Peer-reviewed version available at *Bulletin of the Iranian Mathematical Society* **2019**, *45*, 1821-1829; doi:10.1007/s41980-019-00232-4

CLOSED EXPRESSIONS OF THE FIBONACCI POLYNOMIALS IN TERMS OF TRIDIAGONAL DETERMINANTS

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Abstract. In the paper, the authors find a new closed expression for the Fibonacci polynomials and, consequently, for the Fibonacci numbers, in terms of a tridiagonal determinant.

1. Main results

A tridiagonal matrix (determinant) is a square matrix (determinant) with nonzero elements only on the diagonal and slots horizontally or vertically adjacent the diagonal. See the paper [14] and closely-related references therein.

A square matrix $H=(h_{ij})_{n\times n}$ is called a tridiagonal matrix if $h_{ij}=0$ for all pairs (i,j) such that $|i-j|>1$. On the other hand, a matrix $H=(h_{ij})_{n\times n}$ is called a lower (or an upper, respectively) Hessenberg matrix if $h_{ij}=0$ for all pairs (i,j) such that $i+1<j$ (or $j+1<i$, respectively). See the paper [22] and closely-related references therein.

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2010 *Mathematics Subject Classification.* 11B39, 11B83, 11C20, 11Y55, 15A15, 65F40.

Key words and phrases. closed expression; Fibonacci number; Fibonacci polynomial; tridiagonal determinant; Hessenberg determinant.

This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 March 2017 doi:10.20944/preprints201703.0208.v1

Peer-reviewed version available at Bulletin of the Iranian Mathematical Society 2019, 45, 1821-1829; doi:10.1007/s41980-019-00232-4

In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.

It is general knowledge that the Bernoulli numbers and polynomials B_k and $B_k(u)$ for $k \ge 0$ satisfy $B_k(0) = B_k$ and can be generated respectively by

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!} \quad \text{and} \quad \frac{ze^{uz}}{e^z-1} = \sum_{k=0}^{\infty} B_k(u) \frac{z^k}{k!}$$

for $|z| < 2\pi$. Because the function $\frac{x}{e^x-1} - 1 + \frac{x}{2}$ is odd in $x \in \mathbb{R}$, all of the Bernoulli numbers B_{2k+1} for $k \in \mathbb{N}$ equal 0. In [26, Theorem 1.2], the Bernoulli polynomials $B_k(u)$ for $k \in \mathbb{N}$ were expressed by lower Hessenberg determinants

$$B_k(u) = (-1)^k \left| \frac{1}{\ell+1} \binom{\ell+1}{m} \left[(1-u)^{\ell-m+1} - (-u)^{\ell-m+1}\right] \right|_{1 \le \ell \le k, 0 \le m \le k-1}.$$

Consequently, the Bernoulli numbers B_k for $k \in \mathbb{N}$ can be represented by lower Hessenberg determinants

$$B_k = (-1)^k \left| \frac{1}{\ell+1} \binom{\ell+1}{m} \right|_{1 \le \ell \le k, 0 \le m \le k-1}.$$

For $k \in \{0\} \cup \mathbb{N}$ and $x \in \mathbb{R}$, the Euler numbers E_k and the Euler polynomials $E_k(x)$ can be generated respectively by

$$\frac{2e^{t/2}}{e^t+1} = \sum_{k=0}^{\infty} \frac{E_k}{k!} \left(\frac{t}{2}\right)^k \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{k=0}^{\infty} \frac{E_k(x)}{k!} t^k$$

for $t \in (-\pi, \pi)$. Since the generating function $\frac{2e^{t/2}}{e^t+1}$ of the Euler numbers E_k is even on $(-\pi, \pi)$, then $E_{2k-1} = 0$ for all $k \in \mathbb{N}$. At the website [35], the Euler numbers E_{2k} were represented by lower Hessenberg determinants

$$E_{2k} = (-1)^k (2k)! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \frac{1}{(2k-6)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(2k)!} & \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}, \quad k \in \mathbb{N}.$$

In [34, Theorem 1.1], the Euler numbers E_{2k} for $k \in \mathbb{N}$ were represented by lower Hessenberg and sparse determinants

$$E_{2k} = (-1)^k \left| \binom{i}{j-1} \cos\left((i-j+1)\frac{\pi}{2}\right) \right|_{(2k) \times (2k)}.$$

