INTEGRAL REPRESENTATIONS OF CATALAN NUMBERS AND THEIR APPLICATIONS

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Abstract. In the paper, the authors survey integral representations of the Catalan numbers and the Catalan–Qi function, discuss equivalent relations between these integral representations, supply alternative and new proofs of several integral representations, collect applications of some integral representations, and present sums of several power series whose coefficients involve the Catalan numbers.

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1. Introduction

The Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in various counting problems in combinatorial mathematics. The nth Catalan number can be expressed in terms of the central binomial coefficients $\binom{2n}{n}$ by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}. \tag{1.1}$$

The Catalan numbers C_n were described in the 18th century by Leonhard Euler and are named after the Belgian mathematician Eugéne Charles Catalan. In 1988, it came to light that the Catalan numbers C_n had been used in China by the Mongolian mathematician Ming Antu by 1730. See [18, 19, 20, 22, 23, 24, 25, 26, 64]. In recent years, the Catalan numbers C_n has been analytically generalized and

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studied in [21, 27, 40, 41, 42, 43, 44, 45, 50, 52, 56, 57, 58, 61, 67, 69, 70] and the closely related references therein. For more information on the Catalan numbers C_n, please refer to the monographs [10, 15, 59, 63] and the closely related references therein.

2. Integral representations of the Catalan numbers

In this section, we recall integral representations of the Catalan numbers C_n and their reciprocals $\frac{1}{C_n}$ and sketch their proofs as possible as we can.

2.1. Penson–Sixdeniers' integral representations in 2001.

In 2001, Penson and Sixdeniers [33] established an integral representation by the Mellin transform.

Theorem 2.1 ([33, p. 2, Eq. (10)]). *For $n \ge 0$, the Catalan numbers C_n can be represented by an integral*

$$C_n = \frac{1}{2\pi}\int_0^4 \sqrt{\frac{4-x}{x}}\, x^n \,\mathrm{d}\, x. \tag{2.1}$$

Proof. We rewrite the proof in [33] as follows. The Mellin transform of a real- or complex-valued function $f(x)$ is defined [32, p. 29, Entry 1.14.32] by

$$\mathscr{M}(f;s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}\, x.$$

If $f(x)$ is continuous on $(0,\infty)$ and $\mathscr{M}(f;\sigma+it)$ is integrable on $(-\infty,\infty)$, then the inverse Mellin transform [32, p. 29, Entry 1.14.35] reads that

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\mathscr{M}(f;s)\,\mathrm{d}\, s.$$

Therefore, it is sufficient to compute the inverse Mellin transform

$$f(x) = \mathscr{M}^{-1}\left[\frac{4^{s-1}\Gamma\bigl(s-\frac{1}{2}\bigr)}{\sqrt{\pi}\,\Gamma(s+1)};x\right] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\frac{4^{s-1}\Gamma\bigl(s-\frac{1}{2}\bigr)}{\sqrt{\pi}\,\Gamma(s+1)}\,\mathrm{d}\, s,$$

where the classical Euler gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\, t, \quad \Re(z)>0. \tag{2.2}$$

From the property

$$\mathscr{M}\bigl(x^b f\bigl(ax^h\bigr);s\bigr) = \frac{1}{h}a^{-(s+b)/h}\mathscr{M}\left(f\left(\frac{x+b}{h}\right);s\right)$$

in [62], it follows immediately that

$$\mathscr{M}\left(\frac{1}{\sqrt{x}}f\left(\frac{x}{4}\right);s\right) = 4^{s-1/2}\mathscr{M}\left(f\left(x-\frac{1}{2}\right);s\right). \tag{2.3}$$

Applying in (2.3) the formula

$$\mathscr{M}\bigl[\bigl(1-x^h\bigr)_+^{\alpha-1};s\bigr] = \frac{1}{h}B\left(\alpha,\frac{s}{h}\right), \quad \Re(\alpha), s>0$$

in [9, p. 1102, Section 12.43, Entry 22] and [28, p. 151, Entry 2.2(1)] to $h=1$ and $\alpha=\frac{3}{2}$ yields

$$f(x) = \frac{1}{\pi\sqrt{x}}\left(1-\frac{x}{4}\right)_+^{1/2}, \tag{2.4}$$

where

$$(y)_+^{\alpha-1} = \begin{cases} y^{\alpha-1}, & y > 0; \\ 0, & y < 0, \end{cases}$$

the classical beta function $B(z,w)$ can be defined by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} \,\mathrm{d}\, t = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}} \,\mathrm{d}\, t = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{2.5}$$

for $\Re(z), \Re(w) > 0$. Then the desired integral representation of C_n is proved. □

Theorem 2.2 ([33, p. 3, Eq. (16)]). *For $n \ge 0$, the sequence $n!C_n$ can be represented by*

$$n!C_n = \frac{(2n)!}{(n+1)!} = \int_0^\infty \left[\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{x}}{2}\right) + \frac{1}{\sqrt{\pi x}}e^{-x/4} - \frac{1}{2}\right] x^n \,\mathrm{d}\, x, \tag{2.6}$$

where $\operatorname{erf}(x)$ denotes the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \,\mathrm{d}\, t. \tag{2.7}$$

Proof. We recite the proof in [33] as follows. This follows from applying the formula

$$\int_0^\infty x^{s-1}\left[\int_0^\infty h(y) f\left(\frac{x}{y}\right)\frac{\mathrm{d}\, y}{y}\right] \mathrm{d}\, x = \mathscr{M}(h;s)\mathscr{M}(f;s)$$

in [32, p. 29, Entries 1.14.39 and 1.14.40] to $h(x) = e^{-x}$ and the function $f(x)$ in (2.4). □

By similar arguments, Penson and Sixdeniers [33] also derived

$$(n!)^2 C_n = \frac{(2n)!}{n+1} = \int_0^\infty \left[\frac{e^{\sqrt{x}}}{\sqrt{x}} + \operatorname{Ei}(-\sqrt{x})\right] x^n \,\mathrm{d}\, x$$

and an integral representation of the sequence $\mathrm{B}_n C_n$, where B_n is the Bell numbers [11, 34, 39, 55] and $\operatorname{Ei}(y)$ is the exponential integral function which can be defined by

$$\operatorname{Ei}(y) = -\int_{-x}^\infty \frac{e^{-t}}{t} \,\mathrm{d}\, t.$$

2.2. **Dana-Picard's integral representations in 2005.** In 2005, using a recurrence relation and the telescopic process, Dana-Picard [7] obtained integral representations for the Catalan numbers C_n and their reciprocals $\frac{1}{C_n}$ respectively.

Theorem 2.3 ([7, Proposition 2.1 and Eq. (9)]). *For $n \ge 0$, the Catalan numbers C_n and their reciprocals $\frac{1}{C_n}$ can be represented by*

$$C_n = \frac{1}{\pi} \int_0^2 x^{2n} \sqrt{4 - x^2} \,\mathrm{d}\, x \tag{2.8}$$

and

$$\frac{1}{C_n} = \frac{(2n+3)(2n+2)(2n+1)}{2^{4n+4}} \int_0^2 x^{2n+1} \sqrt{4-x^2} \,\mathrm{d}\, x. \tag{2.9}$$

Proof. Now we sketch the proof in [7]. Let

$$I_n(a)=\int_0^a x^n\sqrt{a^2-x^2}\,\mathrm{d}\,x,\quad n\ge0. \tag{2.10}$$

Then $I_0=\frac{\pi}{4}a^2$ and

$$I_n(a)=a^2\frac{n-1}{n+2}I_{n-2}(a).$$

Using the telescopic method yields

$$I_{2n}(a)=\pi\left(\frac{a}{2}\right)^{2n+2}\frac{(2n)!}{n!(n+1)!}$$

and

$$I_{2n+1}(a)=a^{2n+3}\frac{2^{2n+1}}{(2n+3)(2n+2)(2n+1)}\frac{n!(n+1)!}{(2n!)}.$$

Substituting (1.1) into the above equations and making use of (2.10) result in

$$C_n=\frac{1}{\pi}\left(\frac{2}{a}\right)^{2n+2}I_{2n}(a)=\frac{1}{\pi}\left(\frac{2}{a}\right)^{2n+2}\int_0^a x^{2n}\sqrt{a^2-x^2}\,\mathrm{d}\,x \tag{2.11}$$

and

$$\begin{aligned}\frac{1}{C_n}&=\frac{1}{a^{2n+3}}\frac{(2n+3)(2n+2)(2n+1)}{2^{2n+1}}I_{2n+1}(a)\\&=\frac{1}{a^{2n+3}}\frac{(2n+3)(2n+2)(2n+1)}{2^{2n+1}}\int_0^a x^{2n+1}\sqrt{a^2-x^2}\,\mathrm{d}\,x.\end{aligned} \tag{2.12}$$

Further setting $a=2$ leads to (2.8) and (2.9) immediately. □

2.3. **Dana-Picard's integral representations in 2010 and 2011.** In 2010, using separately three different substitutions, Dana-Picard [5] established the following integral representations for the Catalan numbers C_n and their reciprocals $\frac{1}{C_n}$.

Theorem 2.4 ([5, Proposition 2.1])**.** *For $n\ge0$, the Catalan numbers C_n and their reciprocals $\frac{1}{C_n}$ can be represented by*

$$C_n=\frac{2^{2n+2}}{\pi}\int_0^1 x^{2n}\sqrt{1-x^2}\,\mathrm{d}\,x \tag{2.13}$$

and

$$\frac{1}{C_n}=\frac{(2n+3)(2n+2)(2n+1)}{2^{2n+1}}\int_0^1 x^{2n+1}\sqrt{1-x^2}\,\mathrm{d}\,x. \tag{2.14}$$

Proof. The sketch of the proof in [5] can be written as follows. For $n\ge0$, let

$$A_n=\int_0^1 x^n\sqrt{1-x^2}\,\mathrm{d}\,x.$$

By the substitution $x=\sin u$ for $u\in\left[0,\frac{\pi}{2}\right]$, we can deduce

$$A_n=S_n-S_{n+2},$$

where

$$S_n=\int_0^{\pi/2}\sin^n u\,\mathrm{d}\,u.$$

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Considering the well-known fact that

$$S_{2p}=\frac{\pi}{2^{2p+1}}\frac{(2p)!}{(p!)^2}$$

and using the expression (1.1) derive

$$A_{2p}=\frac{\pi}{2^{2p+2}}\frac{(2p)!}{p!(p+1)!}=\frac{\pi}{2^{2p+2}}C_p \tag{2.15}$$

and

$$A_{2p+1}=\frac{2^{2p}(p!)^2}{(2p+3)(2p+1)!}=\frac{2^{2p+1}}{(2p+3)(2p+2)(2p+1)}\frac{1}{C_p}.$$

Accordingly, we acquire

$$C_p=\frac{2^{2p+2}}{\pi}A_{2p}=\frac{2^{2p+2}}{\pi}\int_0^1 x^{2p}\sqrt{1-x^2}\,\mathrm{d}\,x$$

and

$$\begin{aligned}\frac{1}{C_p}&=\frac{(2p+3)(2p+2)(2p+1)}{2^{2p+1}}A_{2p+1}\\&=\frac{(2p+3)(2p+2)(2p+1)}{2^{2p+1}}\int_0^1 x^{2p+1}\sqrt{1-x^2}\,\mathrm{d}\,x.\end{aligned}$$

The integral representations (2.13) and (2.14) are thus proved. □

Theorem 2.5 ([5, Proposition 3.1])**.** *For $n\ge0$, the Catalan numbers C_n can be represented by*

$$C_n=\frac{2^{2n+2}}{\pi}\int_0^\infty\frac{u^2}{(1+u^2)^{n+2}}\,\mathrm{d}\,u. \tag{2.16}$$

The outline of the proof in [5]. Using the substitution $u^2=\frac{1}{x^2}-1$ produces

$$A_{2p}=\int_0^\infty\frac{u^2}{(1+u^2)^p}\,\mathrm{d}\,u.$$

Combining this with (2.15) yields (2.16). □

The outline of the proof in [6]. It was stated in [14] that

$$\int_0^{\pi/2}\sin^t x\,\mathrm{d}\,x=\frac{\sqrt{\pi}}{2}\frac{\Gamma\bigl(\frac{t+1}{2}\bigr)}{\Gamma\bigl(\frac{t+2}{2}\bigr)},\quad t>-1. \tag{2.17}$$

