [65,66], if the spectrum of T contains a Pisot number as an eigenvalue, the sequence is quasiperiodic; otherwise it is not

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(and then is purely aperiodic). We recall that a Pisot number is a positive algebraic number (i.e., a number that is a solution of an algebraic equation) greater than one, all of whose conjugate elements (the other solutions of the defining algebraic equation) have modulus less than unity [67].

For the family FS(h,ℓ), we have

$$\mathsf{T}_{h,\ell} = \begin{pmatrix} h & 1 \\ \ell & 0 \end{pmatrix}, \tag{12}$$

whose eigenvalues are

$$\tau_{h,\ell}^{(\pm)} = \frac{1}{2}\left(h \pm \sqrt{h^2+4\ell}\right). \tag{13}$$

Incidentally, the largest eigenvalue $\tau_{h,\ell}^{(+)}$, which is often known as the Perron-Frobenius eigenvalue [68], coincides with the ratio $\sigma(h,\ell)$.

The eigenvalues $\tau_{h,1}$ are Pisot numbers, so all the sequences in the Olympic-metal family FS$(h,1)$ are quasiperiodic. In contradistinction, $\tau_{1,\ell}$ are not Pisot numbers and the corresponding non-Olympic-metal systems FS $(1,\ell)$, are aperiodic.

The main differences of these two situations can be appreciated by the nature of their Fourier spectrum [69]. For a specific word of length n, the discrete Fourier transform reads

$$\hat{W}_n(k) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} W(n) \exp\left(\frac{-2\pi i jk}{n}\right), \qquad k = 1, 2 \ldots, n, \tag{14}$$

where $W(j)$ is a numerical array obtained from the word by assigning to each letter of the alphabet a fixed number. This assignment is otherwise arbitrary and does not change any conclusion. In consequence, one could, e.g., use $L \mapsto -1$ and $H \mapsto 1$. The structure factor (or power spectrum) is [70]

$$F_n(k) = |\hat{W}_n(k)|^2. \tag{15}$$

From a rigorous perspective, the only well-established concept attached to the Fourier spectrum is its spectral measure. If $d\nu_n(k) = F_n(k)\,dk$, we will be concerned with the nature of the limit $d\nu(k) = \lim_{n\to\infty} d\nu_n(k)$, which corresponds to an infinite structure and a continuous variable k. Just as any positive measure, $d\nu(k)$ has a unique decomposition

$$d\nu(k) = d\nu_{\mathrm{pp}}(k) + d\nu_{\mathrm{ac}}(k) + d\nu_{\mathrm{sc}}(k) \tag{16}$$

into its pure point, absolutely continuous and singular continuous parts [33]. The pure point part refers to the presence of Bragg peaks; the absolute continuous part is a differentiable function (diffuse scattering), while the singular continuous part it is neither continuous nor does it have Bragg peaks; it shows broad peaks, which are never isolated and, with increasing resolution, split again into further broad.

For example, in random media in the diffusive regime, uncorrelated disorder gives rise to an absolutely continuous diffraction measure. In contradistinction, in the regime of Anderson localization, where exponentially localized states occur at discrete resonant frequencies, the energy spectrum is pure-point. On the other hand, periodic structures exhibit sharp diffraction Bragg peaks (i.e., a pure-point measure) due to their long-range periodic order, but they may also support continuous energy bands.

Interestingly enough, more complex structures exist that display singular continuous diffraction spectra: the primary example of such structures can be deterministically generated according to the aperiodic Thue-Morse sequence [39].

In Fig. 2 we plot the power spectrum $F_n(k)$ for the Olympic- and non-Olympic-metal families. The former, exhibit δ-like Bragg peaks that can be properly labeled in terms of the eigenvalues $\tau_{h,1}$ as

$$k_{m_1 m_2} = \frac{2\pi}{\Lambda_0} m_1 \tau_{h,1}^{m_2}, \tag{17}$$

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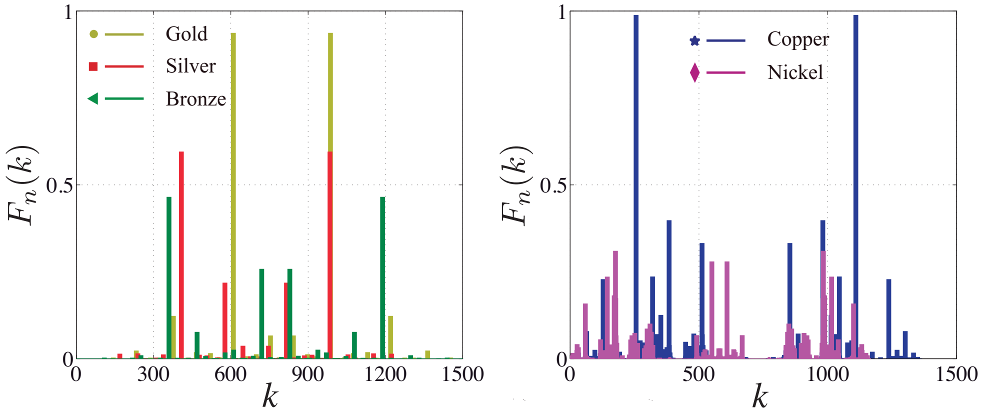

