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NOTES ON A FAMILY OF INHOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In the note, the authors find several simple and explicit forms for a family of inhomogeneous linear ordinary differential equations studied in "D. Lim, *Differential equations for Daehee polynomials and their applications*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1303–1315; Available online at `http://dx.doi.org/10.22436/jnsa.010.04.02`".

1. Motivation and main results

In [3, Theorem 1], it was obtained inductively and recurrently that the differential equations

$$\frac{\partial^n F(t,x)}{\partial t^n}=\frac{1}{t^n}\left[\sum_{i=0}^{n}\left(a_i(n,x)+\frac{b_i(n,x)}{\ln(1+t)}\right)\left(\frac{t}{1+t}\right)^i\right]F(t,x),\quad n\ge0 \tag{1.1}$$

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have the same solution

$$F(t,x)=\frac{\ln(1+t)}{t}(1+t)^x, \tag{1.2}$$

where $a_0(n,x)=(-1)^n n!$ and $b_0(n,x)=0$ for all $n\ge0$,

$$\begin{aligned} a_j(n,x)=(x)_j(-1)^{n-j}(n-j)!\sum_{i_{j-1}=0}^{n-j}\sum_{i_{j-2}=0}^{n-j-i_{j-1}}\\ \dots\sum_{i_i=0}^{n-j-i_{j-1}-\dots-i_2}(n-i_{j-1}-\dots-i_1-j+1) \end{aligned}$$

and

$$\begin{aligned} b_j(n,x)=\sum_{k=0}^{j-1}\prod_{i=0}^{j-1}\frac{(x-i)(-1)^{n-j}(n-j)!}{x-k}\sum_{i_{j-1}=0}^{n-j}\sum_{i_{j-2}=0}^{n-j-i_{i-1}}\\ \dots\sum_{i_1=0}^{n-j-i_{j-1}-\dots-i_2}(n-i_{j-1}-\dots-i_1-j+1) \end{aligned}$$

for $1\le j\le n+1$, and $(x)_n=\prod_{k=0}^{n-1}(x-k)$ is the falling factorial.

It is clear that the above expressions for $a_j(n,x)$ and $b_j(n,x)$ are very difficult to compute by hand or by computer. The derivation of the quantities $a_j(n,x)$ and $b_j(n,x)$ in [3] is much long and tedious.

The aim of this paper is to alternatively supply several new, simple, and explicit expressions for $a_j(n,x)$ and $b_j(n,x)$. In other words, the aim of this paper is to alternatively provide several new, simple, and explicit forms for the family of differential equations in (1.1).

Our main results can be stated as the following theorems.

Theorem 1. *For $n\ge0$, the function $F(t,x)$ defined in (1.2) satisfies*

$$\frac{\partial^n F}{\partial t^n}=(-1)^n\frac{n!}{t^n}\left\{\sum_{i=0}^{n}(-1)^i\frac{(x)_i}{i!}\left[1-\frac{1}{\ln(1+t)}\sum_{k=1}^{n-i}\frac1k\left(\frac{t}{1+t}\right)^k\right]\left(\frac{t}{1+t}\right)^i\right\}F \tag{1.3}$$

and

$$\frac{\partial^n F}{\partial t^n}=\frac{n!}{(1+t)^n}\left\{\sum_{i=0}^{n}(-1)^i\frac{(x)_{n-i}}{(n-i)!}\left[1-\frac{1}{\ln(1+t)}\sum_{k=1}^{i}\frac1k\left(\frac{t}{1+t}\right)^k\right]\left(\frac{1+t}{t}\right)^i\right\}F.$$

Theorem 2. *Let*

$$\Phi(z,s,a)=\sum_{k=0}^{\infty}\frac{z^k}{(a+k)^s},\quad a\ne0,-1,\dots$$

denote the Lerch transcendent. Then

$$\frac{\partial^n F(t,x)}{\partial t^n}=\frac{(-1)^n n!}{(1+t)^{n-x+1}}\sum_{i=0}^{n}(-1)^i\frac{(x)_i}{i!}\Phi\left(\frac{t}{1+t},1,n-i+1\right)$$

and

$$\frac{\partial^n F(t,x)}{\partial t^n}=\frac{n!}{(1+t)^{n-x+1}}\sum_{i=0}^{n}(-1)^i\frac{(x)_{n-i}}{(n-i)!}\Phi\left(\frac{t}{1+t},1,i+1\right)$$

for $n\ge0$.

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2. A lemma

In order to prove our main results in Theorems 1 and 2, we need the following lemma.

Lemma 1 ([7, Theorem 2]). *Let*

$$f(x)=\begin{cases}\dfrac{\ln x}{x-1}, & 0<x\ne1;\\ 1, & x=1.\end{cases}$$

For $n\ge0$, the nth derivative of $f(x)$ can be computed by

$$f^{(n)}(x)=\begin{cases}\dfrac{(-1)^n n!}{(x-1)^{n+1}}\left[\ln x-\displaystyle\sum_{k=1}^{n}\frac1k\left(\frac{x-1}{x}\right)^k\right], & 0<x\ne1;\\ (-1)^n\dfrac{n!}{n+1}, & x=1\end{cases}\tag{2.1}$$

and

$$f^{(n)}(x)=\begin{cases}(-1)^n\dfrac{n!}{x^{n+1}}\Phi\left(\dfrac{x-1}{x},1,n+1\right), & 0<x\ne1;\\ (-1)^n\dfrac{n!}{n+1}, & x=1.\end{cases}\tag{2.2}$$

3. Proofs of Theorems 1 and 2

We now start out to prove our main results in Theorems 1 and 2.