See also [36, p. 16, Eq. (2.18)]. Then it is not difficult to obtain

$$P_n=\int_0^1\frac{x^n}{\sqrt{1-x^2}}\,\mathrm{d}\,x=\int_0^{\pi/2}\sin^n u\,\mathrm{d}\,u=\begin{cases}\dfrac{\pi(2p)!}{2^{2p+1}(p!)^2}, & n=2p;\\[2mm] \dfrac{2^{2p}(p!)^2}{(2p+1)!}, & n=2p+1.\end{cases}$$

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On the other hand, using three irrational substitutions $u^2 = \frac{1}{x^2} - 1$, $u^2 = 1 - x^2$, and $u = \sqrt{\frac{1-x}{1+x}}$ to compute I_n produces

$$\begin{aligned} P_n &= \int_0^\infty \left(1+u^2\right)^{-(n+2)/2} \,\mathrm{d}\, u \\ &= \int_0^1 \left(1-u^2\right)^{(n-1)/2} \,\mathrm{d}\, u \\ &= 2\int_0^1 \frac{(1-u^2)^n}{(1+u^2)^{n+1}} \,\mathrm{d}\, u \end{aligned} \tag{2.18}$$

respectively. By similar argument to the proof of Theorem 2.4 and by the first formula in (2.18), the integral representation (2.16) is verified once again. □

A new proof the formula (2.16). In [9, p. 325], the fourth formula reads that

$$\int_0^\infty \frac{x^{\mu-1}}{(p+qx^\nu)^{n+1}} \,\mathrm{d}\, x = \frac{1}{\nu p^{n+1}} \left(\frac{p}{q}\right)^{\mu/\nu} B\left(1+n-\frac{\mu}{\nu}, \frac{\mu}{\nu}\right)$$

for $0 < \frac{\mu}{\nu} < n+1$ and $p, q \ne 0$. Setting $p = q = 1$, $\mu = 3$, and $\nu = 2$ and replacing n by $n+1$ find

$$\int_0^\infty \frac{x^2}{(1+x^2)^{n+2}} \,\mathrm{d}\, x = \frac{1}{2} B\left(\frac{2n+1}{2}, \frac{3}{2}\right) = \frac{\pi}{2^{2n+2}} C_n,$$

where we used in the last step the observation

$$C_n = \frac{1}{\pi} 2^{2n+1} B\left(n+\frac{1}{2}, \frac{3}{2}\right) \tag{2.19}$$

in [38, Remark 6.2, Eq. (6.1)]. The formula (2.16) is thus proved. □

Theorem 2.6 ([5, Proposition 4.1])**.** *For $n \ge 0$, the Catalan numbers C_n can be represented by*

$$C_n = \frac{2^{2n+5}}{\pi} \int_0^1 \frac{u^2(1-u^2)^{2n}}{(1+u^2)^{2n+3}} \,\mathrm{d}\, u. \tag{2.20}$$

The outline of the proof in [5]. Taking the substitution $u = \sqrt{\frac{1-x}{1+x}}$ concludes

$$A_n = 8\int_0^1 \frac{u^2(1-u^2)^2}{(1+u^2)^{n+3}} \,\mathrm{d}\, u.$$

Combining this for even n with (2.15), we derive the integral presentation (2.20) immediately. □

The outline of the proof in [6]. By same argument as in the proof of Theorem 2.4 and by the third formula in (2.18), the integral representation (2.20) is verified once again. □

2.4. **Dana-Picard–Zeitoun–Qi's integral representations in 2012 and 2016.** In 2012, Dana-Picard and Zeitoun [8] deduced an integral representation for the Catalan numbers C_n, which was corrected and developed by Qi [35] as the following integral representations.

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Theorem 2.7 ([8, Corollary 3.2] and [35, Theorem 3.1]). *For $n \ge 0$ and $a > 0$, the Catalan numbers C_n can be represented by*

$$\begin{aligned} C_n &= \frac{1}{\pi}\frac{4^n}{n+1}\frac{1}{a^{2n+1}}\int_{-a}^{a} x^{2n}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x \\ &= \frac{1}{\pi}\frac{2^{2n+1}}{n+1}\frac{1}{a^{2n}}\int_{0}^{a} \frac{x^{2n}}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x \\ &= \frac{1}{\pi}\frac{2^{2n+1}}{n+1}\int_{0}^{\pi/2} \sin^{2n} x\,\mathrm{d}\,x \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} C_n &= \frac{1}{\pi}\frac{2^{2n+1}}{2n+1}\frac{1}{a^{2n+2}}\int_{-a}^{a} x^{2n+1}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x \\ &= \frac{1}{\pi}\frac{2^{2n+2}}{2n+1}\frac{1}{a^{2n+2}}\int_{0}^{a} \frac{x^{2n+2}}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x \\ &= \frac{1}{\pi}\frac{2^{2n+2}}{2n+1}\int_{0}^{\pi/2} \sin^{2n+2} x\,\mathrm{d}\,x. \end{aligned} \tag{2.22}$$

Proof. We sketch the proof in [35]. Let a be a positive number. For $n \ge 0$, define

$$J_n = \int_{-a}^{a} x^n\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x. \tag{2.23}$$

Then

$$J_n = \frac{1}{2}a^{n+1}\biggl([1+(-1)^n]B\biggl(\frac{1}{2},\frac{n+1}{2}\biggr) + [1+(-1)^{n+1}]B\biggl(\frac{1}{2},\frac{n+2}{2}\biggr)\biggr) \tag{2.24}$$

and

$$J_n = a^{n+1}\pi\left[\frac{1+(-1)^n}{n}\frac{1}{B\bigl(\frac{1}{2},\frac{n}{2}\bigr)} + \frac{1+(-1)^{n+1}}{n+1}\frac{1}{B\bigl(\frac{1}{2},\frac{n+1}{2}\bigr)}\right]. \tag{2.25}$$

The Catalan numbers C_n can be expressed in terms of the beta function $B(x,y)$ by

$$C_n = \frac{1}{\pi}\frac{4^n}{n+1}B\biggl(\frac{1}{2},n+\frac{1}{2}\biggr). \tag{2.26}$$

Taking $n = 2p$ in (2.24) and utilizing (2.26) lead to

$$J_{2p} = a^{2p+1}B\biggl(\frac{1}{2},\frac{2p+1}{2}\biggr) = a^{2p+1}\pi\frac{p+1}{4^p}C_n$$

which is equivalent to

$$C_n = \frac{4^n}{n+1}\frac{1}{a^{2n+1}\pi}J_{2n} = \frac{1}{\pi}\frac{4^n}{n+1}\frac{1}{a^{2n+1}}\int_{-a}^{a} x^{2n}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x.$$

The first formula (2.21) thus follows.

By similar argument to the deduction of (2.26), we can discover

$$C_n = \frac{4^{n+1}}{(2n+1)(2n+2)}\frac{1}{B\bigl(\frac{1}{2},n+1\bigr)}, \quad n \ge 0.$$

Employing this identity and setting $n = 2p+1$ in (2.25) figures out

$$J_{2p+1} = a^{2p+2}\frac{2\pi}{2p+2}\frac{1}{B\bigl(\frac{1}{2},p+1\bigr)} = a^{2p+2}\frac{2\pi}{2p+2}\frac{(2p+1)(2p+2)}{4^{p+1}}C_p$$

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which can be rearranged as

$$C_p = \frac{1}{a^{2p+2}}\frac{1}{\pi}\frac{2^{2p+1}}{2p+1}J_{2p+1} = \frac{1}{\pi}\frac{1}{a^{2p+2}}\frac{2^{2p+1}}{2p+1}\int_{-a}^{a} x^{2p+1}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x.$$

The first formula in (2.22) is thus proved.

The rest integral representations follow from mathematical techniques and changing variable of integration. □

2.5. **Shi–Liu–Qi's integral representation in 2015.** In 2015, by virtue of an integral representation of the gamma function $\Gamma(x)$, Shi, Liu, and Qi [61] established an integral representation for the Catalan function

$$C_x = \frac{4^x\Gamma(x+1/2)}{\sqrt{\pi}\,\Gamma(x+2)}, \quad x>0.$$

Theorem 2.8 ([61, Theorem 1])**.** *For $x \ge 0$, the Catalan function C_x can be represented by*

$$C_x = \frac{e^{3/2}4^x(x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}}\exp\left[\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)\frac{e^{-t/2}-e^{-2t}}{t}e^{-xt}\,\mathrm{d}\,t\right]. \quad (2.27)$$

Proof. This follows straightforwardly from applying the well-known formula

$$\ln\Gamma(z) = \ln\left(\sqrt{2\pi}\,z^{z-1/2}e^{-z}\right) + \int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)e^{-zt}\,\mathrm{d}\,t, \quad \Re(z)>0$$

in [65, (3.22)] to the logarithm of the Catalan function C_x. □

2.6. **Qi–Shi–Liu's integral representations in 2015.** In 2015, by virtue of the Cauchy integral formula in the theory of complex functions, Qi and his two graduates, Shi and Liu, find an integral representation of the generating function $\frac{1}{1+\sqrt{1-4x}}$ for the Catalan numbers C_n. Consequently, they derived an integral representation of the Catalan numbers C_n.

Theorem 2.9 ([54, Theorem 1.4])**.** *The Catalan numbers C_n for $n \ge 0$ can be represented by*

$$C_n = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}}\,\mathrm{d}\,t = \frac{2}{\pi}\int_0^\infty \frac{t^2}{(t^2+1/4)^{n+2}}\,\mathrm{d}\,t. \quad (2.28)$$

Proof. The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^\infty C_n x^n. \quad (2.29)$$

By virtue of the Cauchy integral formula in the theory of complex functions, we discover

$$\frac{1}{1+\sqrt{1-4x}} = \frac{1}{2\pi}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)(t-x+1/4)}\,\mathrm{d}\,t$$

for $x\in\left(-\infty,\frac{1}{4}\right]$. Therefore, it follows that

$$\begin{aligned}C_n &= \frac{1}{n!}\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\frac{2}{1+\sqrt{1-4x}}\\ &= \frac{1}{\pi}\frac{1}{n!}\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\int_0^\infty \frac{\sqrt{t}}{(t+1/4)(t-x+1/4)}\,\mathrm{d}\,t\end{aligned}$$

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$$= \frac{1}{\pi} \int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}} \,\mathrm{d}\, t.$$

Further using the substitution $\sqrt{t} = s$ yields the second integral representation in (2.28). The theorem is thus proved. □

2.7. **Qi's integral representations in 2017.**

Theorem 2.10 ([38, Theorem 3.1 and Remark 6.6])**.** *The Catalan numbers C_n for $n \ge 0$ can be represented by*

$$C_n = \frac{2}{\pi(n+1)} \int_0^2 \frac{x^{2n}}{\sqrt{4-x^2}} \,\mathrm{d}\, x = \frac{2^{2n+1}}{\pi} \int_0^1 \sqrt{\frac{1-t}{t}}\, t^n \,\mathrm{d}\, t. \tag{2.30}$$

Proof. Using the substitution $x = a\sin s$ for $s \in \left[0, \frac{\pi}{2}\right]$ and employing (2.17) for $t = n \ge 0$ reveal

$$I_n(a) = a^{n+2} \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{4\Gamma\left(\frac{n}{2}+2\right)} \tag{2.31}$$

for $a > 0$ and $n \ge 0$. Differentiating with respect to a on both sides of (2.10) gives

$$I_n'(a) = a \int_0^a \frac{x^n}{\sqrt{a^2-x^2}} \,\mathrm{d}\, x. \tag{2.32}$$

On the other hand, differentiating with respect to a on both sides of (2.31) results in

$$I_n'(a) = \frac{\sqrt{\pi}}{4}(n+2)a^{n+1} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}. \tag{2.33}$$

Combining (2.32) with (2.33) and simplifying lead to

$$\int_0^a \frac{x^n}{\sqrt{a^2-x^2}} \,\mathrm{d}\, x = \sqrt{\pi}\, a^n \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} \tag{2.34}$$

for $a > 0$ and $n \ge 0$. The first representation in (2.30) follows from combining

$$C_n = \frac{4^n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad n \ge 0 \tag{2.35}$$

in [15, p. 112, Eq. (5.5)] with (2.34).

