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Article

The Time Domain Transition Radiation as a Vehicle to Probe Qualitatively the Connection between the Elementary Charge, Heisenberg's Uncertainty Principle and the Size of the Universe

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Abstract: The energy, momentum and the action associated with the time domain transition radiation fields are investigated. The results show that for a charged particle moving with speed v, the longitudinal momentum associated with the transition radiation is approximately equal to E/c for values of $1-v/c$ smaller than about 10^{-3} where E is the total radiated energy and c is the speed of light in free space. The action of the transition radiation, defined as the product of the energy dissipated and the duration of the emission, increases as $1-v/c$ decreases and, for an electron, it becomes equal to $h/4\pi$ when $v=c-v_m$ where v_m is the speed associated with the lowest energy state of a particle confined inside the universe. Combining these results with Heisenberg's uncertainty principle, an expression for the electronic charge based on other fundamental physical constants is derived. The best agreement between the experimentally observed electronic charge and the theoretical prediction is obtained when one assumes that the actual size of the universe is about 250 times larger than the visible universe.

Keywords: transition radiation, Heisenberg's uncertainty principle, electronic charge, size of the universe

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1. Introduction

Transition radiation, first predicted by Ginzberg and Frank [1], and later analysed in more details by others [2, 3], is produced when a charged particle, moving with a certain speed, crosses a boundary between two mediums with different dielectric constants. The spectrum of the radiation depends on the dielectric constants of the two media. The same type of radiation is produced when a charged particle is ejected from or impinged on a vacuum-conductor interface. Here we will concentrate on the latter case.

2. Theory

The analysis of transition radiation is usually done in frequency domain but here we will confine our analysis to time domain. Moreover, in analysing the transition radiation the charge is usually considered to be concentrated on a point particle but in the present study we represent it as having a spatial distribution so that the movement of the charge can be represented by a current pulse of finite duration. Of course when the duration of the current pulse approaches zero (i.e. when it is represented by a Dirac delta function) it reduces to the movement of a point charge.

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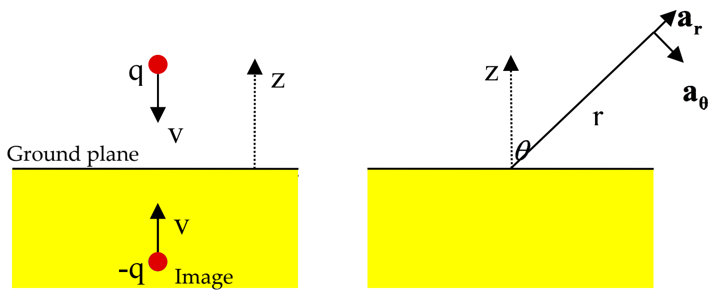

(a) (b)

Figure 1. (a) A charged particle moves along the negative z-axis towards the ground plane. (b) The definition of parameters pertinent to the expression derived for the electric radiation field. In the diagram $\mathbf{a}_r$ and $\mathbf{a}_\theta$ are unit vectors in the radial and increasing θ directions.

Let us consider a charged particle propagating with speed v and incident on a perfectly conducting ground plane. The direction of motion of the charged particle is along the negative z-axis and the conducting plane is located on the x-y plane. The results to be derived would be identical (with field expressions having an opposite sign) if a charged particle moving out of the conducting plane with speed v is considered instead. The process of interaction between the charged particle and the conducting plane is represented by a Gaussian current pulse that moves along the negative z-axis towards the plane and incident on it. This is illustrated in Figure 1a. Transition radiation is produced during the interaction of the current pulse with the conducting plane.

In the analysis the Gaussian current pulse is represented by:

$$i(t) = i_0 e^{-t/2\sigma^2} \tag{1}$$

In the above equation i_0 is a constant having units of Amperes, t is the time and σ is the standard deviation of the Gaussian pulse. The charge, q, associated with this current pulse is given by:

$$q = i_0 \tau \tag{2}$$

In Equation (2), τ is the duration of the current pulse and it is given by:

$$\tau = \sqrt{2\pi\sigma^2} \tag{3}$$

Thus the current pulse can also be written as:

$$i(t) = \frac{q}{\sqrt{2\pi\sigma^2}} e^{-t/2\sigma^2} \tag{4}$$

Observe that as $\sigma \to 0$ the current pulse reduces to a Dirac delta function. Usually the field expressions for transition radiation is given by assuming that the current pulse is a Dirac delta function. Results pertinent to that case can easily be obtained by letting $\sigma \to 0$. The electric radiation field associated with the transition radiation produced by the incident of this current pulse on the conducting plane can be expressed as [4]

$$\mathbf{E}(t) = \frac{i(t - r/c) v \sin\theta}{4\pi\varepsilon_0 c^2 r} \left\{ \frac{1}{1+\beta\cos\theta} + \frac{1}{1-\beta\cos\theta} \right\} \mathbf{a}_\theta \tag{5}$$

where $\beta = v/c$. The geometry relevant to the definition of parameters pertinent to this equation is shown in Figure 1b. Note that $\mathbf{a}_\theta$ is an unit vector in the direction of increasing θ. The first term inside the bracket is generated by the incident of the current pulse on the conducting plane and the second term is generated by its image in the perfectly conducting ground plane. The Poynting vector associated with this radiation field is given by

$$\mathbf{S}(t) = \frac{i(t - r/c)^2 \beta^2 \sin^2\theta}{4\pi^2 \varepsilon_0 c r^2} \frac{1}{\left(1 - \beta^2 \cos\theta^2\right)^2} \mathbf{a}_\theta \tag{6}$$

3. Expression for the total energy dissipated by the transition radiation

First of all, observe from Equation (6) that when the current pulse is represented by a Dirac delta function the energy dissipated by transition radiation per unit solid angle is given by (representing the Dirac delta function by δ)

$$\frac{\Delta W}{\Delta\Omega} = \frac{q^2 \delta(t - r/c)^2 \beta^2 \sin^2\theta}{4\pi^2 \varepsilon_0 c} \frac{1}{\left(1 - \beta^2 \cos\theta^2\right)^2} \tag{7}$$

Performing Fourier transformation the energy dissipated per unit solid angle per unit frequency is given by:

$$\frac{\Delta W}{\Delta\Omega\Delta\omega} = \frac{q^2 \beta^2 \sin^2\theta}{4\pi^2 \varepsilon_0 c} \frac{1}{\left(1 - \beta^2 \cos\theta^2\right)^2} \tag{8}$$

This is the traditional equation that is being used in analysing the transition radiation [1, 2, 3]. However, confining our analysis to the time domain, the total energy lost by the charged particle as transition radiation is given by

$$\Delta U = \frac{\left(\int_{-\infty}^{\infty} i(t - r/c)^2\, dt\right)\beta^2}{4\pi^2 \varepsilon_0 c} \int_0^{\pi/2}\int_0^{2\pi} \frac{\sin^3\theta\, d\theta\, d\varphi}{\left(1 - \beta^2 \cos\theta^2\right)^2} \tag{9}$$

Performing the spatial integral we obtain

$$\Delta U = \frac{\left(\int_{-\infty}^{\infty} i(t - r/c)^2\, dt\right)\beta^2}{2\pi \varepsilon_0 c} \frac{1}{4}\left[\frac{(1+\beta^2)}{\beta^3} \ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{2}{\beta^2}\right] \tag{10}$$

Note that if $(1-\beta) << 1$, i.e. $\beta \approx 1$, the above expression reduces to

$$\Delta U = \frac{\left(\int_{-\infty}^{\infty} i(t - r/c)^2\, dt\right)\beta^2}{4\pi \varepsilon_0 c} \left[\ln\left(\frac{2}{1-\beta}\right) - 1\right] \tag{11}$$

4. Expression for the momentum transported by the transition radiation

Consider the radiation moving in a radial direction associated with the angle θ. The flux of momentum transported by the radiation field moving in that direction is given by

$$\mathbf{M}(t) = \frac{i(t-r/c)^2 \beta^2 \sin^2\theta}{4\pi^2 \varepsilon_0 c^2 r^2} \frac{1}{\left(1-\beta^2\cos\theta^2\right)^2} \mathbf{a}_r \tag{12}$$

Due to symmetry only the z-component of this momentum is not zero and it is given by

$$\Delta \mathrm{p}_z(t) = \frac{i(t-r/c)^2 \beta^2 \sin^2\theta}{4\pi^2 \varepsilon_0 c^2 r^2} \frac{\cos\theta}{\left(1-\beta^2\cos\theta^2\right)^2} \tag{13}$$