Figure 2. Normalized power spectrum for words up to 1500 letters for different generalized Fibonacci sequences. In the left panel, for the Olympic-metal family FS(h,1), with $h = 1, 2$, and 3. In the right panel, for the non-Olympic-metal sequences FS(1,2) and FS(1,3).

with m_1 and m_2 integers and Λ_0 being a suitable average period of the structure. We can verify the existence of incommensurate intervals between peaks, confirming the quasiperiodicity of these arrangements. Moreover, a relevant result, known as the gap-labelling theorem [71], relates the position of the peaks in Eq. (17) with the location of the gaps in the energy spectra of the elementary excitations supported by the structure.

For non-Olympic metals, the global structure looks blurred. Individual Bragg peaks are not separated by well-defined intervals, but tend to cluster forming "broad bands". The strength of the dominant peaks is considerably bigger for the copper, which suggests that the nickel-mean lattice is more disordered than the copper one. A complete account of these issues is outside the scope of this work; however, a thorough analysis [45] shows that these spectra are multifractal and their Fourier-spectral measures are singular continuous ones.

4. Lempel-Ziv complexity

The information encoded in the Fourier power spectrum, although complete, cannot be directly encapsulated in a single number reflecting the physical behavior of the system. To bypass this drawback we propose an alternative measure of the long-range order in terms of the closely related concept of complexity [72].

Of course, any definition of complexity is beholden to the perspective brought to bear upon it. In a broad sense, complexity lies in the difficulty faced in describing system characteristics. For example, the two cases discussed in the Introduction, namely the periodic and the random spatial profiles, are extreme examples of simple models and therefore systems with zero complexity.

A perfect crystal is completely ordered and the components are arranged following stringent rules of symmetry. A small piece of information is enough to describe the perfect crystal and the information stored in this system can be considered minimal. On the other hand, the random profile is completely disordered: the system can be found in any of its accessible states with the same probability and it has therefore a maximum information. These two simple systems are extrema in the scale of order and information. We thus conclude that a sensible definition of complexity must not be made in terms of just order or information.

The Lempel-Ziv (LZ) complexity [58] and its variants are popular measures for characterizing the randomness of a sequence. Indeed, in a way, it gives a clear indication of how hard is it to create the sequence, so it is a sensible magnitude from the experimental viewpoint. For a word $W_n = \{x_1 x_2 \ldots x_n\}$ of length n ($x_i \in \{L, H\}, 1 \leq i \leq n$), a procedure that partitions W_n into non-overlapping blocks is called a parsing. A block starting at position i and ending at position j of W_n is often called a phrase $W_n(i, j)$. The set of phrases generated by a parsing of W_n is denoted by PW_n and the number of phrases by $c(W_n)$.

Assume that a word W_n has been parsed up to position i, so that $PW_n(1, i)$ is the set of phrases generated so far. According to the original LZ procedure, the next phrase $W_n(i+1, j)$ will be the first block which is

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not yet an element of $PW_n(1,i)$. As an illustration, the string $LLHHLLHLHLHLLHHH$ will be parsed as $L|LH|HL|LHL|HLHLL|HHH$.

The LZ complexity $C(W_n)$ is defined as

$$C(W_n) = \frac{1}{n} c(W_n) [\log_2 c(W_n) + 1] . \tag{18}$$

For a random word W_n, it can be shown that $\lim_{n\to\infty} C(W_n)$ gives just the Shannon entropy rate. In consequence, the LZ complexity also quantifies average information quantity in Shannon sense. Note that the normalization in (18) guarantees that $0 \leq C(W_n) \leq 1$ only in the asymptotic limit; for finite n, we can find instances for which C is greater than one. Since C is based on the study of recurrence of patterns in a symbolic sequence, this approach provides a tool for the analysis of complex sequences.

5. Complexity of the generalized Fibonacci quasicrystals

Next, we analyze the different response of the metal families discussed thus far as a function of their LZ complexity. For that purpose, we concentrate in a definite magnitude; the transmittance, which in the presence of a pseudo-bandgap tends to zero.