Proof of Theorem 1. By (2.1), it is straightforward that

$$\begin{aligned}\frac{\partial^n F(t,x)}{\partial t^n}&=\sum_{i=0}^{n}\binom{n}{i}\left[\frac{\ln(1+t)}{t}\right]^{(n-i)}[(1+t)^x]^{(i)}\\
&=\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{n-i}(n-i)!}{t^{n-i+1}}\left[\ln(1+t)-\sum_{k=1}^{n-i}\frac1k\left(\frac{t}{1+t}\right)^k\right](x)_i(1+t)^{x-i}\\
&=\frac{(1+t)^x\ln(1+t)}{t}\frac{(-1)^n n!}{t^n}\sum_{i=0}^{n}\frac{(-1)^i(x)_i}{i!}\left[1-\frac{1}{\ln(1+t)}\sum_{k=1}^{n-i}\frac1k\left(\frac{t}{1+t}\right)^k\right]\left(\frac{t}{1+t}\right)^i\\
&=(-1)^n\frac{n!}{t^n}\left\{\sum_{i=0}^{n}(-1)^i\frac{(x)_i}{i!}\left[1-\frac{1}{\ln(1+t)}\sum_{k=1}^{n-i}\frac1k\left(\frac{t}{1+t}\right)^k\right]\left(\frac{t}{1+t}\right)^i\right\}F(t,x)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^n F(t,x)}{\partial t^n}&=\sum_{i=0}^{n}\binom{n}{i}\left[\frac{\ln(1+t)}{t}\right]^{(i)}[(1+t)^x]^{(n-i)}\\
&=\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^i i!}{t^{i+1}}\left[\ln(1+t)-\sum_{k=1}^{i}\frac1k\left(\frac{t}{1+t}\right)^k\right](x)_{n-i}(1+t)^{x-n+i}\\
&=\frac{\ln(1+t)}{t(1+t)^{n-x}}\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^i i!}{t^i}\left[1-\frac{1}{\ln(1+t)}\sum_{k=1}^{i}\frac1k\left(\frac{t}{1+t}\right)^k\right](x)_{n-i}(1+t)^i\\
&=\frac{F(t,x)}{(1+t)^n}\sum_{i=0}^{n}(-1)^i\frac{n!}{(n-i)!}\left[1-\frac{1}{\ln(1+t)}\sum_{k=1}^{i}\frac1k\left(\frac{t}{1+t}\right)^k\right](x)_{n-i}\left(\frac{1+t}{t}\right)^i\end{aligned}$$

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$$= \frac{n!}{(1+t)^n}\left\{\sum_{i=0}^{n}(-1)^i\frac{(x)_{n-i}}{(n-i)!}\left[1-\frac{1}{\ln(1+t)}\sum_{k=1}^{i}\frac{1}{k}\left(\frac{t}{1+t}\right)^k\right]\left(\frac{1+t}{t}\right)^i\right\}F(t,x).$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. By (2.2), it is immediate that

$$\begin{aligned}
\frac{\partial^n F(t,x)}{\partial t^n} &= \sum_{i=0}^{n}\binom{n}{i}\left[\frac{\ln(1+t)}{t}\right]^{(n-i)}[(1+t)^x]^{(i)}\\
&= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\frac{(n-i)!}{(1+t)^{n-i+1}}\Phi\left(\frac{t}{1+t},1,n-i+1\right)(x)_i(1+t)^{x-i}\\
&= \frac{(-1)^n n!}{(1+t)^{n-x+1}}\sum_{i=0}^{n}(-1)^i\frac{(x)_i}{i!}\Phi\left(\frac{t}{1+t},1,n-i+1\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^n F(t,x)}{\partial t^n} &= \sum_{i=0}^{n}\binom{n}{i}\left[\frac{\ln(1+t)}{t}\right]^{(i)}[(1+t)^x]^{(n-i)}\\
&= \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\frac{i!}{(1+t)^{i+1}}\Phi\left(\frac{t}{1+t},1,i+1\right)(x)_{n-i}(1+t)^{x-n+i}\\
&= \frac{n!}{(1+t)^{n-x+1}}\sum_{i=0}^{n}(-1)^i\frac{(x)_{n-i}}{(n-i)!}\Phi\left(\frac{t}{1+t},1,i+1\right).
\end{aligned}$$

The proof of Theorem 2 is complete. □

4. Remarks

Finally we give several remarks about our main results mentioned and verified above.

Remark 1. Comparing (1.1) with (1.3) reveals that the formulas

$$a_i(n,x) = (-1)^{n-i}\frac{n!}{i!}(x)_i$$

and

$$b_i(n,x) = -a_i(n,x)\sum_{k=1}^{n-i}\frac{1}{k}\left(\frac{t}{1+t}\right)^k = (-1)^{n-i+1}\frac{n!}{i!}(x)_i\sum_{k=1}^{n-i}\frac{1}{k}\left(\frac{t}{1+t}\right)^k$$

should be valid for all $i,n\ge 0$.

Remark 2. The idea of this paper comes from the articles [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the closely related references therein.

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