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Quantum Fisher Information: Theory and Applications

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Abstract

Entanglement is at the heart of quantum technologies such as quantum information and quantum metrology. Providing larger quantum Fisher information (QFI), entangled systems can be better resources than separable systems in quantum metrology. QFI topic is a very active research area and it has many possible usage areas in quantum information domain. In this study, we review quantum Fisher information research with both from theoritical and application perspective.

Keywords: quantum metrology; quantum fisher ınformation; decoherence; optimization

1. Introduction

The Quantum Information Theory and Quantum Computation are hot working areas that are the theoretical basis of Quantum Computers, which are described as computer technology of the future and intended to operate at very high speeds.

Quantum Fisher Information, a version of Fisher Information, developed for quantum systems, has also become a highly studied subject in recent years, as it also measures the sensitivity that systems can provide for phase sensitive tasks [1-32]. In the solution of some problems that are important for Classical Computer Engineering; For example, processing large amounts of data and variable data, data mining, etc. Fisher Information is used [33]. When Quantum Computers are produced it is envisaged that Quantum Fisher Information will be used to solve similar or more complex problems.

Some of the multipartite entagled quantum systems can exceed the highest sensitivity limit that conventional systems can provide. This is called "useful" systems that can exceed the classical limit [34,35]. Which systems are usable under which decoherence has become a fundamental area for quantum technologies [10]. Fisher information is believed to be able to develop a entanglement measure for multi-partite entangled systems and is being studied up to date [36].

The quantum extension of Fisher information resulting from the question of how much data we can obtain from an experiment has been developed by Helstrom (1976) [37] and Holevo (1982) [38]. Quantum Fisher information (Petz et al., 2008) is basically used to calculate the phase sensitivity that systems can provide in quantum technologies. Classical systems, however, provide sensitivity to a certain level (called the "shot noise limit"), while Hyllus et al. (2010) [36] found that all quantum systems can not provide higher phase sensitivity than classical systems. Therefore, which quantum systems can provide better precision than the classical has become an important field of study, and these quantum systems have started to be called "useful".

There is also a standard limit on quantum systems, but with the limitation of the Heisenberg uncertainty principle, It was shown by Lloyd et al. that in some cases, can exceed this quantum limit[35]. It has been shown that only bipartite entangled systems in the EPR / Bell pair structure are not usable and that the usability is a basic or necessary, multi-partite entangled quantum system, but for a sufficient condition the average quantum Fisher knowledge per particle in the system must meet a certain criterion. It was shown by Pezze et al. in 2009 [34]. According to this finding, the ratio of the number of sides (ie the quantum particle forming the system) to the maximum quantum Fisher information the system can give is defined as a parameter as "average quantum Fisher information per particle" for a multi-partite entangled system. The smaller this parameter means the higher the system's phase sensitivity. For classical systems that can give the best results, this parameter is minimum 1. To understand that a given system is "usable", that is, it can be better than the classical one, it must be known that this parameter is less than 1 for that system. This is the necessary and sufficient condition.

Along with this groundbreaking breakthrough in quantum technologies, research on the quantum Fisher knowledge of multi-partite entangled systems and the classification of usable systems have grown avalanche: Xiong et al. (2010) [39] showed that the superposition states of the Dicke systems are more usable than themselves. Numerous studies have been carried out on quantum Fisher information of various superposition states with pure and singular states of GHZ and W systems. Superposition coefficients in such a quantum system of 3 quantum particles, superposition coefficients between superposition and quantum Fisher's information of the relative phases are found by Wang et al. (2012) [3] and quantum Fisher information of a similar system of 4 quantum particles by Ouyang et al.

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2. Fisher Information

Some of the most noteworthy studies on Quantum Computing are some of the studies on Large Data based on the *classic Fisher Information*. *Fisher Information*, in fact, is defined as a physical criterion, while providing some very interesting results in solving problems related to some large data. With an approach called *Extreme Physical Information* [33], it is possible to make various conclusions about these systems statistically. In order to understand this concept, it is necessary to extract and observe the common shareholders of the components in the system.

In fact, the whole universe is made up of all the entities that are moving, collapsing, pulling one, or interacting in different ways. It is known that information about the whole of the systems in the nature is also contained in these systems. Although Fisher Information is conceptually equivalent to Shannon Information, which is one of the cornerstones of the Information Theory, there are some differences between the two. Shannon defines the amount of information in the message by relating the uncertainty of a message to the concept of probability. The total uncertainty in a message is actually equal to the total knowledge in that message.