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Recently, the first author represented the Euler polynomials $E_k(x)$ for $k \ge 0$ by lower Hessenberg determinants

$$E_k(x) = \frac{(-1)^k}{2^k}\begin{vmatrix} 1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0 & 0 \\ x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0 & 0 \\ x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2 & 0 \\ x^{k-1} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2} & 2 \\ x^k & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} & \binom{k}{k-1} \end{vmatrix}.$$

Consequently, the Euler numbers E_k for $k \ge 0$ can be expressed by lower Hessenberg determinants

$$E_k = (-1)^k\begin{vmatrix} 1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2} & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2^2} & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0 & 0 \\ \frac{1}{2^3} & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{2^{k-2}} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2 & 0 \\ \frac{1}{2^{k-1}} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2} & 2 \\ \frac{1}{2^k} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} & \binom{k}{k-1} \end{vmatrix}.$$

For more and detailed information on the Bernoulli numbers B_k, the Bernoulli polynomials $B_k(u)$, the Euler numbers E_k, and the Euler polynomials $E_k(x)$, please refer to recently published papers such as [2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 18, 19, 20, 21, 23, 25, 26, 27, 28, 29, 31, 34] and plenty of closely-related references therein.

It is well-known that the Fibonacci numbers

$$F_n = \frac{\bigl(1+\sqrt{5}\,\bigr)^n - \bigl(1-\sqrt{5}\,\bigr)^n}{2^n\sqrt{5}}$$

for $n \in \mathbb{N}$ form a sequence of integers and satisfy the linear recurrence relation

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

with $F_1 = F_2 = 1$. The first fourteen Fibonacci numbers F_n for $1 \le n \le 14$ are

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad 89, \quad 144, \quad 233, \quad 377.$$

The Fibonacci numbers F_n can be viewed as a particular case $F_n(1)$ of the Fibonacci polynomials

$$F_n(s) = \frac{1}{2^n}\frac{\bigl(s+\sqrt{4+s^2}\,\bigr)^n - \bigl(s-\sqrt{4+s^2}\,\bigr)^n}{\sqrt{4+s^2}}$$

which can be generated by

$$\frac{t}{1-ts-t^2} = \sum_{n=1}^{\infty} F_n(s)t^n = t + st^2 + \bigl(s^2+1\bigr)t^3 + \bigl(s^3+2s\bigr)t^4 + \cdots. \tag{2}$$

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See the monograph [5], the websites [32, 33], and related references therein. In [17, p. 215, Example 1], it was deduced that

$$F_n=\begin{vmatrix}1 & -1 & 0 & 0 & \cdots & 0 & 0\\ 1 & 1 & -1 & 0 & \cdots & 0 & 0\\ 0 & 1 & 1 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & -1\\ 0 & 0 & 0 & 0 & \cdots & 1 & 1\end{vmatrix}_{(n-1)\times(n-1)}, \quad n\in\mathbb{N}.$$

In [15, 16], among other things, it was listed that

$$F_{r+1}(x)=\begin{vmatrix}x & 1 & 0 & \cdots & 0 & 0\\ -1 & x & 1 & \cdots & 0 & 0\\ 0 & -1 & x & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & x & 1\\ 0 & 0 & 0 & \cdots & -1 & x\end{vmatrix}_{r\times r}, \quad F_{r-1}=\begin{vmatrix}1 & 1 & 1 & \cdots & 1 & 1\\ -1 & 0 & 1 & \cdots & 1 & 1\\ 0 & -1 & 0 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & -1 & 0\end{vmatrix}_{r\times r},$$

$$F_{r-1}=|q_{ij}|_{r\times r}, \quad F_{2r}=\begin{vmatrix}1 & 2 & 3 & \cdots & r-1 & r\\ -1 & 1 & 2 & \cdots & r-2 & r-1\\ 0 & -1 & 1 & \cdots & r-3 & r-2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 2\\ 0 & 0 & 0 & \cdots & -1 & 1\end{vmatrix}_{r\times r}, \quad F_{2r+1}=|Q_{ij}|_{r\times r}$$

for $r\in\mathbb{N}$, where

$$q_{ij}=\begin{cases}-1, & i-j+1;\\ i+j+1 \mod 2, & i\le j;\\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad Q_{ij}=\begin{cases}-1, & i=j+1;\\ 2, & i=j;\\ 1, & i<j;\\ 0, & \text{otherwise.}\end{cases}$$

In this paper, motivated by the above closed expressions for the Bernoulli numbers B_k, the Bernoulli polynomials $B_k(u)$, the Euler numbers E_k, the Euler polynomials $E_k(x)$, the Fibonacci numbers F_n, and the Fibonacci polynomials $F_n(s)$, we will find a new closed expression for the Fibonacci polynomials $F_n(s)$ and, consequently, for the Fibonacci numbers F_n, in terms of a tridiagonal determinant.