The second integral representation in (2.30) follows immediately from combining (2.5) and (2.19). The desired proof is complete. □

2.8. **Qi–Akkurt–Yildirim's integral representation.**

In [40, Theorem 1.1], an integral representation

$$C_n = \frac{k 2^{1+2n(1-k)}}{\pi(n+1)} \int_0^2 \frac{x^{(2n+1)k-1}}{\sqrt{2^{2k}-x^{2k}}} \,\mathrm{d}\, x \tag{2.36}$$

for $k > 0$ and $n \in \mathbb{N}$ was established.

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3. The Catalan–Qi function and its integral representations

In 2015, Qi and his coauthors generalized in [53, Remark 1] and its formally published version [58, Eq. (9)] the Catalan numbers C_n as the so-called Catalan–Qi function

$$C(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(z+a)}{\Gamma(z+b)},\quad \Re(a),\Re(b)>0,\quad \Re(z)\ge 0. \tag{3.1}$$

It is clear that

$$C(b,a;z)=\frac{1}{C(a,b;z)}. \tag{3.2}$$

When taking $x=n\in\{0\}\cup\mathbb{N}$, we call the quantities $C(a,b;n)$ the Catalan–Qi numbers. It is easy to see that

$$C\left(\frac{1}{2},2;n\right)=C_n \quad\text{and}\quad C(a,b;n)=\left(\frac{b}{a}\right)^n\frac{(a)_n}{(b)_n} \tag{3.3}$$

for all $n\ge 0$, where

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge 1\\ 1, & n=0\end{cases}$$

is called the rising factorial or the Pochhammer symbol.

It is well known that the Wallis ratio is defined by

$$W_n=\frac{(2n-1)!!}{(2n)!!}=\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1}{\sqrt{\pi}}\frac{\Gamma\bigl(n+1/2\bigr)}{\Gamma(n+1)},\quad n\in\mathbb{N}.$$

Hence, it is easy to see that

$$C_n=\frac{4^n}{n+1}W_n.$$

The Wallis ratio, or say, the ratio of two gamma functions, has been studied and applied by many mathematicians, see [12, 36, 37, 46, 47, 48, 49, 51], for example, and plenty of literature therein.

Now we are in a position to recall and to alternatively prove three integral representations of the Catalan–Qi function $C(a,b;x)$ as follows.

Theorem 3.1 ([50, Eq. (10)])**.** *For $b>a>0$ and $x\ge 0$, the Catalan–Qi function $C(a,b;x)$ has the integral representation*

$$C(a,b;x)=\frac{1}{B(a,b-a)}\left(\frac{b}{a}\right)^x\int_0^\infty\bigl(1-e^{-u}\bigr)^{b-a-1}e^{-(x+a)u}\,\mathrm{d}\,u. \tag{3.4}$$

Proof. This follows from combination of the definition (3.1) and the integral formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=\frac{1}{\Gamma(b-a)}\int_0^\infty\bigl(1-e^{-u}\bigr)^{b-a-1}e^{-(z+a)u}\,\mathrm{d}\,u,\quad b>a\ge 0$$

in [65, p. 67] for the ratio of two gamma functions $\Gamma(z+a)$ and $\Gamma(z+b)$. □

Theorem 3.2 ([50, Theorem 4])**.** *For $b>a>0$ and $x\ge 0$, the Catalan–Qi function $C(a,b;x)$ has integral representations*

$$C(a,b;x)=\left(\frac{a}{b}\right)^{b-1}\frac{1}{B(a,b-a)}\int_0^{b/a}\left(\frac{b}{a}-t\right)^{b-a-1}t^{x+a-1}\,\mathrm{d}\,t \tag{3.5}$$

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and

$$C(a,b;x)=\left(\frac{a}{b}\right)^a\frac{1}{B(a,b-a)}\int_0^\infty\frac{t^{b-a-1}}{(t+a/b)^{x+b}}\,\mathrm{d}\,t. \tag{3.6}$$

An alternative proof. Making use of the last formula in (2.5) and the definition (3.1), we can rewritten the Catalan–Qi function $C(a,b;x)$ as

$$C(a,b;x)=\left(\frac{b}{a}\right)^x\frac{B(b,x+a)}{B(a,x+b)}$$

and

$$C(a,b;x)=\left(\frac{b}{a}\right)^x\frac{B(x+a,b-a)}{B(a,b-a)}. \tag{3.7}$$

Applying (2.5) into the factor $B(x+a,b-a)=B(b-a,x+a)$ in (3.7) leads to

$$\begin{aligned}C(a,b;x)&=\left(\frac{b}{a}\right)^x\frac{1}{B(a,b-a)}\int_0^1 t^{x+a-1}(1-t)^{b-a-1}\,\mathrm{d}\,t\\&=\left(\frac{b}{a}\right)^x\frac{1}{B(a,b-a)}\int_0^{b/a}\left(\frac{a}{b}s\right)^{x+a-1}\left[1-\left(\frac{a}{b}s\right)\right]^{b-a-1}\mathrm{d}\left(\frac{a}{b}s\right)\\&=\left(\frac{a}{b}\right)^{b-1}\frac{1}{B(a,b-a)}\int_0^{b/a}\left(\frac{b}{a}-s\right)^{b-a-1}s^{x+a-1}\,\mathrm{d}\,s\end{aligned}$$

and

$$\begin{aligned}C(a,b;x)&=\left(\frac{b}{a}\right)^x\frac{1}{B(a,b-a)}\int_0^\infty\frac{t^{b-a-1}}{(1+t)^{x+b}}\,\mathrm{d}\,t\\&=\left(\frac{b}{a}\right)^x\frac{1}{B(a,b-a)}\int_0^\infty\frac{(bs/a)^{b-a-1}}{(1+bs/a)^{x+b}}\,\mathrm{d}\left(\frac{b}{a}s\right)\\&=\left(\frac{a}{b}\right)^a\frac{1}{B(a,b-a)}\int_0^\infty\frac{s^{b-a-1}}{(s+a/b)^{x+b}}\,\mathrm{d}\,s\end{aligned}$$

respectively. The proof of Theorem 3.2 is thus complete. □

4. Discussing various integral representations

In this section, we will discuss various integral representations recalled and proved above.

4.1. Discussing (2.1). Applying the substitution $x=4t$ in (2.1), rearranging, and employing the first definition in (2.5) yield

$$\begin{aligned}C_n&=\frac{1}{2\pi}\int_0^1\sqrt{\frac{4-4t}{4t}}\,(4t)^n\,\mathrm{d}(4t)\\&=\frac{2^{2n+1}}{\pi}\int_0^1(1-t)^{1/2}t^{n-1/2}\,\mathrm{d}\,t\\&=\frac{2^{2n+1}}{\pi}B\left(\frac{3}{2},n+\frac{1}{2}\right).\end{aligned}$$

On the other hand, letting $a=\frac{1}{2}$, $b=2$, and $x=n\ge0$ in (3.7) and considering the first relation in (3.3) give

$$C_n=4^n\frac{1}{B\left(\frac{1}{2},\frac{3}{2}\right)}B\left(n+\frac{1}{2},\frac{3}{2}\right)=\frac{2^{2n+1}}{\pi}B\left(\frac{3}{2},n+\frac{1}{2}\right).$$

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As a result, the integral representation (2.1) is a special case of the integral representation (3.5). This can also be verified simpler by taking $a=\frac{1}{2}$, $b=2$, and $x=n\ge 0$ in (3.5).

4.2. **Discussing** (2.6)**.** By (2.35) and $\Gamma(n+1)=n!$, we obtain

$$n!C_n=n!\frac{4^n\Gamma\bigl(n+\frac{1}{2}\bigr)}{\sqrt{\pi}\,(n+1)!}=\frac{4^n\Gamma\bigl(n+\frac{1}{2}\bigr)}{\sqrt{\pi}\,(n+1)}.$$

Combining this with (2.6) and simplifying give

$$\begin{aligned}\Gamma\biggl(n+\frac{1}{2}\biggr)&=\sqrt{\pi}\,(n+1)\int_0^\infty\biggl[\frac{1}{2}\operatorname{erf}\biggl(\frac{\sqrt{x}}{2}\biggr)+\frac{1}{\sqrt{\pi x}}e^{-x/4}-\frac{1}{2}\biggr]\biggl(\frac{x}{4}\biggr)^n\,\mathrm{d}\,x\\&=2\sqrt{\pi}\,(n+1)\int_0^\infty\biggl[\operatorname{erf}\bigl(\sqrt{t}\,\bigr)+\frac{e^{-t}}{\sqrt{\pi t}}-1\biggr]t^n\,\mathrm{d}\,t.\end{aligned}$$

Hence, we guess that

$$\Gamma\biggl(x+\frac{1}{2}\biggr)=2\sqrt{\pi}\,(x+1)\int_0^\infty\biggl[\operatorname{erf}\bigl(\sqrt{t}\,\bigr)+\frac{e^{-t}}{\sqrt{\pi t}}-1\biggr]t^x\,\mathrm{d}\,t,\quad x>-\frac{1}{2}$$

which is equivalent to

$$\Gamma(x)=\sqrt{\pi}\,(2x+1)\int_0^\infty\biggl[\operatorname{erf}\bigl(\sqrt{t}\,\bigr)+\frac{e^{-t}}{\sqrt{\pi t}}-1\biggr]t^{x-1/2}\,\mathrm{d}\,t,\quad x>0.$$

Actually, this can be derived from

$$\int_0^\infty\frac{e^{-t}}{\sqrt{\pi t}}t^{x-1/2}\,\mathrm{d}\,t=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}\,t=\frac{\Gamma(x)}{\sqrt{\pi}},\quad x>0,$$

by the definition (2.2), and

$$\begin{aligned}\int_0^\infty\bigl[\operatorname{erf}\bigl(\sqrt{t}\,\bigr)-1\bigr]t^{x-1/2}\,\mathrm{d}\,t&=\frac{1}{x+1/2}\int_0^\infty\bigl[\operatorname{erf}\bigl(\sqrt{t}\,\bigr)-1\bigr]\frac{\mathrm{d}}{\mathrm{d}\,t}t^{x+1/2}\,\mathrm{d}\,t\\=-\frac{1}{x+1/2}\int_0^\infty\bigl[\operatorname{erf}\bigl(\sqrt{t}\,\bigr)-1\bigr]'t^{x+1/2}\,\mathrm{d}\,t&=-\frac{1}{x+1/2}\int_0^\infty\frac{e^{-t}}{\sqrt{\pi}\,\sqrt{t}}t^{x+1/2}\,\mathrm{d}\,t\\&=-\frac{2}{\sqrt{\pi}\,(2x+1)}\int_0^\infty e^{-t}t^x\,\mathrm{d}\,t=-\frac{2\Gamma(x+1)}{\sqrt{\pi}\,(2x+1)},\quad x>-\frac{1}{2},\end{aligned}$$

by integration by part and the definition (2.7). In a word, we proved the integral representation (2.6) alternatively.

4.3. **Discussing Theorems 2.3 and 2.4.** By the substitution $x=2t$, the integral representations (2.8) and (2.9) reduce to (2.13) and (2.14). This can also be showed by letting $a=1$ in (2.8) and (2.9). Consequently, the integral representations (2.8) and (2.9) are respectively equivalent to (2.13) and (2.14).

By the substitution $x=\sqrt{t}$ in (2.13) and by the first definition in (2.5), we obtain

$$\begin{aligned}C_n&=\frac{2^{2n+2}}{\pi}\int_0^1 t^n\sqrt{1-t}\,\frac{1}{2\sqrt{t}}\,\mathrm{d}\,t\\&=\frac{2^{2n+1}}{\pi}\int_0^1 t^{n-1/2}\sqrt{1-t}\,\mathrm{d}\,t=\frac{2^{2n+1}}{\pi}B\biggl(n+\frac{1}{2},\frac{3}{2}\biggr).\end{aligned}$$

Accordingly, the integral representation (2.13) is a special case of the integral representation (3.5) and is equivalent to (2.1).