Entropy was also first used by *Shannon* in data communications in computer science. Therefore, according to the notion that in the literature *Shannon Entropy* is the entropy part of the logarithm of the symbols in the mean value of the shortest probabilities to encode a message. So if we have roughly 256 characters in our alphabet, we divide the logarithm of this number ($\log_2 256 = 8$) into the entropy of the message. So the more change in the message, the more we need to code it. In other words, if our alphabet is 256 characters but we are sending only one character then we need 0/256 different coding (0 bite) since it is entropy 0. Or, similarly, we send $256/8 = 8$ bits in this case if we are sending the same frequency in every letter.

Fisher Information is basically the measure of the amount of information according to a known parameter in a statistical distribution. According to Fisher Information, the opposite of information is uncertainty. The *Exploratory Data Analysis* technique is described by way of Fisher Information. Some of the big data problems in the real world that can be explained by this technique can be listed as follows [33]:

- Population size estimation / analysis
- Analysis of stock market movements and financial data, estimation
- Analysis of data in social media applications

- Detection and modeling of the growth in the cancerous region for various types of cancer in humans
- Estimation and analysis of optimal financial investment values
- Various modeling and estimation applications in molecular biology and bioinfomatics

Classical Fisher information is used extensively in the solution of these kinds of problems, which are often confronted in today's world. It is envisaged that the quantum Fisher information, which is the type developed for the quantum systems of this information, will be actively used to solve similar and more complex large data problems that are expected to be solved in that architecture together with the production of Quantum Computers.

3. Quantum Fisher Information

Quantum Fisher Information (QFI) is a very useful concept for analyzing situations that require phase sensitivity. This feature has attracted attention and extends the classical Fisher Information. Especially for systems with a higher QFI value, the accuracy is more clearly achieved; For example, clock synchronization [40] and quantum frequency standards [41]. Although some of the pure entangled systems may exceed the classical limit, this does not apply to all entangled systems [36]. The interaction between the quantum system and the environment not only reduces entanglement but also reduces the system's Quantum Fisher Information, in general. So we can say that researching quantum systems on QFI is important for the progress of quantum technologies. In recent studies, a single parameter, χ^2 parameter, phase sensitivity was added to measure only the self-knowledge of the system under investigation [11]. Since a condition of $\chi^2 < 1$ is not provided for a general quantum system, it is understood that the system has multiple entanglement and this system provides better phase accuracy than a separable system. These quantum systems are called "useful" systems in the literature. For two-level N-particle quantumsystems, the Cramer-Rao limit is defined by the following formula [37,38]:

$$\Delta\phi_{QCB} \equiv \frac{1}{\sqrt{N_m F}} \tag{1}$$

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where N_m is the number of experiments on the system being measured and F is the Quantum Fisher Information value. We can write 3-dimensional vectors normalized in the nth direction of angular momentum operators, J_n, Pauli matrices as follows:

$$J_{\vec{n}} = \sum_{\alpha=x,y,z} \frac{1}{2} n_\alpha \sigma_\alpha \tag{2}$$

For J_n, the Fisher Information of the ρ quantum system can be expressed in a symmetric matrix C [10]:

$$F(\rho, J_{\vec{n}}) = \sum_{i \neq j} \frac{2(p_i - p_j)^2}{p_i + p_j} |\langle i|J_{\vec{n}}|j\rangle|^2 = \vec{n} C \vec{n}^T \tag{3}$$

where p_i and $|i\rangle$ represent the eigenvalues and eigenvectors of the ρ system, respectively, and the matrix C is defined as

$$C_{kl} = \sum_{i \neq j} \frac{(p_i - p_j)^2}{p_i + p_j} [\langle i|J_k|j\rangle\langle j|J_i|i\rangle + \langle i|J_i|j\rangle\langle j|J_k|i\rangle] \tag{4}$$

The largest F value between the N options is selected and averaged over N particles. The Fisher Information value is calculated as the greatest eigenvalue of the C matrix. This definition is expressed by the equation:

$$\overline{F}_{\max} = \frac{1}{N} \max_{\vec{n}} F(\rho, J_{\vec{n}}) = \frac{\lambda_{\max}}{N} . \tag{5}$$

4. Quantum Metrology and Quantum Fisher Information Applications

Quantum mechanical systems provide the appropriate environment for us to acquire knowledge on many issues related to the physical world. Among these informations, the ones obtained from measuring devices are very important. Quantum mechanics limits the clarity of these measuring devices to the Heisenberg Uncertainty Principle and the Margolus-Levitin Theorem. Quantum mechanics provides various strategies to pass the standard quantum limit, which is Semi-Classical limits, and the shot noise limit. Scientists and engineers have started to build strategies to increase the accuracy of the measurements of many different types of effects of the concepts of squeezing and entanglement, starting with increasing the sensitivity of interferometric devices and position measurements.