To keep irrelevant details apart, we deal with normal incidence at a definite wavelength. For definiteness, for the practical realization of the layers the materials are chosen to be cryolite (Na_3AlF_6) and zinc selenide (ZnSe), with refractive indices $n_L = 1.34$ and $n_H = 2.568$, respectively, at $\lambda = 0.65$ μm. This is a standard choice and, although it is a simple example, it allows one to work out easily the details of the method, which can be immediately extended to other media and spectral regions.

To make a fair benchmarking we have to fix the thicknesses in a consistent manner. To this end, we impose that the average transmittance over all the generations in the same family is minimal. To obtain these optimal values, the two thickness are varied independently from 0.02 μm to 0.30 μm. This range is in turn subdivided into 50 even intervals, so all in all we get a set of 50×50 equal rectangles. The center of each rectangle is picked as an initial guess and we seek for a best point by using a random permutation of the two thicknesses and an iterative search with fixed (positive or negative) increment.

In the next optimization step, we apply a quasi-Newton algorithm to improve the points of the previous exploration. Finally, the best of the 2500 local optima are taken as the global optimum thicknesses.

We have also included the periodic crystal $[LH]^N$, for which the optimal solution is known to be

$$n_L \frac{d_L}{\lambda} = \frac{1}{4}, \qquad n_H \frac{d_H}{\lambda} = \frac{1}{4} . \tag{19}$$

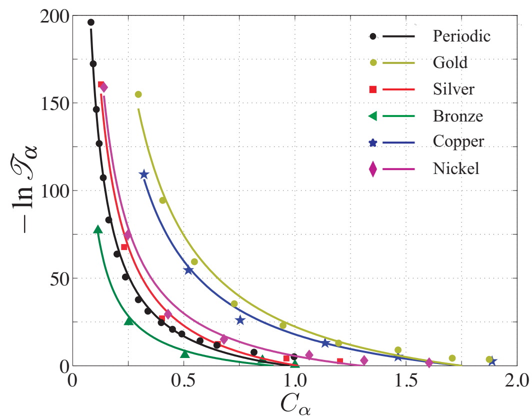

Figure 3. Behavior of the transmittance as a function of the LZ complexity for the periodic system and the generalized Fibonacci families indicated in the inset. We consider generations up to 1600 letters.

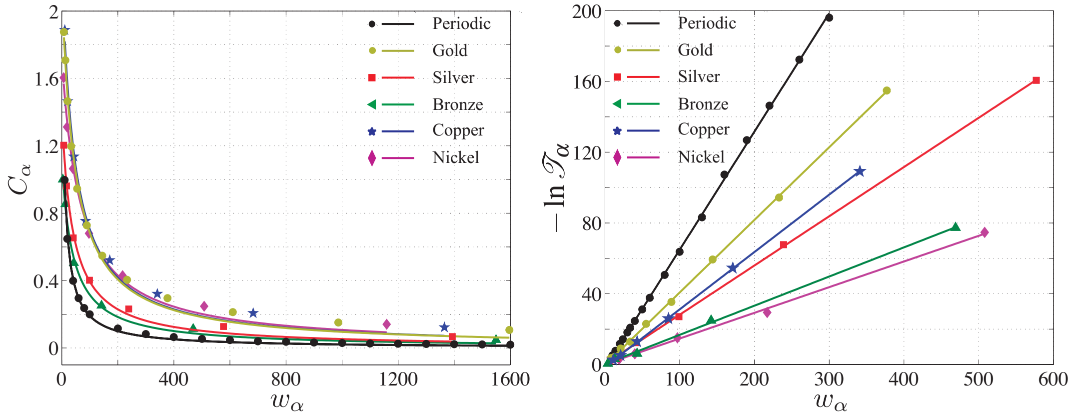

Figure 4. (Left) LZ complexity versus the number of letters in the words of several generations for the periodic systems and the generalized Fibonacci quasicrystals indicated in the insets. (Right) Transmittance for the same cases as before, as a function of the number of letters w_α.

The resulting values, in the dimensionless form nd/λ, are presented in Table 1. In general, all the families present $n_L d_L/\lambda$ lesser than 0.25, except the bronze. The values of $n_H d_H/\lambda$ are close to 0.25, except for the gold and silver.

In Fig. 3 we plot the logarithm of the transmittance $-\ln \mathscr{T}_\alpha$ for all the families as a function of the corresponding LZ complexity. In all the cases, we have computed the LZ complexity using an efficient implementation in Matlab [73]. We have restricted the generations in each family to those giving values of $\mathscr{T}$ in the specified limits, which limits the number of letters to 1600. For a fixed value of the complexity, the gold family is always the best, whereas the bronze is the worst. Surprisingly, the periodic Bragg reflector performs badly in this sense. This can be intuitively understood with a simple glance at Fig. 1: the degree of organization of the systems differs in a remarkable way. In other words, the number of effective interfaces between media L and H may be significantly different from the total number of layers.