Similarly, by the substitution $x=\sqrt{t}$ in (2.14) and by the first definition in (2.5), we acquire

$$\begin{aligned} \frac{1}{C_n} &= \frac{(2n+3)(2n+2)(2n+1)}{2^{2n+2}} \int_0^1 t^n\sqrt{1-t}\,\mathrm{d}\,t \\ &= \frac{(2n+3)(2n+2)(2n+1)}{2^{2n+2}} B\left(n+1,\frac{3}{2}\right). \end{aligned} \tag{4.1}$$

This implies that the integral representations (2.9) and (2.14) for reciprocals of the Catalan numbers C_n can be alternatively verified by using (2.35) and (2.5) in sequence as follows:

$$\begin{aligned} \frac{1}{C_n} &= \frac{\sqrt{\pi}\,\Gamma(n+2)}{4^n\Gamma\left(n+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\,(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\Gamma(n+1)\Gamma\left(\frac{3}{2}\right)}{4^n\Gamma\left(\frac{3}{2}\right)\Gamma\left(n+\frac{5}{2}\right)} \\ &= \frac{(2n+3)(2n+2)(2n+1)}{2^{2n+2}} B\left(n+1,\frac{3}{2}\right) \\ &= \frac{(2n+3)(2n+2)(2n+1)}{2^{2n+2}} \int_0^1 t^n\sqrt{1-t}\,\mathrm{d}\,t \\ &= \frac{(2n+3)(2n+2)(2n+1)}{2^{2n+1}} \int_0^1 x^{2n+1}\sqrt{1-x^2}\,\mathrm{d}\,x \\ &= \frac{(2n+3)(2n+2)(2n+1)}{2^{4n+4}} \int_0^2 x^{2n+1}\sqrt{4-x^2}\,\mathrm{d}\,x \\ &= \frac{(2n+3)(2n+2)(2n+1)}{2^{2n+1}} \frac{1}{a^{2n+3}} \int_0^a x^{2n+1}\sqrt{a^2-x^2}\,\mathrm{d}\,x \\ &= \frac{(2n+3)(2n+2)(2n+1)}{2^{2n+2}} \frac{1}{a^{n+3/2}} \int_0^a t^n\sqrt{a-t}\,\mathrm{d}\,t \end{aligned}$$

for $a>0$ and $n\ge 0$.

4.4. **Discussing** (2.16)**.** Using the substitution $u=\sqrt{t}$ in (2.16) and considering the second expression (2.5) produce

$$\begin{aligned} \frac{2^{2n+2}}{\pi}\int_0^\infty \frac{u^2}{(1+u^2)^{n+2}}\,\mathrm{d}\,u &= \frac{2^{2n+2}}{\pi}\int_0^\infty \frac{t}{(1+t)^{n+2}}\frac{1}{2\sqrt{t}}\,\mathrm{d}\,t \\ &= \frac{2^{2n+1}}{\pi}\int_0^\infty \frac{t^{1/2}}{(1+t)^{n+2}}\,\mathrm{d}\,t = \frac{2^{2n+1}}{\pi} B\left(\frac{3}{2},n+\frac{1}{2}\right) = C_n. \end{aligned}$$

Hence, the integral representation (2.16) is proved once again.

4.5. **Discussing** (2.20)**.** Letting $t=\frac{1-u^2}{1+u^2}$ in the integral of (2.20) gives

$$\begin{aligned} \int_0^1 \frac{u^2(1-u^2)^{2n}}{(1+u^2)^{2n+3}}\,\mathrm{d}\,u &= \int_0^1 \frac{1-t}{1+t}\left(\frac{1+t}{2}\right)^3 t^{2n}\frac{1}{(1+t)^2}\sqrt{\frac{1+t}{1-t}}\,\mathrm{d}\,t \\ &= \frac{1}{8}\int_0^1 t^{2n}\sqrt{1-t^2}\,\mathrm{d}\,t = \frac{1}{16}\int_0^1 s^{n-1/2}\sqrt{1-s}\,\mathrm{d}\,s \\ &= \frac{1}{16}B\left(\frac{3}{2},n+\frac{1}{2}\right) = \frac{1}{16}\frac{\pi}{2^{2n+1}}C_n = \frac{\pi}{2^{2n+5}}C_n. \end{aligned}$$

The integral representation (2.20) is thus proved again.

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4.6. **Discussing** (2.27)**.** Currently we do not find any application of the integral representation (2.27) and do not derive any property of the Catalan numbers C_n from the integral representation (2.27).

4.7. **Discussing** (2.28)**.** By the substitution $t=\frac{u}{4}$ in the first integral of (2.28) and comparing with the second integral in (2.5) gives

$$\begin{aligned}\int_0^\infty \frac{\sqrt{t}\,}{(t+1/4)^{n+2}}\,\mathrm{d}\,t&=\frac14\int_0^\infty \frac{\sqrt{u/4}\,}{(u/4+1/4)^{n+2}}\,\mathrm{d}\,u\\&=2^{2n+1}\int_0^\infty \frac{\sqrt{u}\,}{(1+u)^{n+2}}\,\mathrm{d}\,u=2^{2n+1}B\biggl(\frac32,n+\frac12\biggr)=\pi C_n.\end{aligned}$$

Thus, the integral representations in (2.28) are proved alternatively.

When changing the variable of integration by $t=\frac{u}{2}$ in the last representation in (2.28), we can recover the integral representation (2.16).

4.8. **Discussing** (2.30)**.** The first integral in (2.30) can be computed as

$$\begin{aligned}\int_0^2\frac{x^{2n}}{\sqrt{4-x^2}\,}\,\mathrm{d}\,x&=\int_0^2\frac{\bigl(2\sqrt{t}\,\bigr)^{2n}}{\sqrt{4-\bigl(2\sqrt{t}\,\bigr)^2}\,}\,\mathrm{d}\bigl(2\sqrt{t}\,\bigr)=2^{2n}\int_0^1\frac{t^n}{\sqrt{1-t}\,}\frac{1}{2\sqrt{t}\,}\,\mathrm{d}\,t\\&=2^{2n-1}\int_0^1 t^{n-1/2}(1-t)^{-1/2}\,\mathrm{d}\,t=2^{2n-1}B\biggl(n+\frac12,\frac12\biggr).\end{aligned}$$

Then from (2.26) it follows that

$$\int_0^2\frac{x^{2n}}{\sqrt{4-x^2}\,}\,\mathrm{d}\,x=2^{2n-1}\pi\frac{n+1}{4^n}C_n$$

which can be rewritten as (2.30).

4.9. **Discussing** (2.36)**.** The first integral representation (2.30) is a special case of the one (2.36). Actually, the paper [40] was motivated by the article [38].

4.10. **Discussing** (3.4)**.** By the substitution $e^{-u}=t$ in (3.4) and by the first integral in (2.5), we can see that the expressions (3.4) and (3.7) are equivalent to each other.

4.11. **Discussing** (3.5) **and** (3.6)**.** When $a=\frac12$, $b=2$, and $x=n\ge0$, the integral representations (3.5) and (3.6) reduce to (2.1) and (2.28) respectively.

Letting $a=\frac12$, $b=2$, and $x=n\ge0$ in (3.7) results in the expression (2.19).

4.12. **The beta function and reciprocals of the Catalan numbers.** By (2.35), the identity $\Gamma\bigl(\frac12\bigr)=\sqrt{\pi}\,$, and the recurrence relation $\Gamma(x+1)=x\Gamma(x)$, it is easy to see that

$$\begin{aligned}\frac{1}{C_n}=\frac{\sqrt{\pi}\,\Gamma(n+2)}{4^n\Gamma\bigl(n+\frac12\bigr)}&=\frac{\bigl(n+\frac12\bigr)(n+1)}{4^n}\frac{\Gamma\bigl(\frac12\bigr)\Gamma(n+1)}{\Gamma\bigl(n+\frac32\bigr)}\\&=\frac{(2n+1)(n+1)}{2^{2n+1}}B\biggl(\frac12,n+1\biggr)\end{aligned}\tag{4.2}$$

which is different from the one in (4.1). Indeed, the Catalan numbers C_n and their reciprocals $\frac{1}{C_n}$ can also be represented in terms of the beta functions $B\bigl(n+\ell-\frac12,m+\frac12\bigr)$ and $B\bigl(n+\ell,m+\frac12\bigr)$ for $\ell,m\in\mathbb{N}$ respectively.

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5. Applications of integral representations

Most of the above integral representations can be applied to discover properties of the Catalan numbers C_n. Now we recall some known applications of several integral representations of the Catalan numbers C_n.

5.1. The integral representation (2.1) was applied in the proof of [42, Theorem 5.1] to discover the identity

$$\sum_{\ell=0}^{\lfloor(j-1)/2\rfloor}(-1)^\ell\binom{j-\ell-1}{\ell}C_{i-\ell-1}=\frac{j}{i}\binom{2i-j-1}{i-1},\quad i\ge j\ge 1.$$

This identity generalizes

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}C_{n-k}=1$$

obtained in [68, p. 2187, Theorem 2, Eq. (15b)].

5.2. The representation (2.20) was applied in [31, p. 10] to compute several infinite series whose general terms involve binomial coefficients.

5.3. Recall from [30, pp. 372–373] and [66, p. 108, Definition 4] that a sequence $\{\mu_n\}_{0\le n\le\infty}$ is said to be completely monotonic if its elements are non-negative and its successive differences are alternatively non-negative, that is,

$$(-1)^k\Delta^k\mu_n\ge 0$$

for $n,k\ge 0$, where

$$\Delta^k\mu_n=\sum_{m=0}^{k}(-1)^m\binom{k}{m}\mu_{n+k-m}.$$

Recall from [66, p. 163, Definition 14a] that a completely monotonic sequence $\{a_n\}_{n\ge0}$ is minimal if it ceases to be completely monotonic when a_0 is decreased.

Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n$ and $\mu=(\mu_1,\mu_2,\dots,\mu_n)\in\mathbb{R}^n$. A sequence λ is said to be majorized by μ (in symbols $\lambda\preceq\mu$) if

$$\sum_{\ell=1}^{k}\lambda_{[\ell]}\le\sum_{\ell=1}^{k}\mu_{[\ell]},\quad k=1,2,\dots,n-1\quad\text{and}\quad\sum_{\ell=1}^{n}\lambda_\ell=\sum_{\ell=1}^{n}\mu_\ell,$$

where $\lambda_{[1]}\ge\lambda_{[2]}\ge\dots\ge\lambda_{[n]}$ and $\mu_{[1]}\ge\mu_{[2]}\ge\dots\ge\mu_{[n]}$ are respectively the components of λ and μ in decreasing order. A sequence λ is said to be strictly majorized by μ (in symbols $\lambda\prec\mu$) if λ is not a permutation of μ. For example,

$$\Bigg(\underbrace{\frac1n,\dots,\frac1n}_{n}\Bigg)\prec\Bigg(\underbrace{\frac1{n-1},\dots,\frac1{n-1}}_{n-1},0\Bigg)\prec\Bigg(\underbrace{\frac1{n-2},\dots,\frac1{n-2}}_{n-2},0,0\Bigg)\prec\cdots\\
\prec\bigg(\frac13,\frac13,\frac13,0,\dots,0\bigg)\prec\bigg(\frac12,\frac12,0,\dots,0\bigg)\prec(1,0,\dots,0).$$

For more information on the theory of majorization and its applications, please refer to monographs [13, 29] and the closely related references therein.

Applying the integral representation (2.28), we can obtain properties and inequalities of the Catalan numbers C_n. Some of them can be recited as follows.