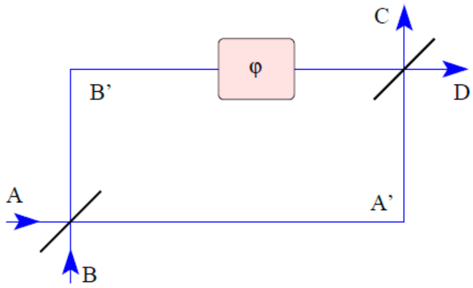

Figure.1- Mach-Zehnder interferometer [35]

Although the feasibility of some quantum techniques is still very futuristic, in the present case the derivation of the entangled system states and the various operations on it can be partially achieved, even at the initial stage. If we look at the examples in the literature, the quantum mechanical systems give results as much as the square root of the N particle used for the accuracy of measurement. Technically, even the derivation of N = 5 or 6 particle entangled systems is quite complicated. On the other hand, it is technically possible to construct and measure the classical systems consisting of millions of particles. As Quantum Technologies evolve, it is clear that the concepts of entanglement and sqeezing also have more pronounced effects on the accuracy of measurement and on the development of sensitivity.

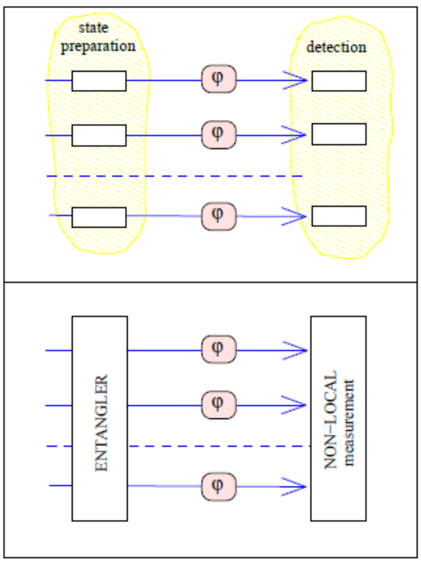

Figure 2- Comparison of Classical and Quantum Strategies [35]

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Especially when we study the studies on the measurement of space-time geometry with quantum systems, it is inevitable that quantum mechanical systems are useful as unique tools in understanding the facts about the known universe.

In their current work [35] scientists have found that they have found a lower limit for single parameter identification operations when working on the Quantum Metrology infrastructure. They have shown that this lower limit is always an accessible boundary for both unitary and non-unitary processes. Calculation of the quantum boundary is a very challenging process for the best system condition in case of noise. The increasing difficulty of numerical analysis in case of an increase in the resources to be calculated. These calculations will not depend on the initial system state and will not require optimization according to all possible Kraus notations. Instead, it is sufficient for the Kraus operators to select the class that best suits the physical condition for the examined system state.

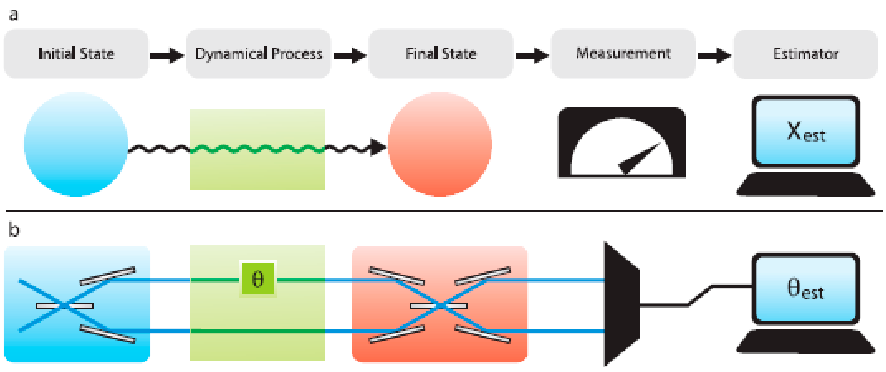

Figure 3- System setup for quantum parameter estimation [35]

(a) Preparation of an arbitrary dynamic process for an unknown parameter x

(b) System setup for viewing a phase transition in order Θ within an interferometer

The power of the method described here can be explained by two examples of problems described in the context of Quantum Metrology: Estimation of the interferometer phase and determination of the conversion frequency for atomic spectral survey, etc.