The main results of this paper can be stated as the following theorem.

Theorem 1. *For $n \in \mathbb{N}$, the Fibonacci polynomials $F_n(s)$ can be expressed as*

$$F_n(s) = \frac{1}{n!}\begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\cdot 1 & 2s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3\cdot 2 & 3s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (n-2)s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & (n-1)(n-2) & (n-1)s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n(n-1) & ns \end{vmatrix} \tag{3}$$

and, consequently,

$$F_n = \frac{1}{n!}\begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\cdot 1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3\cdot 2 & 3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & (n-1)(n-2) & n-1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n(n-1) & n \end{vmatrix}. \tag{4}$$

2. A lemma

In order to supply a concise proof of Theorem 1, we need the following lemma which can be found in [1, p. 40, Exercise 5)], [24, Section 2.2, p. 849], [26, p. 94], [30, Remark 6], and [34, Lemma 2.1].

Lemma 1. *Let $u(x)$ and $v(x) \ne 0$ be two differentiable functions, let $U_{(n+1)\times 1}(x)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(x)$ be an $(n+1)\times n$ matrix whose elements*

$$v_{i,j}(x) = \begin{cases} \dbinom{i-1}{j-1} v^{(i-j)}(x), & i-j \ge 0; \\ 0, & i-j < 0 \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(x)|$ denote the lower Hessenberg determinant of the $(n+1)\times(n+1)$ lower Hessenberg matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{bmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{bmatrix}.$$

Then the n-th derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{v^{n+1}(x)}. \tag{5}$$

3. Proof of Theorem 1

Now we start out to concisely prove Theorem 1 by virtue of the formula (5).

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Applying $u(t)=t$ and $v(t)=1-ts-t^2$ to the formula (5) yields

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\biggl(\frac{t}{1-ts-t^2}\biggr)=\frac{(-1)^n}{(1-ts-t^2)^{n+1}}\begin{vmatrix}
t & 1-ts-t^2 & 0 & 0 & \cdots & 0 & 0 & 0\\
1 & -(2t+s)\binom{1}{0} & 1-ts-t^2 & 0 & \cdots & 0 & 0 & 0\\
0 & -2\binom{2}{0} & -(2t+s)\binom{2}{1} & 1-ts-t^2 & \cdots & 0 & 0 & 0\\
0 & 0 & -2\binom{3}{1} & -(2t+s)\binom{3}{2} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & -(2t+s)\binom{n-2}{n-3} & 1-ts-t^2 & 0\\
0 & 0 & 0 & 0 & \cdots & -2\binom{n-1}{n-3} & -(2t+s)\binom{n-1}{n-2} & 1-ts-t^2\\
0 & 0 & 0 & 0 & \cdots & 0 & -2\binom{n}{n-2} & -(2t+s)\binom{n}{n-1}
\end{vmatrix}$$

$$\to(-1)^n\begin{vmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\
1 & -s\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 & 0\\
0 & -2\binom{2}{0} & -s\binom{2}{1} & 1 & \cdots & 0 & 0 & 0\\
0 & 0 & -2\binom{3}{1} & -s\binom{3}{2} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & -s\binom{n-2}{n-3} & 1 & 0\\
0 & 0 & 0 & 0 & \cdots & -2\binom{n-1}{n-3} & -s\binom{n-1}{n-2} & 1\\
0 & 0 & 0 & 0 & \cdots & 0 & -2\binom{n}{n-2} & -s\binom{n}{n-1}
\end{vmatrix}$$

as $t\to0$ for $n\in\mathbb{N}$. By the generating function in (2), we obtain that

$$F_n(s)=\frac{1}{n!}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\biggl(\frac{t}{1-ts-t^2}\biggr)$$

$$=\frac{(-1)^n}{n!}\begin{vmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\
1 & -s\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 & 0\\
0 & -2\binom{2}{0} & -s\binom{2}{1} & 1 & \cdots & 0 & 0 & 0\\
0 & 0 & -2\binom{3}{1} & -s\binom{3}{2} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & -s\binom{n-2}{n-3} & 1 & 0\\
0 & 0 & 0 & 0 & \cdots & -2\binom{n-1}{n-3} & -s\binom{n-1}{n-2} & 1\\
0 & 0 & 0 & 0 & \cdots & 0 & -2\binom{n}{n-2} & -s\binom{n}{n-1}
\end{vmatrix}$$

which can be rewritten as the expression (3).