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Theorem 5.1 ([54, Theorem 1.4]). *The sequence* $\left\{\frac{C_n}{4^n}\right\}_{n\ge0}$ *is completely monotonic and minimal.*

Theorem 5.2 ([54, Theorem 1.4]). *If* $m\ge1$ *and* $a_0,a_1,\dots,a_m$ *be non-negative integers, then*

$$\left(\frac{C_{a_0}}{4^{a_0}}\right)^{m-1}\frac{C_{\sum_{k=0}^m a_k}}{4^{\sum_{k=0}^m a_k}}\ge\prod_{k=1}^m\frac{C_{a_0+a_k}}{4^{a_0+a_k}} \tag{5.1}$$

and

$$\left|\frac{C_{a_i+a_j}}{4^{a_i+a_j}}\right|_m\ge0, \tag{5.2}$$

where $|e_{kj}|_m$ *denotes a determinant of order* m *with elements* e_{kj}*.*

Theorem 5.3 ([54, Theorem 1.5]). *Let* $m\in\mathbb{N}$ *and let* n *and* a_k *for* $1\le k\le m$ *be non-negative integers. Then the Catalan numbers* C_n *satisfy*

$$\left|(-1)^{a_i+a_j}\mathcal{C}_{n+a_i+a_j}\right|_m\ge0 \tag{5.3}$$

and

$$\left|\mathcal{C}_{n+a_i+a_j}\right|_m\ge0, \tag{5.4}$$

where

$$\mathcal{C}_\ell=\ell!C_\ell,\quad \ell\ge0. \tag{5.5}$$

Theorem 5.4 ([54, Theorem 1.6]). *Let* $m\in\mathbb{N}$ *and let* λ *and* μ *be two* m*-tuples of non-negative integers such that* $\lambda\preceq\mu$*. Then*

$$\left|\prod_{i=1}^m\mathcal{C}_{n+\lambda_i}\right|\le\left|\prod_{i=1}^m\mathcal{C}_{n+\mu_i}\right|, \tag{5.6}$$

where $\mathcal{C}_\ell$ *is defined by* (5.5)*. Consequently,*

1. *the infinite sequence* $\{\mathcal{C}_n\}_{n\ge0}$ *is logarithmically convex,*
2. *the inequality*

$$\mathcal{C}_{\ell+k}^n\le\mathcal{C}_{\ell+n}^k\mathcal{C}_\ell^{n-k} \tag{5.7}$$

is valid for $\ell\ge0$ *and* $n>k>0$*.*

Theorem 5.5 ([54, Theorem 1.7]). *If* $\ell\ge0$*,* $n\ge k\ge m$*,* $k\ge n-k$*, and* $m\ge n-m$*, then*

$$\frac{C_{\ell+k}C_{\ell+n-k}}{C_{\ell+m}C_{\ell+n-m}}\ge\frac{(\ell+m)!(\ell+n-m)!}{(\ell+k)!(\ell+n-k)!}. \tag{5.8}$$

For $n,m\in\mathbb{N}$ *and* $\ell\ge0$*, let*

$$\mathcal{G}_{n,m,\ell}=\mathcal{C}_{\ell+n+2m}(\mathcal{C}_\ell)^2-\mathcal{C}_{\ell+n+m}\mathcal{C}_{\ell+m}\mathcal{C}_\ell-\mathcal{C}_{\ell+n}\mathcal{C}_{\ell+2m}\mathcal{C}_\ell+\mathcal{C}_{\ell+n}(\mathcal{C}_{\ell+m})^2,$$
$$\mathcal{H}_{n,m,\ell}=\mathcal{C}_{\ell+n+2m}(\mathcal{C}_\ell)^2-2\mathcal{C}_{\ell+n+m}\mathcal{C}_{\ell+m}\mathcal{C}_\ell+\mathcal{C}_{\ell+n}(\mathcal{C}_{\ell+m})^2,$$
$$\mathcal{I}_{n,m,\ell}=\mathcal{C}_{\ell+n+2m}(\mathcal{C}_\ell)^2-2\mathcal{C}_{\ell+n}\mathcal{C}_{\ell+2m}\mathcal{C}_\ell+\mathcal{C}_{\ell+n}(\mathcal{C}_{\ell+m})^2,$$

where $\mathcal{C}_\ell$ *is defined by* (5.5)*. Then*

$$\mathcal{G}_{n,m,\ell}\ge0,\quad \mathcal{H}_{n,m,\ell}\ge0, \tag{5.9}$$

$$\mathcal{H}_{n,m,\ell}\lesseqqgtr\mathcal{G}_{n,m,\ell}\quad\text{when } m\lesseqgtr n, \tag{5.10}$$

and

$$\mathcal{I}_{n,m,\ell}\ge\mathcal{G}_{n,m,\ell}\ge0\quad\text{when } n\ge m. \tag{5.11}$$

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5.4. Recall from [30, Chapter XIII], [60, Chapter 1], and [66, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \le (-1)^k f^{(k)}(x) < \infty$ on I for all $k \ge 0$. It is known [66, p. 161, Theorem 12b] that a function f is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform $f(t) = \int_0^\infty e^{-ts} \,\mathrm{d}\,\mu(s)$ of a positive measure μ defined on $[0,\infty)$ such that the above integral converges on $(0,\infty)$.

By virtue of the integral representation (3.5), we obtain asymptotic expansions and complete monotonicity related to the Catalan–Qi function.

Theorem 5.6 ([50, Theorem 4.2])**.** *For $b > a > 0$, we have*

$$C(a,b;x) = \frac{1}{B(a,b-a)}\left(\frac{b}{a}\right)^x \sum_{k=0}^{\infty}(-1)^k \frac{\langle b-a-1\rangle_k}{k!}\frac{1}{x+a+k}, \tag{5.12}$$

where

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is the falling factorial. Consequently, the function

$$(-1)^{\lfloor b-a\rfloor}\left[\left(\frac{a}{b}\right)^x C(a,b;x) - \frac{1}{B(a,b-a)}\sum_{k=0}^{N}(-1)^k \frac{\langle b-a-1\rangle_k}{k!}\frac{1}{x+a+k}\right] \tag{5.13}$$

for $N \in \{0\} \cup \mathbb{N}$ and $b > a > 0$ is completely monotonic in $x \in [0,\infty)$, where $\lfloor x\rfloor$ denotes the floor function whose value is the largest integer less than or equal to x.

For more information and details on applications of the integral representations (2.28) and (3.5), please refer to [27, 41, 42, 43, 44, 45, 50, 52, 56, 57, 58] and the closely related references therein.

6. Power series whose coefficients involve Catalan numbers

In this section, we recall some results on sums of power series whose coefficients involve the Catalan numbers C_n or the Catalan–Qi numbers $C_(a,b;n)$.

6.1. In 2012, Koshy and Gao [16] proved the following theorem.

Theorem 6.1 ([16])**.** *For $|x| < 4$, we have*

$$\sum_{n=0}^{\infty}\frac{x^n}{C_n} = \begin{cases} 1 + \dfrac{x(4-x)^{3/2} + 6x(4-x)^{1/2} + 24\sqrt{x}\arcsin\frac{\sqrt{x}}{2}}{(4-x)^{5/2}}, & 0 \le x < 4; \\ 1 - \dfrac{|x|(4-x)^{3/2} + 6\sqrt{|x|(4-x)} + 24\sqrt{|x|}\ln\frac{\sqrt{-x}+\sqrt{4-x}}{2}}{(4-x)^{5/2}}, & -4 < x \le 0. \end{cases} \tag{6.1}$$

Proof. We reformulate the proof by Koshy and Gao in [16] as follows. Denote

$$f(x) = \sum_{n=0}^{\infty}\frac{x^n}{C_n}. \tag{6.2}$$

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Then

$$f'(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{C_n}=\sum_{n=0}^{\infty}\frac{n+1}{C_{n+1}}x^n.$$

Since $\frac{n+2}{C_n}=\frac{4n+2}{C_{n+1}}$, by the recurrence relation, this yields

$$\sum_{n=0}^{\infty}\frac{n+2}{C_n}x^n=\sum_{n=0}^{\infty}\frac{4n+2}{C_{n+1}}x^n,$$

$$\sum_{n=0}^{\infty}\frac{n}{C_n}x^n+2\sum_{n=0}^{\infty}\frac{x^n}{C_n}=\sum_{n=0}^{\infty}\frac{4(n+1)}{C_{n+1}}x^n-2\sum_{n=0}^{\infty}\frac{x^n}{C_{n+1}},$$

$$xf'(x)+2f(x)=4f'(x)-\frac{2}{x}[f(x)-1],$$

and

$$x(x-4)f'(x)+2(x+1)f(x)=2. \tag{6.3}$$

For $x\ne0$, set $g(x)=\left|\frac{4-x}{x}\right|^{3/2}$. Then $\frac{g'(x)}{g(x)}=-\frac{6}{x(4-x)}$. This implies that

$$[x(x-4)g(x)]'=2(x+1)g(x). \tag{6.4}$$

Multiplying (6.3) by $g(x)$, we obtain

$$x(x-4)f'(x)g(x)+2(x+1)f(x)g(x)=2g(x).$$

Using (6.4), this can be rewritten as

$$[x(x-4)f(x)g(x)]'=2g(x).$$

Using (6.4) again gives

$$\begin{aligned}\{x(x-4)[f(x)-1]g(x)\}'&=[x(x-4)f(x)g(x)]'-[x(x-4)g(x)]'\\&=2g(x)-2(x+1)g(x)=-2xg(x).\end{aligned}$$

Consequently,

$$x(x-4)[f(x)-1]g(x)=-2\int xg(x)\,\mathrm{d}\,x+\alpha_1,$$

$$f(x)=1+\frac{2\int xg(x)\,\mathrm{d}\,x-\alpha_1}{x(4-x)g(x)},$$

where α_1 is a constant.

For $0<x<4$, we have

$$\begin{aligned}\int xg(x)\,\mathrm{d}\,x&=\int x\left(\frac{4-x}{x}\right)^{3/2}\mathrm{d}\,x=\int\frac{(4-x)^{3/2}}{x^{1/2}}\,\mathrm{d}\,x\\&=2\int\bigl(4-u^2\bigr)^{3/2}\,\mathrm{d}\,u\quad\bigl(x=u^2\bigr)\\&=\frac12u\bigl(4-u^2\bigr)^{3/2}+3u\bigl(4-u^2\bigr)^{1/2}+12\arcsin\frac{u}{2}+\alpha_2\\&=\frac12\sqrt{x}\,(4-x)^{3/2}+3\sqrt{x}\,(4-x)^{1/2}+12\arcsin\frac{\sqrt{x}}{2}+\alpha_2,\end{aligned}$$

where α_2 is also a constant. Therefore, we have

$$f(x)=1+\frac{\sqrt{x}\,(4-x)^{3/2}+6\sqrt{x}\,(4-x)^{1/2}+24\arcsin\frac{\sqrt{x}}{2}+2\alpha_2-\alpha_1}{x(4-x)\bigl(\frac{4-x}{x}\bigr)^{3/2}}$$

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$$= 1 + \frac{x(4-x)^{3/2} + 6x(4-x)^{1/2} + 24\sqrt{x}\arcsin\frac{\sqrt{x}}{2} + \alpha\sqrt{x}}{(4-x)^{5/2}},$$

where $\alpha = 2\alpha_2 - \alpha_1$. Since $f(0) = 1 = f'(0)$, we have $\alpha = 0$. Thus, the desired result for $0 < x < 4$ is proved.

For $-4 < x < 0$, by similar argument to the above, we acquire

$$\int xg(x)\,\mathrm{d}\,x = \frac{1}{2}\sqrt{|x|}\,(4-x)^{3/2} + 3\sqrt{|x|(4-x)} + 12\ln\bigl(\sqrt{|x|} + \sqrt{|4-x|}\bigr) + \alpha_3$$

and

$$f(x) = 1 - \frac{|x|(4-x)^{3/2} + 6\sqrt{|x|(4-x)} + 24\sqrt{|x|}\,\ln\frac{\sqrt{-x}+\sqrt{4-x}}{\alpha_4}}{(4-x)^{5/2}}.$$

From $f(0) = 1 = f'(0)$, we can determine $C_4 = 2$. The desired result is thus proved. □

6.2. In 2014, Beckwith and Harbor [4] proposed a problem: show that

$$\sum_{n=0}^{\infty}\frac{2^n}{C_n} = 5 + \frac{3}{2}\pi \quad\text{and}\quad \sum_{n=0}^{\infty}\frac{3^n}{C_n} = 22 + 8\sqrt{3}\,\pi.$$

In 2016, Abel [1] answered this problem by proving a general result below.

