Quantum Computing, Computer Engineering, Information Theory, Entanglement

Detecting Violation of Bell Inequalities using LOCC Maximized Quantum Fisher Information and Entanglement Measures

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Abstract

The violation of Bell's theorem is a very simple way to see that there is no underlying classical interpretation of quantum mechanics. The measurements made on the photons shows that light signal (information) could travel between them, hence completely eliminating any chance that the result was due to anything other than entanglement. Entanglement has been studied extensively for understanding the mysteries of non-classical correlations between quantum systems. It was found that violation of Bell's inequalities could be trivially calculated and for sets of nonmaximally entangled states of two qubits, comparing these entanglement measures may lead to different entanglement orderings of the states. On the other hand, although it is not an entanglement measure and not monotonic under local operations, due to its ability of detecting multipartite entanglement, quantum Fisher information (QFI) has recently received an intense attraction generally with entanglement in the focus. In this work, we visit violation of Bell's inequalities problem with a different approach. Generating a thousand random quantum states and performing an optimization based on local general rotations of each qubit, we calculate the maximal QFI for each state. We analyze the maximized QFI in comparison with violation in Bell's inequalities and we make similar comparison of this violation with commonly studied entanglement measures, negativity and relative entropy of entanglement. We show that there are interesting orderings for system states.

Keywords: entanglement; relative entropy of entanglement; negativity; bell inequalities violation; quantum fisher information; optimization

Introduction

Entanglement studies in bipartite and multi-partite forms are popular topics since the seminal ERP paper [1]. Algebraically, entanglement may be defined as inseparability of quantum states. Quantum nonlocality can be considered as a correlation that cannot be explained by local hidden-variable theories. Bell-type inequalities [2, 3] are generally used to express these kind of nonlocalities quantitatively [4]. In this paper our focus will be observation of the violation of Bell’s inequality in the form CHSH inequality [3].

For two qubits, Bell inequalities can be violated only if their states are entangled. However, as shown by Werner [5], there are entangled states that can still exhibit correlations which do not violate any Bell inequality for any possible local measurements; that is, unless a sequence of measurements, or several copies, or other more sophisticated scenarios are applied [4]. Therefore, a natural question can be raised as to how much entangled states can be without violating the CHSH inequality or, more generally, for any fixed degree of the CHSH violation.

Concurrence and negativity entanglement measures are based on the eigenvalues of the density matrix (after some transformations for Concurrence). REE is based on the distance of the state to the closest separable state. Although the value of such measures for separable states turns to be 0, and 1 for maximally entangled states, it was found that this

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is not the case for the states in between [6-9,45-47]: There are pairs of states that have equal values of one measure but different values of another measure.

While detecting entanglement, quantum Fisher information (QFI) has been found to be a useful tool [10]. In particular, if a state exceeds the shot noise level (SNL) that the best separable states can achieve, which is 1, then that state is entangled. Therefore, besides providing information for the phase sensitivity of the state with respect to SU(2) rotations in the context of quantum metrology [11-21]; QFI has received an intense attention with the entanglement in the focus [22-29]. There are attempts to find a QFI based general entanglement measure that can quantify not only bipartite but also multipartite states and even bound entangled states [26]. Since QFI provides information for the sensitivity of a state with respect to changes in the state, including unitary operations, it is naturally non-invariant with respect to changes in general. Therefore when comparing the QFI of a state with entanglement measures (which are invariant with respect to unitary rotations), it would be plausible to perform an optimization to find the maximal QFI of the state over all possible local unitary rotations.

Since QFI plays an important role in various aspects of entanglement, in this work, we extend the state ordering problem for quantifying CHSH violation which was studied in the context of the standard entanglement measures, with the maximized QFI. We generate a thousand two-qubit states, calculate their negativity values, run a simulation for obtaining the relative entropy of entanglement values, calculate QFI and maximize the QFI with respect to general local unitary operations and finally analyze these results. We study the relation between the CHSH violation and negativity (and other entanglement measures) for arbitrary two-qubit states analogously to the comparisons of the CHSH violation with REE [30]. We find different result classes and report them.

It is worth noting an increasing interest in developing device-independent approaches to entanglement testing and quantifying, which are based on various Bell inequality violations. We believe that our work may contribute to this lively field.