The proposed method provides a transition from the Heisenberg limit to an asymptotic shot noise like behavior in the application of optical interference. However, a definite effect of this method is that even if there is a small amount of noise, the development at the shot noise threshold remains at a certain multiplier and does not change the behavior of $\frac{1}{\sqrt{N}}$.

The described method can be explained as a generic calculation tool for estimating parameters and non-integrative dynamics, which can occur in everywhere and every kind of problem where appropriate.

5. Interferometry and Quantum Fisher Information

Measurement protocols using multi-port devices created by injecting multiple photon Fock system conditions that may work in phase estimation work have been theoretically established. The results have been tested on real multi-port devices and are available on many possible photon quantum systems.

In some studies [4], a protocol including photon counting detection with input of Fock system states was defined, and the measurement of this protocol with the standard quantum limit was simulated [4].

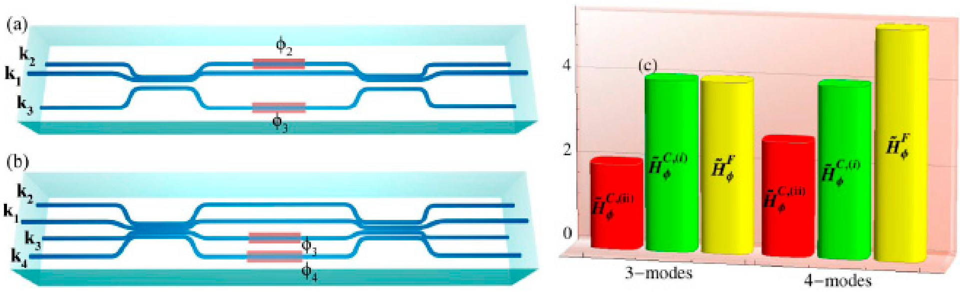

Figure 4- 3 and 4-port interferometer's quantum Fisher information results [4]

Other current studies aim to examine the two-mode interference tools from the perspective of Quantum Information Theory. The problem to be solved is to determine if all the N particule pure entangled system states can reach the sub-pulse noise sensitivity when optimized with local operations on the particles in these interference vehicles. Quantum Fisher Information is defined in the Cramér-Rao theory, which defines a limit on optimal sensitivity through F_Q, and is used in the study. For $F_Q > N$, the sub-shot noise sensitivity can be reached with a central limit.

A general two-system linear interferometer such as the Mach-Zehnder interferometer tool has been studied on maximum Fisher information. The literature has contributed to the fact that Quantum Fisher Information should be optimized for the first time in these studies. It has been stated that optimization processes can be performed directly on mixed system conditions and experimentally [4].

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6. Decoherence and Quantum Fisher Information

Recent studies [10] investigate the variation of the maximal QFI value for the GHZ system states in three different channels of decoherence. These channels are listed as Amplitude Damping Channel (ADC), Phase Damping Channel (PDC) and Depolarising Channel (DPC), respectively. Sudden loss cases can be seen in both measurement and spin compression, but there is a special case for maximal QFI. The value obtained for the ADC shows a sudden increase after a *p* value as can be seen in the graph below.