Theorem 6.2 ([1, 4])**.** *For* $0 \le x < 4$, *we have*

$$\sum_{n=0}^{\infty}\frac{x^n}{C_n} = 1 - \frac{x(x-10)}{(4-x)^2} + \frac{24\sqrt{x}}{(4-x)^{5/2}}\arctan\sqrt{\frac{x}{4-x}}. \tag{6.5}$$

Proof. We slightly modify the proof in [1] as follows. Using the beta integral

$$\int_0^1 t^m(1-t)^n\,\mathrm{d}\,t = \frac{m!n!}{(m+n+1)!}$$

gives

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{C_n} &= 1 + \sum_{n=1}^{\infty}n(n+1)\frac{(n-1)!n!}{(2n)!}x^n\\ &= 1 + \sum_{n=1}^{\infty}n(n+1)x^n\int_0^1 t^{n-1}(1-t)^n\,\mathrm{d}\,t\\ &= 1 + \int_0^1\sum_{n=1}^{\infty}n(n+1)x^nt^{n-1}(1-t)^n\,\mathrm{d}\,t\end{aligned}$$

for $|x| < 4$. Further using

$$\sum_{n=1}^{\infty}n(n+1)z^n = \frac{2z}{(1-z)^3}$$

produces

$$\sum_{n=0}^{\infty}\frac{x^n}{C_n} = 1 + 2x\int_0^1\frac{1-t}{[1-xt(1-t)]^3}\,\mathrm{d}\,t.$$

Direct calculation of the integral yields the result (6.5). □

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6.3. The editorial comment in [1] listed the formulas

$$\sum_{n=0}^{\infty}\frac{1}{C_n}=2+\frac{4\pi}{9\sqrt{3}}, \qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{C_n}=\frac{14}{25}-\frac{24\sqrt{5}}{125}\ln\frac{1+\sqrt{5}}{2},$$

$$\sum_{n=0}^{\infty}\frac{(-2)^n}{C_n}=\frac{1}{3}-\frac{1}{3\sqrt{3}}\ln\bigl(2+\sqrt{3}\bigr), \qquad \sum_{n=0}^{\infty}\frac{(-3)^n}{C_n}=\frac{10}{49}-\frac{36}{49\sqrt{21}}\ln\frac{5+\sqrt{21}}{2}.$$

The editorial comment in [1] also pointed out that the result (6.1) had existed in [16], that the sum

$$\sum_{n=0}^{\infty}\frac{x^n}{C_n}=2\frac{\sqrt{4-x}\,(8+x)+12\sqrt{x}\,\arctan\frac{\sqrt{x}}{\sqrt{4-x}}}{\sqrt{(4-x)^5}} \tag{6.6}$$

can be found on the website `http://planetmath.org/`, and that the problem by Beckwith and Harbor [4] can be solved easily from

$$\sum_{n=1}^{\infty}\frac{2^n}{\binom{2n}{n}}=\frac{\pi}{2}+1, \qquad \sum_{n=1}^{\infty}\frac{n2^n}{\binom{2n}{n}}=\pi+3,$$

$$\sum_{n=1}^{\infty}\frac{3^n}{\binom{2n}{n}}=\frac{4\pi\sqrt{3}}{3}+3, \qquad \sum_{n=1}^{\infty}\frac{n3^n}{\binom{2n}{n}}=\frac{20\pi\sqrt{3}}{3}+18$$

which are special cases of the general formula in [17, p. 452, Theorem] below.

Theorem 6.3 ([17, p. 452, Theorem]). *For $|x|<1$, we have*

$$\frac{2x\arcsin x}{\sqrt{1-x^2}}=\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{m\binom{2m}{m}}. \tag{6.7}$$

Proof of (6.7). Making use of the familiar Gregory series

$$t\arctan t=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^{2m}}{2m-1}$$

and setting $t=\frac{x}{\sqrt{1-x^2}}$ yields $\arctan t=\arcsin x$ and

$$\begin{aligned}\frac{x}{\sqrt{1-x^2}}\arcsin x&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}x^{2m}}{(2m-1)(1-x^2)^m}\\&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m-1}\sum_{j=0}^{\infty}(-1)^j\binom{-m}{j}x^{2(j+m)}\\&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m-1}\sum_{j=0}^{\infty}\binom{m+j-1}{j}x^{2(j+m)}\\&=\sum_{r=1}^{\infty}x^{2r}\sum_{m=1}^{r}\frac{(-1)^{m-1}(r-1)!}{(m-1)!(r-m)!(2m-1)}.\end{aligned}$$

Using Wallis' integral

$$\int_0^{\pi/2}(\sin\theta)^{2r-1}\,\mathrm{d}\,\theta=\frac{2\cdot4\cdot6\cdots(2r-2)}{1\cdot3\cdot5\cdots(2r-1)}$$

results in

$$r\binom{2r}{r}\sum_{\nu=0}^{r-1}(-1)^\nu\binom{r-1}{\nu}\frac{1}{2\nu+1}=r\binom{2r}{r}\int_0^1\sum_{\nu=0}^{r-1}(-1)^\nu\binom{r-1}{\nu}y^{2\nu}\,\mathrm{d}\,y$$
$$=r\binom{2r}{r}\int_0^1(1-y^2)^{r-1}\,\mathrm{d}\,y$$
$$=r\binom{2r}{r}\int_0^{\pi/2}\sin^{2r-1}\theta\,\mathrm{d}\,\theta$$
$$=2^{2r-1}.$$

The sum (6.7) is thus proved. □

From (6.7), Lehmer [17] also derived

$$2(\arcsin x)^2=\sum_{m=1}^\infty\frac{(2x)^{2m}}{m^2\binom{2m}{m}},\quad \sum_{m=1}^\infty\frac{(2x)^{2m}}{m^3\binom{2m}{m}}=4\int_0^x\frac{(\arcsin y)^2}{y}\,\mathrm{d}\,y$$

and gave a recursive formula for

$$\sum_{m=1}^\infty\frac{m^{k-2}(2x)^{2m}}{\binom{2m}{m}}.$$

Lehmer [17, p. 454] pointed out that there are no known sum for interesting series of the form

$$\sum_{m=1}^\infty\frac{1}{m^k\binom{2m}{m}}$$

for $k\ge5$.

6.4. In 2016, motivated by the above-mentioned problem posed by Beckwith and Harbor [4], Amdeberhan and his four coauthors [3] also proposed a general problem: find a closed-form formula for the series in (6.2). They obtained the sum

$$\sum_{n=0}^\infty\frac{z^n}{C_n}={}_2F_1\left(1,2;\frac12;\frac z4\right)=\frac{2(z+8)}{(4-z)^2}+\frac{24\sqrt z}{(4-z)^{5/2}}\arcsin\frac{\sqrt z}{2},\quad |z|<4 \tag{6.8}$$

by several methods, where ${}_2F_1$ is the classical hypergeometric function which is a special case of the generalized hypergeometric series

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^\infty\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{z^n}{n!} \tag{6.9}$$

defined for complex numbers $a_i\in\mathbb{C}$ and $b_i\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, for positive integers $p,q\in\mathbb{N}$, and in terms of the rising factorial

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge1;\\ 1, & n=0.\end{cases}$$

We observe that the formulas (6.5) and (6.6) are the same one, that the sums (6.1) and (6.8) are the same one, and that, since

$$\arctan\sqrt{\frac{x}{4-x}}=\arcsin\frac{\sqrt x}{2}$$

for $0\le x<4$, the four sums (6.5) to (6.8) are essentially the same one.

7. Sums of some new series

By applying some of the above-mentioned integral representations of the Catalan numbers C_n, we now construct some new finite and infinite power series.

7.1. Sums of two finite and infinite series.

Making use of the integral representations (2.13) and (2.14), (2.16) and (2.20), (2.21) and (2.22), we can find the following finite and infinite power series involving the Catalan numbers C_n.

Theorem 7.1. *For $k \ge 0$, we have the finite sums*

$$\sum_{n=0}^{k} \frac{C_n}{2^{2n}} = \frac{2}{\pi}\left[B\left(\frac{1}{2},\frac{1}{2}\right) - B\left(\frac{1}{2}, k+\frac{3}{2}\right)\right]$$

and

$$\sum_{n=0}^{k} \frac{2^{2n}}{(n+1)(2n+1)(2n+3)}\frac{1}{C_n} = \frac{1}{2\pi}\left[B\left(\frac{1}{2},\frac{1}{2}\right) - B\left(\frac{1}{2}, k+\frac{3}{2}\right)\right].$$

Consequently, we have the infinite series

$$\sum_{n=0}^{\infty} \frac{C_n}{2^{2n}} = 2 \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{2^{2n}}{(n+1)(2n+1)(2n+3)}\frac{1}{C_n} = 1. \tag{7.1}$$

Proof. Dividing the integral representations (2.13) and (2.14) and summing up over $0 \le n \le k$ give

$$\sum_{n=0}^{k} \frac{C_n}{2^{2n+2}} = \frac{1}{\pi}\int_0^1 \left(\sum_{n=0}^{k} x^{2n}\right)\sqrt{1-x^2}\,\mathrm{d}\,x = \frac{1}{\pi}\int_0^1 \frac{1-x^{2(k+1)}}{1-x^2}\sqrt{1-x^2}\,\mathrm{d}\,x$$
$$= \frac{1}{\pi}\int_0^1 \left(1-x^{2k+2}\right)\left(1-x^2\right)^{-1/2}\mathrm{d}\,x = \frac{1}{2\pi}\int_0^1 \left(t^{-1/2}-t^{k+1/2}\right)(1-t)^{-1/2}\,\mathrm{d}\,t$$
$$= \frac{1}{2\pi}\left[B\left(\frac{1}{2},\frac{1}{2}\right) - B\left(k+\frac{3}{2},\frac{1}{2}\right)\right] \to \frac{1}{2\pi}B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}$$

and

$$\sum_{n=0}^{k} \frac{2^{2n+1}}{(2n+3)(2n+2)(2n+1)}\frac{1}{C_n} = \int_0^1 \left(\sum_{n=0}^{k} x^{2n+1}\right)\sqrt{1-x^2}\,\mathrm{d}\,x$$
$$= \int_0^1 \frac{x\left(1-x^{2k+2}\right)}{1-x^2}\sqrt{1-x^2}\,\mathrm{d}\,x = \frac{1}{2}\int_0^1 \left(1-t^{k+1}\right)(1-t)^{-1/2}\,\mathrm{d}\,t$$
$$= \frac{1}{2}\left[B\left(\frac{1}{2},1\right) - B\left(\frac{1}{2},k+2\right)\right] \to \frac{1}{2}B\left(\frac{1}{2},1\right) = 1$$

as $k \to \infty$.