Results

For the problem setup defined in this paper's scope, first we reproduced the comparison result between Negativity and $M(\rho)$ values [31]. In Figure 1, this result can be visualised. From the definitions, while $M(\rho) > 1$ the CHSH inequalities are violated. We run the Monte Carlo simulation for 1 million random states (dots above the black line).

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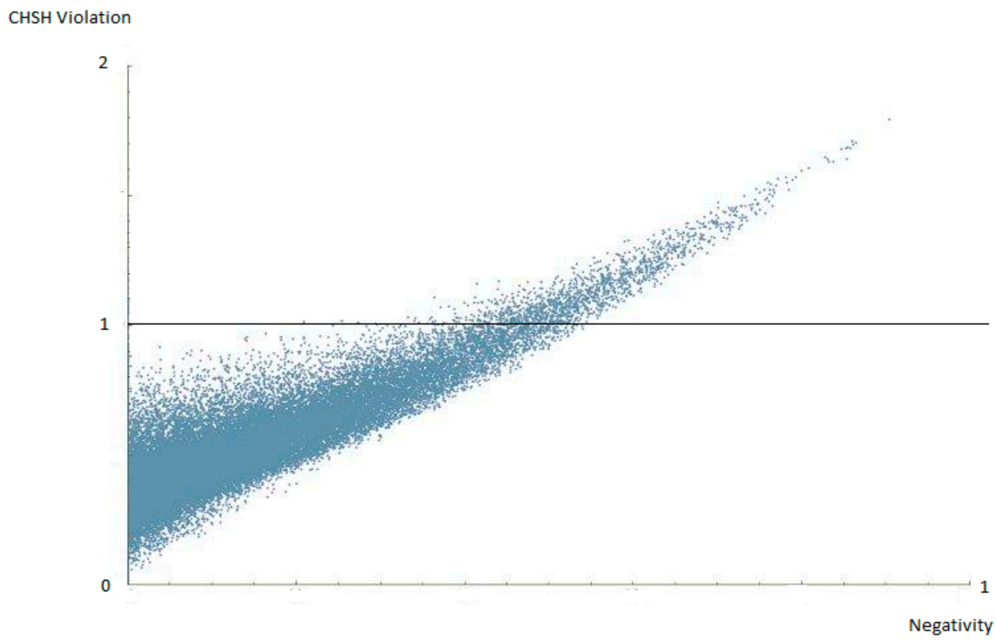

Figure 1 | Comparison of Negativity vs *M(ρ)* (A measure for CHSH violation) values for 1 million random states

Secondly, we looked at the similar comparison between CHSH violation and Relative Entropy of Entanglement (REE). Similarly while $M(\rho) > 1$ the CHSH inequalities are violated. We run the Monte Carlo simulation for 1000 random states (dots above the blue line). In Figure 2, this results can be viewed. Due to results that we obtained: For this 1000 random states set while a system's REE > 0.2589, CHSH violation is strictly observed. For 0.1367 < REE < 0.2589 both cases are possible. For REE < 0.1367 CHSH violation does not occur. It can be understood that for REE value a threshold may be set for observing the CHSH violation.