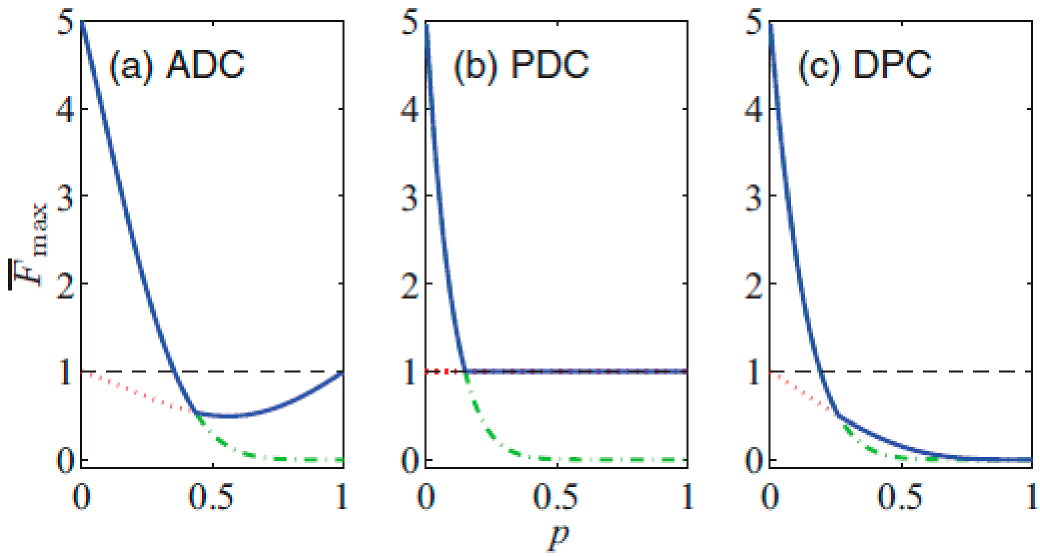

Figure 5 – ADC, PDC ve DPC value changes according to p

In other recent studies [6] they have calculated quantum Fisher Information values for a system state which is a superposition of N-particle multi-entangled W and GHZ system states and examined the changes in values. It was found that the values obtained for the mean QFI per particle (RMQFI) are inversely proportional to the number of particles and are a sharp peak between 0.6 and 0.8. The behavior of the RMQFI value has changed for the values of N from 2 to 10. The evaluated system state is defined by the following equations:

$$|GHZ_N\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}} \tag{6}$$

$$|\psi_N\rangle = \alpha|W_N\rangle + \beta|GHZ_N\rangle \tag{7}$$

here α and β has the following relation:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{8}$$

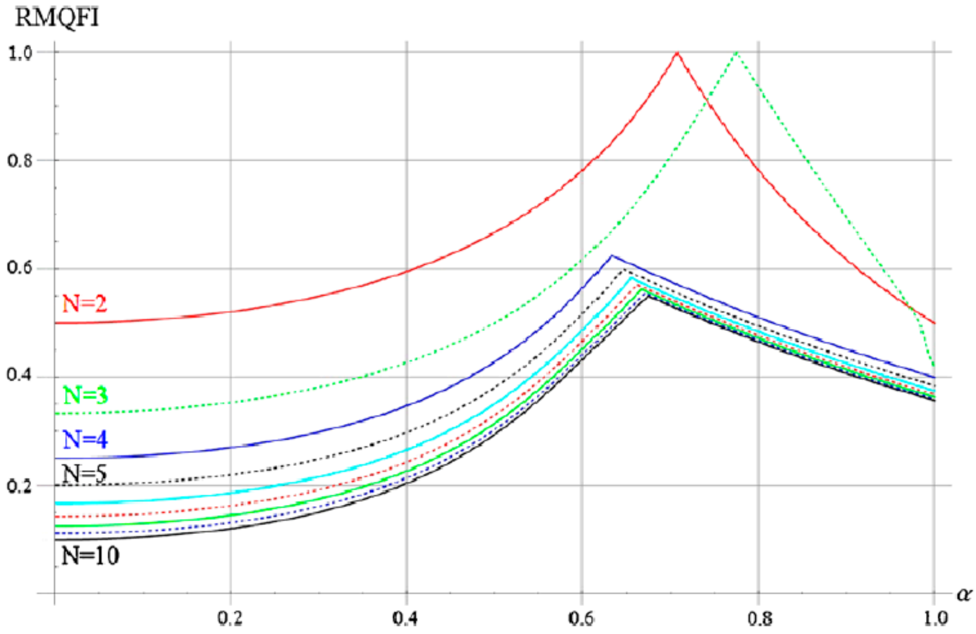

Figure 6 – Explained changes in RMQFI [6]

Also in recent studies [7] investigated how the KFB values change for a system state consisting of more than one particle and having a random relative phase, which is the superposition of the GHZ and 2 W system states. If the number of particles goes from 3 to 4, the extraordinary change in QFI value can be noted. In addition, dependence on QFI decreased as the number of particles increased. The change in QFI value was analyzed according to the relative change for the mentioned system condition.