Thus the total momentum transported by the radiation in the z-direction is given by

$$\Delta P_z = \frac{\left(\int_{-\infty}^{\infty} i(t-r/c)^2 dt\right)\beta^2}{4\pi^2 \varepsilon_0 c^2} \int_0^{\pi/2}\int_0^{2\pi} \frac{\sin^3\theta\cos\theta d\theta d\varphi}{\left(1-\beta^2\cos\theta^2\right)^2} \tag{14}$$

After performing the spatial integral the above equation can be written as

$$\Delta P_z = \frac{\left(\int_{-\infty}^{\infty} i(t-r/c)^2 dt\right)\beta^2}{2\pi \varepsilon_0 c^2} \left[\frac{1}{2\beta^2}\ln\frac{1}{\beta} + \frac{1}{2}\ln\frac{\beta}{1-\beta} - \frac{1}{2\beta}\right] \tag{15}$$

Note that if $(1-\beta) << 1$, i.e. $\beta \approx 1$, the above expression for the momentum reduces to

$$\Delta P_z = \frac{\left(\int_{-\infty}^{\infty} i(t-r/c)^2 dt\right)}{4\pi \varepsilon_0 c^2} \left[\ln\frac{1}{1-\beta} - 1\right] \tag{16}$$

Comparison of Equation (16) with Equation (11) shows that when $|\ln(1-\beta)| \gg \ln 2$ the vertical component of the momentum transported by the transition radiation reduces to

$$\Delta P_z \approx \Delta U / c \tag{17}$$

This indicates that when $|\ln(1-\beta)| \gg \ln 2$ the radiation, and hence the energy transported, is directed mostly along the z-axis.

5. The action associated with the transition radiation

The action associated with the transition radiation is defined as the product of the duration over which the transition radiation is generated and the total energy transported by it. In order to evaluate this quantity let us first perform the time integral associated with the Equation 11. Once this is done the result is given by

$$\Delta U = \frac{q^2\beta^2}{16\pi^{3/2}\varepsilon_0 c\sigma} \left[\frac{(1+\beta^2)}{\beta^3}\ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{2}{\beta^2}\right] \tag{18}$$

Then the action associated with the transition radiation is given by

$$\tau \Delta U = \frac{\tau q^2\beta^2}{16\pi^{3/2}\varepsilon_0 c\sigma} \left[\frac{(1+\beta^2)}{\beta^3}\ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{2}{\beta^2}\right] \tag{19}$$

Substituting for τ from Equation 3 we obtain

$$\tau\,\Delta U = \frac{\sqrt{2}q^2\beta^2}{16\pi\varepsilon_0 c}\left[\frac{(1+\beta^2)}{\beta^3}\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{2}{\beta^2}\right] \tag{20}$$

Observe that the action is independent of the duration of the pulse. For this reason it is also valid for the case when $\sigma \to 0$ and, therefore, for the case when the charge is represented by a Dirac delta function i.e. a point particle. Note also that in the case when $\beta < 1$ and $(1-\beta) \ll 1$ the Equation (20) reduces to

$$\tau\,\Delta U = \frac{\sqrt{2}q^2}{8\pi\varepsilon_0 c}\left[\ln\left(\frac{1+\beta}{1-\beta}\right)-1\right] \tag{21}$$

Note that the action goes to infinity when $\beta \to 1$. In other words, the action becomes infinity as the speed of the particle reaches the speed of light. The next question that one has to answer is the following: How close does the speed of a particle can approach the speed of light? In the next section we will show that the speed of the particle cannot reach a speed which is arbitrarily close to the speed of light but there is an upper bound to the speed which is less than the speed of light.