All the curves can be properly fitted to the functional form

$$-\ln \mathscr{T}_\alpha = a_0 + \frac{a_1}{C_\alpha}, \tag{20}$$

which corresponds to shifted hyperbolas. The resulting coefficients a_0 and a_1 for each family can be found in Table 2. In some cases (notably, the bronze) the cloud of points is not large enough to appraise at a glance the goodness of the fit. Nonetheless, the correlation coefficients r are always better than 0.99, which ensures the validity of (20). Note also that for hypothetical values of C_α greater than 2, this fitting would eventually lead to transmittances $\mathscr{T}_\alpha$ greater than one, which has no physical meaning. In the realistic range considered in the plot, the hyperbolas never intersect.

Table 1. Thicknesses giving minimum of the average transmittance over all the generations in a the generalized Fibonacci quasicrystals indicated in the first column.

System	Metal	$n_L d_L/\lambda$	$n_H d_H/\lambda$
Periodic	—	0.2500	0.2500
FS(1,1)	Gold	0.2251	0.3417
FS(2,1)	Silver	0.2311	0.3809
FS(3,1)	Bronze	0.3416	0.2459
FS(1,2)	Copper	0.1249	0.2501
FS(1,3)	Nickel	0.2451	0.2277

Table 2. Parameters involved in the fittings in Eqs. (20), (21), and (23).

System	Metal	a_0	a_1	b_0	b_1	c_0	c_1
Periodic	—	−17.917	17.301	0.54422	0.04718	−1.82860	0.66697
FS(1,1)	Gold	−29.779	52.116	0.46257	0.00998	−0.35174	0.41042
FS(2,1)	Silver	−22.297	22.590	0.71580	0.01821	0.29576	0.27829
FS(3,1)	Bronze	−10.953	9.847	0.90351	0.02264	0.28379	0.16478
FS(1,2)	Copper	−23.922	41.833	0.44177	0.00980	−1.27180	0.32427
FS(1,3)	Nickel	−18.687	24.282	0.57597	0.00876	0.16579	0.14514

It is interesting to check how the complexity depends on the number of letters (in our case, layers) w_α in each word W_α. The results are shown in Fig. 4. The curves can be fitted to the functional form

$$C_\alpha = \frac{1}{b_0 + b_1 w_\alpha}, \tag{21}$$

which is again an hyperbola, but now with a w_α offset. The corresponding parameters can be found in Table 2, and the Pearson correlation coefficients are always greater than 0.99.

For a fixed value of w_α, the periodic sequence is has the lower complexity, which seems intuitively obvious. The gold and the non-Olympic series (copper and nickel) appear almost superimposed, whereas silver and bronze are in an intermediate situation.

In the limit $\alpha \to \infty$, the LZ complexity C_α tends to zero for all the sequences, as it is directly seen in the figure and is evident from the fitting (21). This is consistent with the asymptotic behavior

$$C_\alpha \sim \frac{(\ln w_\alpha)^2}{w_\alpha \ln \Phi}, \tag{22}$$

which has been found for the Fibonacci quasicrystals [74].

Finally, we can merge these two last results, schematized in Eqs. (20) and (21), to check the dependence of the transmittance with the number of layers. The final results reads

$$-\ln \mathscr{T}_\alpha = c_0 + c_1 w_\alpha . \tag{23}$$

Again the associated parameters are given in Table 2 and they are consistent with the previous ones.

The results are presented in Fig. 4. The correlation coefficients (always greater than 0.999) confirm that (23) is indeed a good approximation. Note, in passing, that this implies that the reflectance approaches the unity exponentially with the number of layers, as one would expect from a bandgap. The periodic stack has the biggest slope, followed by the Olympic-metal family. Unexpectedly, the copper performs better than one could anticipate, while the nickel is the worst. To sum up, as the number of letters is concerned, periodicity always beats both quasiperiodic and aperiodic orders.

As a final remark, we mention that we have performed similar analysis for other quasicrystals, especially for generalized Thue-Morse sequences. We do not include the details here for the sake of conciseness. The conclusions are much the same as the ones presented thus far.

6. Conclusions

We have exploited the notion of Lempel-Ziv complexity to explore in a systematic way the performance of generalized Fibonacci sequences. This LZ complexity gives a direct of how difficult is to create the system and is of direct interpretation for the experimentalist. What we have discovered is that these sequences can perform much better than the photonic crystals, while providing more versatility. We think that this constitutes a unique fact that might open avenues for quasicrystals.

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The authors declare no conflict of interest.

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