The expression (4) follows from taking the limit $s\to1$ on both sides of the expression (3). The proof of Theorem 1 is complete.

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4. Remarks

Finally, we list several remarks on Lemma 1, Theorem 1, and others as follows.

Remark 1. As showed, for example, in recently published papers [24, 26, 30, 34] and in this paper, the formula (5) in Lemma 1 is an effectual tool to express some quantities in mathematics as lower Hessenberg determinants.

Remark 2. It is easy to see that, from the expressions (3) and (4), we can recover the recurrence relation

$$F_n(s) = sF_{n-1}(s) + F_{n-2}(s)$$

in [33] and the recurrence relation (1) for $n \geq 3$.

Remark 3. The expressions (3) and (4) can be rearranged as

$$F_n(s) = \frac{1}{n!}\begin{vmatrix} \binom{2}{1}s & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2\binom{3}{1} & \binom{3}{2}s & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{4}{2} & \binom{4}{3}s & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{5}{3} & \binom{5}{4}s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3}s & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2}s & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1}s \end{vmatrix}$$

and

$$F_n = \frac{1}{n!}\begin{vmatrix} \binom{2}{1} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2\binom{3}{1} & \binom{3}{2} & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\binom{4}{2} & \binom{4}{3} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2\binom{5}{3} & \binom{5}{4} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-3} & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & \binom{n-1}{n-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}$$

for $n \geq 2$.

Remark 4. The Bernoulli numbers B_n, the Bernoulli polynomials $B_n(u)$, the Euler numbers E_n, and the Euler polynomials $E_n(x)$ can be expressed as different closed forms. Let $S(n, m)$ stand for the Stirling numbers of the second kind which can be computed by

$$S(n,m) = \frac{1}{m!}\sum_{\ell=1}^{m}(-1)^{m-\ell}\binom{m}{\ell}\ell^n, \quad 1 \leq m \leq n.$$

In [26, Theorem 1.1], the Bernoulli polynomials $B_n(u)$ were expressed as

$$B_n(u) = \sum_{k=1}^{n} k! \sum_{r+s=k} \sum_{\ell+m=n} (-1)^m \binom{n}{\ell} \frac{\ell!}{(\ell+r)!} \frac{m!}{(m+s)!} \times \left[\sum_{i=0}^{r}\sum_{j=0}^{s}(-1)^{i+j}\binom{\ell+r}{r-i}\binom{m+s}{s-j}S(\ell+i,i)S(m+j,j)\right]u^{m+s}(1-u)^{\ell+r}$$

for $n \in \mathbb{N}$. Consequently, the Bernoulli numbers B_n can be represented as

$$B_n = \sum_{i=1}^{n} (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i,i), \quad n \in \mathbb{N}.$$

In [34, Theorem 1.3], the expressions

$$E_n = 1 + \sum_{k=1}^{n} \frac{(k+1)!}{2^k} S(n,k) \sum_{\ell=1}^{k} (-1)^\ell \frac{2^\ell}{\ell+1} \binom{\ell+1}{k-\ell}$$

and

$$E_n = 1 + \sum_{\ell=1}^{n} (-1)^\ell \frac{1}{\ell+1} \sum_{k=0}^{n-\ell} \frac{(k+\ell+1)!}{2^k} \binom{\ell+1}{k} S(n,k+\ell)$$

for $n \in \mathbb{N}$ were acquired. In [10, Theorem 3.1], the Euler polynomials $E_n(x)$ were represented by

$$E_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[\sum_{\ell=1}^{n-k+1} \frac{(-1)^{\ell-1}(\ell-1)!}{2^{\ell-1}} S(n-k+1,\ell)\right] x^k, \quad n \in \mathbb{N}.$$

Consequently, the Euler numbers E_{2n} for $n \in \mathbb{N}$ can be calculated by

$$E_{2n} = 4^n \sum_{k=0}^{2n} \left[\sum_{\ell=1}^{2n-k+1} \frac{(-1)^{\ell-1}(\ell-1)!}{2^{\ell-1}} S(2n-k+1,\ell)\right] \frac{(-1)^k}{2^k} \binom{2n}{k}.$$

In mathematics, different forms have different implications. In other words, from different forms of mathematical quantities, one can read out different meanings and significance. Therefore, expressing the Fibonacci numbers F_n and the Fibonacci polynomials $F_n(s)$ in terms of tridiagonal determinants is meaningful and significant.

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