6. The upper bound of the speed that can be achieved by a particle confined in the current universe

All particles in nature are restricted by a maximum speed, which is the speed of light in free space. So in principle, any particle can reach a speed which is less than but infinitesimally close to the speed of light. This means that there is no upper limit to the exponent k which is defined by the expression

$$(1-\beta) = 10^{-k} \tag{22}$$

That is, the value of the parameter k is unbounded. On the other hand Heisenberg's uncertainty principle dictates that both the momentum and the location of a particle cannot be measured to infinite accuracy. That is, if the speed of the particle is well defined, i.e. the momentum is well defined, there is a large uncertainty in the location of the particle. When one confines the particle to a given volume, the maximum uncertainty associated with the location of the particle is specified and that in turn determines, through the Heisenberg's uncertainty principle, the minimum uncertainty with which the momentum of the particle, and hence the speed of the particle, can be specified. As the size of the region that confines the particle decreases the uncertainty associated with its speed increases. This fact is utilized in text books to illustrate that an electron cannot be confined inside a nucleus because its speed and its energy becomes so large that the electron cannot be confined inside the nucleus. In the same way, the uncertainty principle makes it impossible for us to define the momentum and the speed of a particle located inside the universe to an infinite accuracy. Is it possible for us to say that one can determine the speed of the particle to an infinite accuracy? Thus, in any experiment, real or gedanken, carried out inside the universe, the speed of a particle confined inside the universe cannot be specified to an accuracy better than the one dictated by the uncertainty principle. Consider a particle of mass m moving with a defined speed v along the z-axis. Let us denote the diameter of the universe by D_u. Since the particle is confined inside the universe, the uncertainty in the location of the particle along the x and y directions is D_u. This is the case because the particle has to be somewhere inside the universe. The Heisenberg's uncertainty principle demands that the minimum uncertainty in the momentum of the particle in the x and y directions to be about $h/4\pi D_u$. Since the momentum in these directions is at least of the order of the uncertainty in momentum, the minimum speed of the particle along the x and y axis, say v_{m}, is given by the equation

$$D_u m v_m \approx h/4\pi \tag{23}$$

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One can use non-relativistic momentum in deriving the above equation because for known masses this speed is extremely small in comparison to the speed of light. Thus the minimum speed along the x and y axis is given by

$$v_{\mathrm{m}} \approx \frac{h}{4\pi D_u m} \tag{24}$$

This, of course, is an order of magnitude estimation but it is sufficient for the problem under consideration. In the case of an electron this is given by

$$v_{\mathrm{m}e} = \frac{c\alpha a_0}{D_u} \tag{25}$$

In Equation (25), α is the fine structure constant and a_0 is the Bohr radius. Since the limiting speed in free space is the speed of light, the maximum speed $v_{\max}$ that can be realized by a particle moving along the z-axis is given by

$$c = \sqrt{v_{\max}^2 + 2v_m^2} \tag{26}$$

For speeds $v_{\max}$ comparable to the speed of light the above equation gives

$$v_{\max} = c(1 - v_m / c) \tag{27}$$

Note that in deriving Equation 27 we have assumed that $(1 + v_{\max} / c) \approx 2$. Since there are no preferred directions in space and since the above argument can be applied to a particle moving along in any direction in space, one can conclude that the minimum value of $(1-\beta)$ that can be realized by a particle confined inside the universe is given by

$$(1-\beta)_{\min} = \frac{h}{4\pi mcD_u} \tag{28}$$

In the case of an electron the minimum value of $(1-\beta)$ is given by

$$(1-\beta)_{\min} = \alpha a_0 / \mathrm{D}_u \tag{29}$$

It is of interest to note that an electron moving with this maximum speed will have an energy of about 10^{16} GeV.

7. Upper limit of the action associated with the transition radiation generated by a charged particle

Due to confinement inside the universe, the minimum value of $(1-\beta)$ that can be realized by a particle is given by Equation 28 and for an electron it is given by Equation 29. Thus, the maximum action associated with the transition radiation, A_m, is obtained by substituting the expression for $(1-\beta)_{\min}$ given in Equation 28 into Equation 21. This is given by

$$A_m = \frac{\sqrt{2}q^2}{8\pi\varepsilon_0 c}\left[\ln\left(\frac{8\pi mD_u c}{h}\right) - 1\right] \tag{30}$$

Since the first term inside the bracket is much larger than unity one can write this as

$$A_m = \frac{\sqrt{2}q^2}{8\pi\varepsilon_0 c\sigma}\left[\ln\left(\frac{8\pi mD_u c}{h}\right)\right] \tag{31}$$