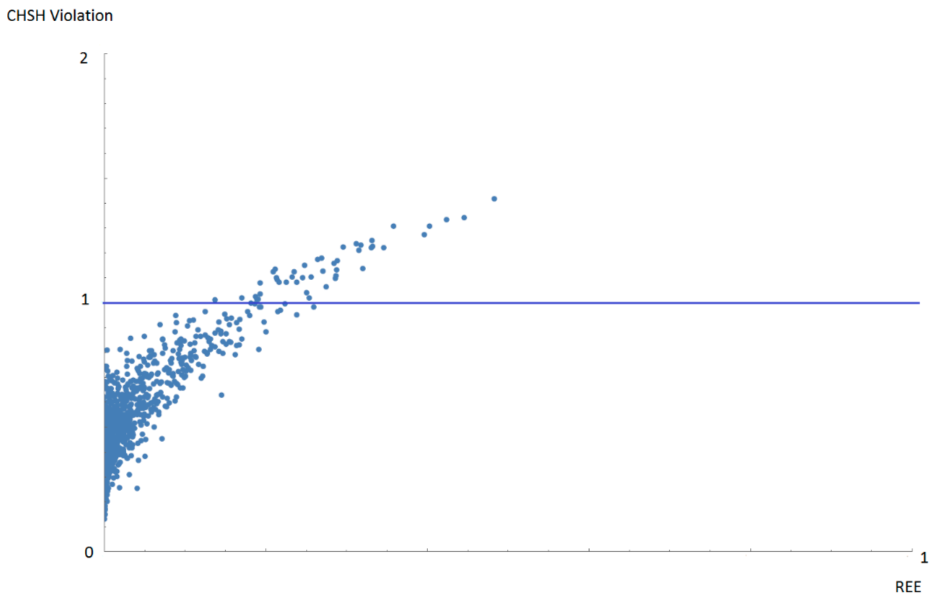

Figure 2 | Comparison of REE vs *M(ρ)* (A measure for CHSH violation) values for 1000 random states

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Finally, for these 1000 random states set, we looked at the comparison between CHSH violation and Quantum Fisher Information (QFI). We defined a LOCC Maximization procedure for calculating Maximized QFI values. The procedure may be read in methods section. In Figure 3, it is possible to visualize these comparisons. Purple dots are indicating the comparison between QFI and CHSH Violation and green dots are expressing the comparison between Maximized QFI and CHSH. For state ordering problems, it was shown in [32] that Maximized QFI is providing more meaningful results.

We obtain 3 classes of results in this Maximized QFI and CHSH Violation comparison:

- Class 1: $M(\rho)$ > Maximized QFI and $M(\rho) > 1$. For these states in this class CHSH violation is observed.
- Class 2: $M(\rho)$ > Maximized QFI and $M(\rho) < 1$. For these states in this class CHSH violation is not observed.
- Class 3: $M(\rho)$ < Maximized QFI and $M(\rho) < 1$. For these states also in this class CHSH violation is not observed.

More surprisingly, there is no Class 4 conforming the case: $M(\rho)$ < Maximized QFI and $M(\rho) > 1$.

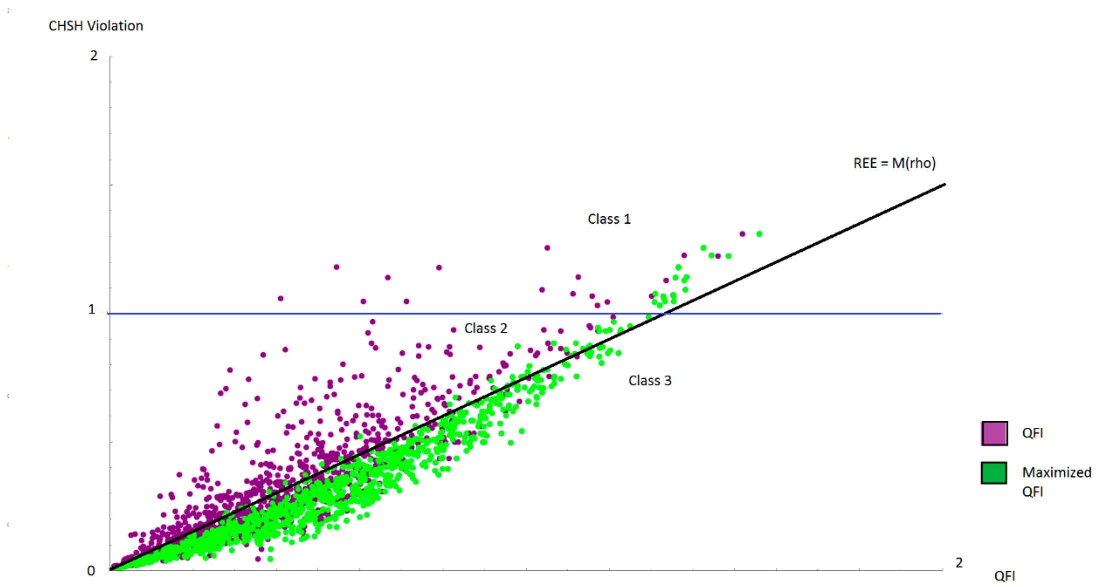

Figure 3 | Comparison of QFI vs $M(\rho)$ (A measure for CHSH violation) values for same 1000 random states (Purple QFI, Green Maximized QFI)