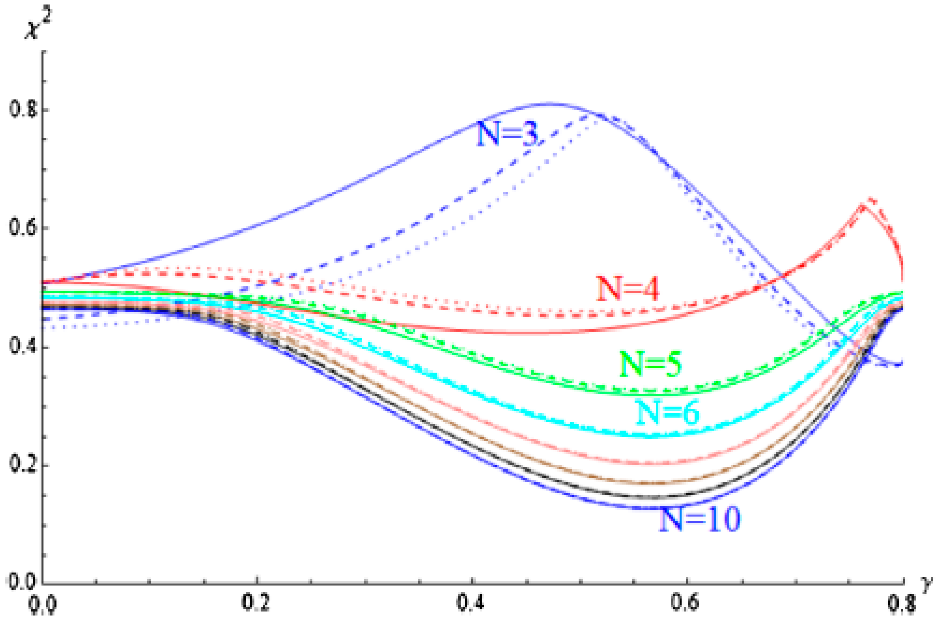

Figure 7 – Changes explained in the previous paragraph for value α = 0.6 [7]

Specifically, it was found that the change of another system state, which we have called a 3-particle original W system state and a 3-particle "W-like" system state, ignoring 1 particle from an original 4-particle W system state. We have found that when we expose each particle to the same decoference channel in these two system states, the original W system state under the depolarized channel is better than the other, but the W-like system state is more interesting under the channel that widens or reduces the phase [22].

We have selected a set of publications in this domain but decoherence and QFI relation is still an active topic in the field.

7. Optimizing Quantum Fisher Information

In a recent study, it was showed that QFI can be optimized for having more meaningful results in comparison with entanglement measures [21,23]. It was performed a maximization via general rotations of each qubit in the Euler representation

$$U_{Rot}(\alpha,\beta,\gamma) = U_x(\alpha)U_z(\beta)U_x(\gamma) \tag{9}$$

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where the rotations about axes are defined as $U_j(\alpha) = \exp(-i\alpha\frac{\sigma_j}{2})$, $j \in \{x, z\}$, with arbitrary three angles for each qubit between $[0, 2\pi]$, each with steps of θ degrees, resulting $O((\frac{2\pi}{\theta})^6)$ QFI calculations. We have found that choosing the steps as $\theta = \frac{\pi}{2}$ is sufficient for a good optimization such that the picture, whereas narrowing the steps could possibly result a better optimization, with the cost of an increase in the running time of the simulation.

According to the results that we obtained, for the chosen a thousand random states, %98 of the QFI values are maximized and %99 of the QFI values are minimized with $\theta = \frac{\pi}{2}$ and the all the rest with $\theta = \frac{\pi}{3}$. [21]

8. Conclusion

Quantum Fisher Information is not an entanglement measure criterion, but it gives us an idea that we need to obtain very important predictions especially for situations where phase sensitivity is necessary. In the present studies, it was determined that Fisher Information was used unoptimized in current research as open field. In this study, we reviewed QFI related topics with an analytical approach.

The most basic result obtained from this is the fact that quite significant ordering relations can be obtained if the QFI values are maximized. This result suggests that the gore LOCC gore maximized QFI values in terms of entanglement and the system state ordering is more useful for the problem. It is predicted that the ranking results obtained from this can be much more interesting results especially if the multi-partite entangled systems are examined. We think that it will be possible to start by examining two qutrit or more leveled systems especially in future studies. One additional open problem may be resolved in a multi-partite entanglement measure based on QFI. Providing larger quantum Fisher information (QFI), entangled systems can be better resources than separable systems for quantum metrology applications.

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