Similarly, from (2.16) and (2.20), it follows that

$$\sum_{n=0}^{k} \frac{C_n}{2^{2n+2}} = \frac{1}{\pi}\int_0^\infty \sum_{n=0}^{k} \frac{u^2}{(1+u^2)^{n+2}}\,\mathrm{d}\,u = \frac{1}{\pi}\int_0^\infty \left[\frac{1}{1+u^2} - \frac{1}{(1+u^2)^{k+2}}\right]\mathrm{d}\,u$$
$$= \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \frac{t^{-1/2}}{(1+t)^{k+2}}\,\mathrm{d}\,t = \frac{1}{2} - \frac{1}{2\pi}B\left(\frac{1}{2},k+\frac{3}{2}\right) \to \frac{1}{2}$$

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and

$$
\begin{aligned}
\sum_{n=0}^{k}\frac{C_n}{2^{2n+5}}&=\frac{1}{\pi}\int_0^1\frac{u^2}{(1+u^2)^3}\sum_{n=0}^{k}\biggl(\frac{1-u^2}{1+u^2}\biggr)^{2n}\operatorname{d}u\\
&=\frac{1}{\pi}\int_0^1\frac{u^2}{(1+u^2)^3}\frac{1-\bigl(\frac{1-u^2}{1+u^2}\bigr)^{2k+2}}{1-\bigl(\frac{1-u^2}{1+u^2}\bigr)^{2}}\operatorname{d}u\\
&=\frac{1}{\pi}\int_0^1\frac{1-t}{1+t}\biggl(\frac{1+t}{2}\biggr)^3\frac{1-t^{2k+2}}{1-t^2}\frac{1}{(1+t)^2}\sqrt{\frac{1+t}{1-t}}\operatorname{d}t\\
&=\frac{1}{8\pi}\int_0^1\frac{1-t^{2k+2}}{\sqrt{1-t^2}}\operatorname{d}t=\frac{1}{16\pi}\int_0^1\frac{1-s^{k+1}}{\sqrt{1-s}}\frac{1}{\sqrt{s}}\operatorname{d}s\\
&=\frac{1}{16\pi}\biggl[\int_0^1(1-s)^{-1/2}s^{-1/2}\operatorname{d}s-\int_0^1s^{k+1/2}(1-s)^{-1/2}\operatorname{d}s\biggr]\\
&=\frac{1}{16\pi}\biggl[B\biggl(\frac12,\frac12\biggr)-B\biggl(\frac12,k+\frac32\biggr)\biggr]\to\frac{1}{16}
\end{aligned}
$$

as $k\to\infty$. The proof of Theorem 7.1 is complete. □

7.2. **Sums of three finite series.** Applying the last integral expressions in (2.21) and (2.22), we can obtain sums of three new finite series.

Theorem 7.2. *For $k\ge0$, we have*

$$
\sum_{n=0}^{k}\frac{n+1}{2^{2n}}C_n=\frac{2}{B\bigl(\frac12,k+1\bigr)},
$$

$$
\sum_{n=0}^{k}\frac{2n+1}{2^{2n}}C_n=2\biggl[\frac{1}{B\bigl(\frac12,k+2\bigr)}-1\biggr],
$$

and

$$
\sum_{n=0}^{k}\frac{2^{2n}}{(2n+1)(n+1)}\frac{1}{C_n}=(k+1)B\biggl(\frac12,k+1\biggr)-1.
$$

When $k\to\infty$, these three series diverge.

Proof. Applying the last expressions in (2.21) and (2.22) yields

$$
\begin{aligned}
\sum_{n=0}^{k}\frac{n+1}{2^{2n}}C_n&=\frac{2}{\pi}\int_0^{\pi/2}\sum_{n=0}^{k}\sin^{2n}x\operatorname{d}x=\frac{2}{\pi}\int_0^{\pi/2}\frac{1-\sin^{2k+2}x}{\cos^2x}\operatorname{d}x\\
&=\frac{2}{\pi}\frac{\sqrt{\pi}\,\Gamma\bigl(k+\frac32\bigr)}{\Gamma(k+1)}=\frac{2}{B\bigl(\frac12,k+1\bigr)}\to\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=0}^{k}\frac{2n+1}{2^{2n}}C_n&=\frac{4}{\pi}\int_0^{\pi/2}\sum_{n=0}^{k}\sin^{2n+2}x\operatorname{d}x\\
&=\frac{4}{\pi}\int_0^{\pi/2}\tan^2x\bigl(1-\sin^{2k+2}x\bigr)\operatorname{d}x=\frac{4}{\pi}\biggl[\frac{\sqrt{\pi}\,\Gamma\bigl(k+\frac52\bigr)}{\Gamma(k+2)}-\frac{\pi}{2}\biggr]\\
&=4\biggl[\frac{\Gamma\bigl(k+\frac52\bigr)}{\sqrt{\pi}\,\Gamma(k+2)}-\frac12\biggr]=2\biggl[\frac{1}{B\bigl(\frac12,k+2\bigr)}-1\biggr]\to\infty
\end{aligned}
$$

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as $k \to \infty$.

From (4.2), it follows that

$$\frac{2^{2n+1}}{(2n+1)(n+1)}\frac{1}{C_n} = B\left(\frac{1}{2}, n+1\right) = \int_0^1 (1-t)^{-1/2} t^n \,\mathrm{d}\, t.$$

Summing up over n from 0 to k leads to

$$\begin{aligned}\sum_{n=0}^{k} \frac{2^{2n+1}}{(2n+1)(n+1)}\frac{1}{C_n} &= \int_0^1 (1-t)^{-1/2} \sum_{n=0}^{k} t^n \,\mathrm{d}\, t \\ = \int_0^1 (1-t)^{-1/2}\frac{1-t^{k+1}}{1-t} \,\mathrm{d}\, t &= \int_0^1 (1-t)^{-3/2}\left(1-t^{k+1}\right) \mathrm{d}\, t \\ = 2\int_0^1 \left(1-t^{k+1}\right)\left[(1-t)^{-1/2}\right]' \mathrm{d}\, t &= -2+2(k+1)\int_0^1 t^k (1-t)^{-1/2} \,\mathrm{d}\, t \\ &= 2\left[(k+1)B\left(\frac{1}{2}, k+1\right) - 1\right] \to \infty\end{aligned}$$

as $k \to \infty$. The proof of Theorem 7.2 is complete. □

7.3. **Sums of three infinite power series.** Now we use (6.8) to derive sums of three infinite power series involving the reciprocal of the Catalan numbers C_n.

Theorem 7.3. *The reciprocals* $\frac{1}{C_n}$ *of the Catalan numbers* C_n *satisfy*

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)C_n} = \sum_{n=0}^{\infty} \frac{z^n}{\binom{2n}{n}} = \frac{4}{4-z} + \frac{4\sqrt{z}}{(4-z)^{3/2}} \arcsin\frac{\sqrt{z}}{2}, \quad |z|<4, \tag{7.2}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{(2n+1)C_n} = \frac{2}{4-z} + \frac{8}{\sqrt{z}\,(4-z)^{3/2}} \arcsin\frac{\sqrt{z}}{2}, \quad |z|<4, \tag{7.3}$$

and

$$\sum_{n=0}^{\infty} \frac{z^n}{(2n+1)(n+1)C_n} = \frac{4}{\sqrt{z(4-z)}} \arcsin\frac{\sqrt{z}}{2}, \quad |z|<4.$$

Proof. Integrating on both sides of (6.8) from 0 to t with $|t|<4$ yields

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)C_n} &= \int_0^t \frac{24}{(4-z)^2} \,\mathrm{d}\, z - \int_0^t \frac{2}{4-z} \,\mathrm{d}\, z + \int_0^t \frac{24\sqrt{z}}{(4-z)^{5/2}} \arcsin\frac{\sqrt{z}}{2} \,\mathrm{d}\, z \\ &= \frac{6t}{4-t} + 2\ln(4-t) - 4\ln 2 \\ &\quad + \int_0^{\arcsin\frac{\sqrt{t}}{2}} \frac{24\sqrt{4\sin^2 s}}{\left(4-4\sin^2 s\right)^{5/2}} 8\sin s\cos s \arcsin\frac{\sqrt{4\sin^2 s}}{2} \,\mathrm{d}\, s \\ &= \frac{6t}{4-t} + 2\ln(4-t) - 4\ln 2 + 12\int_0^{\arcsin\frac{\sqrt{t}}{2}} \frac{\sin^2 s}{\cos^4 s} s \,\mathrm{d}\, s \\ &= \frac{6t}{4-t} + 2\ln(4-t) - 4\ln 2 + 4\int_0^{\arcsin\frac{\sqrt{t}}{2}} s\left(\tan^3 s\right)' \mathrm{d}\, s \\ &= \frac{6t}{4-t} + 2\ln(4-t) - 4\ln 2 + 4\arcsin\frac{\sqrt{t}}{2} \tan^3 \arcsin\frac{\sqrt{t}}{2}\end{aligned}$$

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$$
\begin{aligned}
&- 4\int_0^{\arcsin\frac{\sqrt{t}}{2}} \tan^3 s \,\mathrm{d}\, s\\
&= \frac{6t}{4-t} + 2\ln(4-t) - 4\ln 2 + \frac{4t^{3/2}}{(4-t)^{3/2}}\arcsin\frac{\sqrt{t}}{2}\\
&- 4\int_0^{\arcsin\frac{\sqrt{t}}{2}} \bigl(\tan s\sec^2 s - \tan s\bigr)\,\mathrm{d}\, s\\
&= \frac{6t}{4-t} + 2\ln(4-t) - 4\ln 2 + \frac{4t^{3/2}}{(4-t)^{3/2}}\arcsin\frac{\sqrt{t}}{2}\\
&- \frac{4t}{8-2t} + 4\biggl[\ln 2 - \frac{1}{2}\ln(4-t)\biggr]\\
&= \frac{4t}{4-t} + \frac{4t^{3/2}}{(4-t)^{3/2}}\arcsin\frac{\sqrt{t}}{2}.
\end{aligned}
$$

The equality (7.2) is thus proved.

The formula (6.8) can be rewritten as

$$
\sum_{n=0}^{\infty}\frac{z^{2n}}{C_n} = \frac{2(z^2+8)}{(4-z^2)^2} + \frac{24z}{(4-z^2)^{5/2}}\arcsin\frac{z}{2}, \quad |z|<2.
$$

Integrating on both sides of the above equality gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)C_n} &= \int_0^t\biggl[\frac{2(z^2+8)}{(4-z^2)^2} + \frac{24z}{(4-z^2)^{5/2}}\arcsin\frac{z}{2}\biggr]\,\mathrm{d}\, z\\
&= \frac{3t}{4-t^2} + \frac{1}{4}\ln\frac{2+t}{2-t} + 3\int_0^{\arcsin(t/2)}\frac{\sin u}{\cos^4 u}u\,\mathrm{d}\, u\\
= \frac{3t}{4-t^2} &+ \frac{1}{4}\ln\frac{2+t}{2-t} + \frac{8}{(4-t^2)^{3/2}}\arcsin\frac{t}{2} - \int_0^{\arcsin(t/2)}\frac{1}{\cos^3 u}\,\mathrm{d}\, u\\
= \frac{3t}{4-t^2} &+ \frac{1}{4}\ln\frac{2+t}{2-t} + \frac{8}{(4-t^2)^{3/2}}\arcsin\frac{t}{2} - \frac{t}{4-t^2} - \frac{1}{4}\ln\frac{2+t}{2-t}\\
&= \frac{2t}{4-t^2} + \frac{8}{(4-t^2)^{3/2}}\arcsin\frac{t}{2}
\end{aligned}
$$

which can be rewritten as (7.3).

Since $\frac{1}{(2n+1)(n+1)} = \frac{2}{2n+1} - \frac{1}{n+1}$, by (7.2) and (7.3), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{x^n}{(2n+1)(n+1)C_n} &= \sum_{n=0}^{\infty}\biggl[\frac{2x^n}{(2n+1)C_n} - \frac{x^n}{(n+1)C_n}\biggr]\\
= \sum_{n=0}^{\infty}\frac{2x^n}{(2n+1)C_n} &- \sum_{n=0}^{\infty}\frac{x^n}{(n+1)C_n} = \frac{4}{\sqrt{z(4-z)}}\arcsin\frac{\sqrt{z}}{2}.
\end{aligned}
$$

The proof of Theorem 7.3 is complete. □

7.4. **A new proof for the sum of a power series.** Now we supply a new proof for the following conclusion in [3, pp. 115–116, Section 6].