Discussion

It was found that the comparison between REE and CHSH violation measure is very interesting and meaningful. It may be possible that for REE value a threshold may be set for observing the CHSH violation. According to the comparison shown in Results section for the REE measure values greater a certain value (REE > 0.2589) implicates CHSH violation. Inverse case is also observed (REE < 0.1367). But interestingly, for a range of REE values (0.1367 < REE < 0.2589) both cases are possible. Due to our current knowledge REE, as a natural entanglement measure, is also useful for detecting CHSH violation for certain cases shown here.

Our second conclusion is about QFI's position in this CHSH violation observation problem. Like state ordering problems, in this problem Maximized QFI is providing meaningful results. We obtained 3 classes of results in this Maximized QFI and CHSH Violation comparison that are explained in more detail in Results section. But more interestingly, there is no 4^{th} class having the case $M(\rho)$ < Maximized QFI and $M(\rho) > 1$. This conclusion is also noted.

This finding opened new insights in our understanding the entanglement and CHSH violation connection. Regarding the detection of multipartite entanglement and providing information for the phase sensitivity of a state with respect to SU(2) rotations, quantum Fisher information (QFI) is a practical tool and is being studied within the context of entanglement [10,26].

Incorporating QFI, we have extended the CHSH violation observation problem of general two qubit systems. Since QFI is not monotonic under LOCC, we have maximized QFI with respect to general Euler rotations of each qubit. We have chosen the maximal values of QFI and compared these values with the values obtained by the defined CHSH violation measure.

On the other hand, effect of maximizing QFI becomes significant for revealing the dynamics of comparison of REE with the other two measures, in the comparison of QFI with these measures. An interesting result is that QFI of almost all of the generated random states were both maximized and minimized under LOCC via general Euler rotations, even with large steps (each being $\frac{\pi}{2}$) when scanning the whole space, $[0, 2\pi]$. The few states that remained unchanged were both maximized and minimized as the step size was chosen to be $\frac{\pi}{3}$.

Observing that QFI of all the random states are both maximized and minimized via local SU(2) operations might open new questions for finding non-maximally mixed but QFI non-invariant states under rotations in a similar vein to quantum discord (QD), since QD of SU(2) invariant states has been recently studied [33,34].

We believe that our results can be useful for understanding the relation between QFI and entanglement not only in the bipartite but also in the multipartite settings.

Methods

Violating Bell inequality: Measure of CHSH violation

The CHSH inequality for a two-qubit state $\rho \equiv \rho_{AB}$ can be written as [35,36]:

$$\left| Tr(\rho \mathrm{B}_{CHSH}) \right| \leq 2 \tag{1}$$

in terms of the CHSH operator

$$B_{CHSH} = \vec{a}.\vec{\sigma} \otimes (\vec{b} + \vec{b}').\vec{\sigma} + \vec{a}'.\vec{\sigma} \otimes (\vec{b} - \vec{b}').\vec{\sigma} \tag{2}$$

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where $\vec{a}$, $\vec{a}'$ and $\vec{b}$, $\vec{b}'$ are unit vectors describing the measurements on side A (Alice) and B (Bob), respectively. A shown by Horodecki et al. [37,38] by optimizing the vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$, $\vec{b}'$, the maximum possible average value of the Bell operator for the state ρ is given by

$$\max_{B_{CHSH}} \left| Tr\left(\rho B_{CHSH}\right) \right| = 2\sqrt{M\left(\rho\right)} \quad (3)$$

where $M(\rho) = \max_{j<k} \{h_j + h_k\} \leq 2$, and h_j *(j=1,2,3)* are the eigenvalues of the matrix $U=T^T T$ constructed from the correlation matrix *T* and its transpose T^T. [31]

In other words, $M(\rho) = \tau_1 + \tau_2$ where τ_1 and τ_2 are maximum eigenvalues of symmetric *U* matrix.

It can be showed that

$$M\left(\rho\right) = \tau_1 + \tau_2 = \max\{c_1^2 + c_2^2, c_1^2 + c_3^2, c_2^2 + c_3^2\} \quad (4)$$

Now our question become calculating $\tau_1 + \tau_2$ for any random states / density matrix.