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In the case of an electron the maximum action associated with the transition radiation, A_{em}, is given by

$$A_{me} = \frac{\sqrt{2}e^2}{8\pi\varepsilon_0 c}\left[\ln\left(\frac{2D_u}{\alpha a_0}\right)\right] \tag{32}$$

If we substitute numerical values to the parameters in the above equation we find that

$$A_{me} \approx h/4\pi \tag{33}$$

This is a remarkable result and possible reasons for this observation are discussed in the next section

8. Discussion

The results presented earlier, namely, that the maximum action of the transition radiation associated with the electronic charge is equal to $h/4\pi$ can be explained by appealing again to the uncertainty principle. Consider a charged particle moving with speed v along the z-axis towards the perfectly conducting plane. Its speed is well defined but its location along the z-axis is known only to an accuracy comparable to the size of the universe. At a certain time the charged particle will approach the conducting plane and just at the moment it strikes the plane it gives rise to the transition radiation. Let us attempt to detect the location of the charged particle just before it hits the plane using transition radiation. The transition radiation allows the location of the charged particle only to an accuracy of about τ/v. This is the uncertainty in distance over which the transition radiation is emitted. During this process the momentum of the charged particle will become uncertain by an amount $\Delta U/c$, the longitudinal momentum that is being lost to the transition radiation. These two quantities should satisfy the uncertainty principle, and therefore,

$$\tau v \frac{\Delta E}{c} \geq \frac{h}{4\pi} \tag{34}$$

For values of v close to the speed of light it reduces to equation

$$\tau \Delta U \geq \frac{h}{4\pi} \tag{35}$$

Another way to look at the problem is to utilize the fact that the origin of a radiation pulse can be located only to an accuracy of its effective wavelength. In the present case the duration of the radiation field is τ and the effective wavelength associated with it is about τc. Thus the uncertainty in the location of the origin of transition radiation is about τc which, when used in the uncertainty principle, will give rise to Equation (35). Now, substituting relevant parameters to the left side of Equation (35) we obtain

$$\frac{\sqrt{2}q^2}{8\pi\varepsilon_0 c}\ln\left(\frac{1}{1-\beta}\right) \geq \frac{h}{4\pi} \tag{36}$$

Since the left hand side of this equation has its largest value when $(1-\beta) = (1-\beta)_{\min}$ which in turn is equal to v_m/c, the smallest value of the charge that can be associated with the transition radiation is given by

$$q^2 \approx \frac{\sqrt{2}h\varepsilon_0 c}{\ln\left(\frac{8\pi m D_u c}{h}\right)} \tag{37}$$

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If we substitute for the mass m in Equation (37) the rest mass of an electron, the above equation should give the smallest value of the charge that is associated with a particle having that rest mass. From this we find that the electronic charge is given by

$$q^2 \approx \frac{\sqrt{2}\, h\varepsilon_0 c}{\ln\left(\frac{2D_u}{\alpha a_0}\right)} \tag{38}$$

All the parameters in the above equation are known. A relationship which is qualitatively similar to the one given in Equation (38) was derived previously by Cooray and Cooray [5] by studying the electromagnetic radiation fields of long antennas. Now, the exact size of the universe is not known but the size of the visible universe is about 4 x 10^{26} m. If we substitute these parameters in Equation (38) we obtain $q \approx \pm 1.64 \times 10^{-19}$ C. An exact fit to the electronic charge, i.e. $q = \pm 1.602 \times 10^{-19} C$, is obtained when the size of the universe is about 250 times the visible universe. In this respect, it is encouraging to note that a recent study indicates that the real universe could be actually 250 times larger than the observable universe [6].

9. Conclusions

Expressions for the time domain fields and the associated energy, momentum and action of transition radiation generated when a charged particle incident on a perfectly conducting ground plane are given. The expressions depend on the charge and the speed of propagation of the charged particles. After showing that there is an upper limit, which is less than the speed of light in free space, to the maximum speed that can be achieved by the charged particle, the action of the transition radiation pertinent to this limiting speed is used to estimate the charge of an electron by appealing to the time energy uncertainty principle. The estimated charge lies within a few percent of the electronic charge if the size of the universe is assumed to be equal to the size of the observable universe. If the actual size of the universe is about 250 times larger than the observable universe, the estimated charge coincide exactly with the measured electronic charge.

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