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Theorem 7.4 ([3, Section 6])**.** *For $x\ge0$, we have*

$$\sum_{n=0}^\infty\frac{1}{C_n}\frac{x^n}{n!}=1+\frac14x+\frac{\sqrt{\pi}}{8}(x+6)\sqrt{x}\,e^{x/4}\operatorname{erf}\biggl(\frac{\sqrt{x}}{2}\biggr). \tag{7.4}$$

Proof. In [58, Theorem 1.5], it was obtained that

$$\sum_{n=0}^\infty C(a,b;n)\frac{x^n}{n!}={}_1F_1\biggl(a;b;\frac{b}{a}x\biggr). \tag{7.5}$$

Letting $a=2$ and $b=\frac12$ in (7.5) gives

$$\sum_{n=0}^\infty\frac{1}{C_n}\frac{x^n}{n!}={}_1F_1\biggl(2;\frac12;\frac{x}{4}\biggr).$$

Since

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}e^{-x^2}\sum_{n=0}^\infty\frac{2^kz^{2n+1}}{(2n+1)!!}$$

see [9, p. 889, 8.253] or [32, p. 162, 7.6.2], it is straightforward to verify that

$${}_1F_1\biggl(2;\frac12;\frac{x}{4}\biggr)=1+\frac14x+\frac{\sqrt{\pi}}{8}(x+6)\sqrt{x}\,e^{x/4}\operatorname{erf}\biggl(\frac{\sqrt{x}}{2}\biggr).$$

The proof of Theorem 7.4 is thus complete. □

7.5. **More sums of series involving Catalan or Catalan–Qi numbers.** Except [58, Theorem 1.5], some series such as

$${}_2F_1\biggl(a,1;b;\frac{bt}{a}\biggr)=\sum_{n=0}^\infty C(a,b;n)t^n,\quad a,b>0; \tag{7.6}$$

$$\sum_{n=1}^\infty\biggl(\frac{a}{b}\biggr)^nC(a,b;n)=\frac{a}{b-a-1},\quad b>a+1>1; \tag{7.7}$$

and

$$\sum_{n=0}^\infty C(a,b;n)\frac{x^{2n}}{(2n)!}={}_1F_2\biggl(a;\frac12,b;\frac{b}{4a}x^2\biggr),\quad a,b>0 \tag{7.8}$$

were also established in the papers [27, Theorem 1] and [50, Theorem 10].

In [67], among other things, it was obtained that

$$\sum_{n=0}^\infty\frac{(n+1)(2n)!!C_n}{4^n(2n+1)^2(2n+1)!!}=\frac78\zeta(3)$$

and

$$\sum_{n=0}^\infty\frac{x^n}{C_n}=\frac{24\sqrt{-x}}{(4-x)^{5/2}}\ln\biggl(\frac{\sqrt{-x}+\sqrt{4-x}}{2}\biggr)+\frac{2x}{(4-x)^2}+1,\quad x\in(-4,0],$$

where $\zeta(z)$ denotes the Riemanian zeta function

$$\zeta(s)=\sum_{k=1}^\infty\frac{1}{k^s},\quad \Re(s)>1.$$

8. An alternative proof of the formula (6.7)

Substituting (1.1) into the left-hand side of (6.7) and making use of the identities in (3.2) and (3.3) give

$$\begin{aligned}
h(x^2) &= \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m(m+1)C_m} = \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m(m+1)C\left(\frac12,2;m\right)}\\
&= \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m(m+1)} C\left(2,\frac12,;m\right) = \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m(m+1)}\left(\frac14\right)^m \frac{(2)_m}{\left(\frac12\right)_m}\\
&= \sum_{m=1}^{\infty} \frac{(2)_m}{\left(\frac12\right)_m} \frac{x^{2m}}{m(m+1)} = \sum_{m=1}^{\infty} \frac{(2)_m}{\left(\frac12\right)_m} \frac{\left(x^2\right)^m}{m(m+1)}\\
&= \sum_{m=1}^{\infty} \frac{(2)_m}{\left(\frac12\right)_m} \frac{\left(x^2\right)^m}{m} - \frac{1}{x^2}\sum_{m=1}^{\infty} \frac{(2)_m}{\left(\frac12\right)_m} \frac{\left(x^2\right)^{m+1}}{m+1} \triangleq h_1(x^2) - \frac{1}{x^2}h_2(x^2).
\end{aligned}$$

Differentiation and utilization of (6.9) reveal

$$\begin{aligned}
h_1'(t) &= \sum_{m=1}^{\infty} \frac{(2)_m}{\left(\frac12\right)_m} t^{m-1} = \frac1t\left[\sum_{m=0}^{\infty} \frac{(2)_m}{\left(\frac12\right)_m} t^m - 1\right] = \frac1t\left[\sum_{m=0}^{\infty} \frac{(2)_m m!}{\left(\frac12\right)_m} \frac{t^m}{m!} - 1\right]\\
&= \frac1t\left[\sum_{m=0}^{\infty} \frac{(2)_m(1)_m}{\left(\frac12\right)_m} \frac{t^m}{m!} - 1\right] = \frac1t\left[{}_2F_1\left(1,2;\frac12;t\right) - 1\right]
\end{aligned}$$

and

$$h_2'(t) = \sum_{m=1}^{\infty} \frac{(2)_m}{\left(\frac12\right)_m} t^m = \sum_{m=1}^{\infty} \frac{(2)_m(1)_m}{\left(\frac12\right)_m} \frac{t^m}{m!} = {}_2F_1\left(1,2;\frac12;t\right) - 1.$$

Accordingly, we obtain

$$\begin{aligned}
h'(t) &= \left[h_1(t) - \frac1t h_2(t)\right]' = h_1'(t) - \frac{th_2'(t) - h_2(t)}{t^2}\\
&= \frac1t\left[{}_2F_1\left(1,2;\frac12;t\right) - 1\right] - \frac{h_2'(t)}{t} + \frac{h_2(t)}{t^2}\\
&= \frac1t\left[{}_2F_1\left(1,2;\frac12;t\right) - 1\right] - \frac1t\left[{}_2F_1\left(1,2;\frac12;t\right) - 1\right] + \frac{h_2(t)}{t^2} = \frac{h_2(t)}{t^2}.
\end{aligned}$$

This implies that

$$\left[t^2h'(t)\right]' = h_2'(t) = {}_2F_1\left(1,2;\frac12;t\right) - 1.$$

Combining this with the right equality in (6.8) leads to

$$\begin{aligned}
\left[t^2h'(t)\right]' + 1 &= \frac{2(4t+8)}{(4-4t)^2} + \frac{24\sqrt{4t}}{(4-4t)^{5/2}}\arcsin\frac{\sqrt{4t}}{2}\\
&= \frac{t+2}{2(1-t)^2} + \frac{3\sqrt{t}}{2(1-t)^{5/2}}\arcsin\sqrt{t}\,.
\end{aligned}$$

Integrating with respect to t over $[0,x]$ for $0<x<1$ yields

$$x^2h'(x) + x = \int_0^x \left[\frac{t+2}{2(1-t)^2} + \frac{3\sqrt{t}}{2(1-t)^{5/2}}\arcsin\sqrt{t}\right] \,\mathrm{d}\,t$$

$$
\begin{aligned}
&= \frac{3x}{2(1-x)} + \frac{1}{2}\ln(1-x) + 3\int_0^{\sqrt{x}} \frac{s^2}{(1-s^2)^{5/2}} \arcsin s \,\mathrm{d}\, s \\
&= \frac{3x}{2(1-x)} + \frac{1}{2}\ln(1-x) + 3\int_0^{\arcsin\sqrt{x}} \frac{u\sin^2 u\cos u}{(1-\sin^2 u)^{5/2}} \,\mathrm{d}\, u \\
&= \frac{3x}{2(1-x)} + \frac{1}{2}\ln(1-x) + 3\int_0^{\arcsin\sqrt{x}} \frac{u\sin^2 u}{\cos^4 u} \,\mathrm{d}\, u \\
&= \frac{3x}{2(1-x)} + \frac{1}{2}\ln(1-x) + \int_0^{\arcsin\sqrt{x}} u\bigl(\tan^3 u\bigr)' \,\mathrm{d}\, u \\
&= \frac{3x}{2(1-x)} + \frac{1}{2}\ln(1-x) + \arcsin\sqrt{x}\,\tan^3\arcsin\sqrt{x} \\
&\quad - \int_0^{\arcsin\sqrt{x}} \tan^3 u \,\mathrm{d}\, u \\
&= \frac{3x}{2(1-x)} + \frac{1}{2}\ln(1-x) + \frac{x^{3/2}}{(1-x)^{3/2}}\arcsin\sqrt{x} \\
&\quad - \int_0^{\arcsin\sqrt{x}} \bigl(\tan u\sec^2 u - \tan u\bigr) \,\mathrm{d}\, u \\
&= \frac{3x}{2(1-x)} + \frac{1}{2}\ln(1-x) + \frac{x^{3/2}}{(1-x)^{3/2}}\arcsin\sqrt{x} \\
&\quad - \frac{1}{2}\sec^2\arcsin\sqrt{x} - \ln\cos\arcsin\sqrt{x} + \frac{1}{2} \\
&= \frac{2x+1}{2(1-x)} + \frac{1}{2}\ln(1-x) + \frac{x^{3/2}}{(1-x)^{3/2}}\arcsin\sqrt{x} \\
&\quad - \frac{1}{2(1-x)} - \frac{1}{2}\ln\bigl(1-x\bigr) \\
&= \frac{x}{1-x} + \frac{x^{3/2}}{(1-x)^{3/2}}\arcsin\sqrt{x}\,.
\end{aligned}
$$

Furthermore, similarly integrating gives

$$
\begin{aligned}
h(t) &= \int_0^t \frac{1}{x^2}\left[\frac{x}{1-x} + \frac{x^{3/2}}{(1-x)^{3/2}}\arcsin\sqrt{x} - x\right] \mathrm{d}\, x \\
&= \int_0^t \left[\frac{1}{1-x} + \frac{1}{x^{1/2}(1-x)^{3/2}}\arcsin\sqrt{x}\right] \mathrm{d}\, x \\
&= -\ln(1-t) + \int_0^{\arcsin\sqrt{t}} \frac{2s\sin s\cos s}{\bigl(1-\sin^2 s\bigr)^{3/2}\sin s} \,\mathrm{d}\, s \\
&= -\ln(1-t) + 2\int_0^{\arcsin\sqrt{t}} \frac{s}{\cos^2 s} \,\mathrm{d}\, s \\
&= -\ln(1-t) + 2\arcsin\sqrt{t}\,\tan\arcsin\sqrt{t} - 2\int_0^{\arcsin\sqrt{t}} \tan s \,\mathrm{d}\, s \\
&= -\ln(1-t) + 2\sqrt{\frac{t}{1-t}}\arcsin\sqrt{t} + 2\ln\cos\arcsin\sqrt{t}
\end{aligned}
$$

$$= 2\sqrt{\frac{t}{1-t}} \arcsin\sqrt{t}\,.$$

The proof of the formula (6.7) is complete.

9. Remarks

Finally we list several remarks on closely related results.

Remark 9.1. It seems that there are close and similar ideas in [3, 4] and that the paper [3] is almost an expanded version of [4]. Great minds think alike!

Remark 9.2. In [17, p. 452, Theorem], it was established that

$$\sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x\arcsin x}{\sqrt{1-x^2}}, \quad |x|<1.$$

This can be rearranged as

$$\sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m(m+1)C_m} = \frac{2x\arcsin x}{\sqrt{1-x^2}}, \quad |x|<1.$$

Remark 9.3. Letting $a=\frac{1}{2}$ and $b=2$ in (7.6) and comparing with (2.29) leads to

$${}_2F_1\left(\frac{1}{2},1;2;4x\right) = \frac{1-\sqrt{1-4x}}{2x}, \quad |x| \le \frac{1}{4}.$$

This can also be deduced from the formula

$${}_2F_1(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,\mathrm{d}\,t, \quad \Re(c)>\Re(b)>0$$

in [2, p. 558, 15.3.1] and [9, 9.111].

Remark 9.4. Letting $a=\frac{1}{2}$ and $b=2$ in (7.7) gives

$$\sum_{n=1}^{\infty} \frac{C_n}{4^n} = 1$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n+4)!!} = \frac{1}{2}.$$

Remark 9.5. Taking $a=2$ and $b=\frac{1}{2}$ in (7.8) results in

$$\sum_{n=0}^{\infty} \frac{1}{C_n}\frac{x^{2n}}{(2n)!} = {}_1F_2\left(2;\frac{1}{2},\frac{1}{2};\frac{x^2}{16}\right) = 1+\frac{\pi}{16}x\left[x\boldsymbol{L}_{-1}\left(\frac{x}{2}\right)+6\boldsymbol{L}_0\left(\frac{x}{2}\right)\right],$$

where

$$\boldsymbol{L}_\nu = \left(\frac{z}{2}\right)^{\nu+1}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{2n}$$

denotes the modified Struve function, see [32, p. 228, 11.2.2].

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