In order to quantify the violation of CHSH inequality one can use *M(ρ)* or equivalently

$$B\left(\rho\right) = \sqrt{\max\left[0, M\left(\rho\right) - 1\right]} \quad (5)$$

where *B=0* means CHSH inequality is not violated and *B=1* means is maximally violated. [31]

Negativity

Negativity can be considered a quantitative version of the Peres-Horodecki criterion [39,40]. The negativity for a two-qubit state σ is defined as [6,26,41]:

$$N(\rho) = 2\sum_i \max(0, -\mu_i) \quad (6)$$

where μ_i is the negative eigenvalues of the partial transpose of ρ. Negativity also ranges between 0 for a separable state and 1 for a maximally entangled state. As shown by Vidal and Werner [41], the negativity is an entanglement monotone which means that it can be considered as a useful measure of entanglement.

Relative Entropy of Entanglement (REE)

Relative Entropy of Entanglement (REE) of a given state *σ,* which is defined by Vedral et al [42,43] as the minimum of the quantum relative entropy $S(\rho||\sigma) = Tr(\rho log\rho - \rho log\sigma)$ taken over the set D of all separable states σ, namely

$$E(\rho) = \min_{\rho \in D} S\left(\rho \,||\, \sigma\right) = S(\rho \,||\, \bar{\sigma}) \quad (7)$$

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where $\bar{\sigma}$ denotes a separable state closest to ρ. In general, REE is calculated numerically using the methods described in [43,44].

Quantum Fisher Information

Defining the fictitious angular momentum operators on each qubit in each direction,

$$J_{\vec{n}} = \sum_{\alpha=x,y,z} \frac{1}{2} n_\alpha \sigma_\alpha \quad \textbf{(8)}$$

and the quantum Fisher information of a state ρ of N particles with eigenvalues p_i and the associated eigenvectors $|i\rangle$ in each direction as $F(\rho, J_{\vec{n}})$, the maximal mean quantum Fisher information of the state can be found as

$$\bar{F}_{\max} = \frac{1}{N} \max_{\vec{n}} F(\rho, J_{\vec{n}}) = \frac{\lambda_{\max}}{N}. \quad \textbf{(9)}$$

where $\lambda_{\max}$ is the largest eigenvalue of the 3x3 symmetric matrix C, of which elements can be calculated by [28]

$$C_{kl} = \sum_{i \neq j} \frac{(p_i - p_j)^2}{p_i + p_j} \left[\langle i | J_k | j \rangle \langle j | J_l | i \rangle + \langle i | J_l | j \rangle \langle j | J_k | i \rangle \right], \quad \textbf{(10)}$$

where $k, l \in \{x, y, z\}$.

Random State Density Matrix Generation

In this work, the density matrices of the random states are generated as follows:

$$\rho = VPV^{\dagger} \quad \textbf{(11)}$$

Where $P = diag(\lambda_i)$ diagonal matrix of eigenvalues and V is the unitary matrix [6]. To get such random state we have used the methods in the QI package [39].

Optimization of QFI

We have performed the maximization via general rotations of each qubit in the Euler representation

$$U_{Rot}(\alpha, \beta, \gamma) = U_x(\alpha) U_z(\beta) U_x(\gamma) \quad \textbf{(12)}$$

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where the rotations about axes are defined as $U_j(\alpha) = \exp(-i\alpha\frac{\sigma_j}{2})$, $j \in \{x, z\}$, with arbitrary three angles for each qubit between $[0, 2\pi]$, each with steps of θ degrees, resulting $O((\frac{2\pi}{\theta})^6)$ QFI calculations. We have found that choosing the steps as $\theta = \frac{\pi}{2}$ is sufficient for a good optimization such that the picture, whereas narrowing the steps could possibly result a better optimization, with the cost of an increase in the running time of the simulation.

According to the results that we obtained, for the chosen a thousand random states, %98 of the QFI values are maximized and %99 of the QFI values are minimized with $\theta = \frac{\pi}{2}$ and the all the rest with $\theta = \frac{\pi}{3}$. [32]

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