Several Explicit and Recursive Formulas for the Generalized Motzkin Numbers

Feng Qi

Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

Bai-Ni Guo

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

Abstract. In the paper, the authors find two explicit formulas and recover a recursive formula for the generalized Motzkin numbers. Consequently, the authors deduce two explicit formulas and a recursive formula for the Motzkin numbers, the Catalan numbers, and the restricted hexagonal numbers respectively.

1. Introduction

The Motzkin numbers $M_n$ enumerate various combinatorial objects. In 1977, Donaghey and Shapiro [3] gave fourteen different manifestations of the Motzkin numbers $M_n$. In particular, the Motzkin numbers $M_n$ give the numbers of paths from $(0, 0)$ to $(n, 0)$ which never dip below the $x$-axis $y = 0$ and are made up only of the steps $(1, 0), (1, 1),$ and $(1, -1)$.

The first seven Motzkin numbers $M_n$ for $0 \leq n \leq 6$ are 1, 1, 2, 4, 9, 21, 51. All the Motzkin numbers $M_n$ can be generated by

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} = \frac{1}{1 - x + \sqrt{1 - 2x - 3x^2}} = \sum_{k=0}^{\infty} M_k x^k.$$
In 2007, Mansour et al [10] introduced the \((u,l,d)\)-Motzkin numbers \(m_{n}^{(u,l,d)}\) and obtained [10, Theorem 2.1] that
\[
m_{n}^{(u,l,d)} = m_{n}^{(1,l,ud)},
\]
and
\[
m_{n}^{(u,l,d)} = \frac{1}{n} \sum_{j=0}^{n/2} \frac{1}{j+1} \binom{2j}{j} \left( \frac{n}{2j} \right) \left( \frac{ud}{j^2} \right)^j.
\]
From (1) and (2), it is easy to see that
\[
m_{n}^{(u,l,d)} = m_{n}^{(d,l,u)}.
\]
In 2014, Sun [21] generalized the Motzkin numbers \(M_{n}\) to
\[
M_{n}(a,b) = \left\lfloor \frac{n}{2} \right\rfloor \sum_{k=0}^{n/2} \binom{n}{2k} C_k a^{n-2k} b^k
\]
for \(a, b \in \mathbb{N}\) in terms of the Catalan numbers
\[
C_{n} = \frac{1}{n+1} \binom{2n}{n}
\]
and established the generating function
\[
M_{a,b}(x) = \frac{1 - ax - \sqrt{(1 - ax)^2 - 4bx^2}}{2bx^2} = \sum_{k=0}^{\infty} M_k(a,b) x^k,
\]
where \(|\lambda|\) denotes the floor function defined by the largest integer less than or equal to \(\lambda \in \mathbb{R}\). Wang and Zhang pointed out [22] that
\[
M_n(1,1) = M_n, \quad M_n(2,1) = C_{n+1}, \quad \text{and} \quad M_n(3,1) = H_n,
\]
where \(H_n\) denote the restricted hexagonal numbers described by Harary and Read [5].

For more information on many results, applications, and generalizations of the Motzkin numbers \(M_{n}\), please refer to the papers [3, 7, 8, 21, 22] and closely related references therein. For more information on many results, applications, and generalizations of the Catalan numbers \(C_{n}\), please refer to the monograph [6], the papers [9, 14, 15, 20], the survey article [12], and closely related references therein.

Comparing (1) with (5) reveals that \(M_k(a,b)\) and \(m_{n}^{(u,l,d)}\) are equivalent to each other and satisfy
\[
M_k(a,b) = m_{n}^{(1,a,b)} = m_{k}^{(b,n,1)} \quad \text{and} \quad m_{k}^{(n,l,d)} = M_k(l, ud).
\]
Therefore, it suffices to consider the generalized Motzkin numbers \(M_k(a,b)\), rather than the \((u,l,d)\)-Motzkin numbers \(m_{n}^{(u,l,d)}\), in this paper.

By the second relation in (7), one can reformulated the formula (2) as
\[
M_n(a,b) = a^n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{j+1} \binom{2j}{j} \left( \frac{n}{2j} \right) \left( \frac{b}{a^2} \right)^j.
\]
Substituting (4) into (3) recovers (8) once again.
In 2015, Wang and Zhang [22, Theorem 1] combinatorially obtained, among other things, the recursive formula

$$M_{n+2}(a, b) = aM_{n+1}(a, b) + b \sum_{\ell=0}^{n} M_{\ell}(a, b)M_{n-\ell}(a, b), \quad n \geq 0. \quad (9)$$

In this paper, we will find two explicit formulas, different from (8), and recover the recursive formula (9) for the generalized Motzkin numbers $M_n(a, b)$. Consequently, we will derive two explicit formula and a recursive formula for the Motzkin numbers $M_n$, the Catalan numbers $C_n$, and the restricted hexagonal numbers $H_n$ respectively.

We can state our main results as the following three theorems.

**Theorem 1.** For $n \geq 0$, we can compute the generalized Motzkin numbers $M_n(a, b)$ by

$$M_n(a, b) = \frac{1}{26} \left(\frac{4b-a^2}{2a}\right)^{n+2} \sum_{\ell=0}^{n+2} \left(\frac{2a^2}{4b-a^2}\right)^{\ell} \frac{(2\ell-3)!!}{\ell!} \frac{\ell}{(n-\ell+2)!},$$

where $\binom{p}{q} = 0$ for $q > p \geq 0$ and the double factorial of negative odd integers $-(2n+1)$ is

$$[-(2n+1)]!! = \frac{(-1)^n}{(2n-1)!} = (-1)^n \frac{2^n n!}{(2n)!}, \quad n = 0, 1, \ldots.$$

Consequently, we can compute the Motzkin numbers $M_n$ and the restricted hexagonal numbers $H_n$ respectively by

$$M_n = \frac{9}{8} \left(\frac{3}{2}\right)^{n+2} \sum_{\ell=0}^{n+2} \left(\frac{2}{3}\right)^{\ell} \frac{(2\ell-3)!!}{\ell!} \frac{\ell}{(n-\ell+2)!}$$

and

$$H_n = \frac{(-1)^n}{2} \frac{25}{72} \left(\frac{5}{6}\right)^{n+2} \sum_{\ell=0}^{n+2} (-1)^\ell \left(\frac{18}{5}\right)^{\ell} \frac{(2\ell-3)!!}{\ell!} \frac{\ell}{(n-\ell+2)!}.$$  

**Theorem 2.** For $n \geq 0$, we can compute the generalized Motzkin numbers $M_n(a, b)$ by

$$M_n(a, b) = -\frac{(a-2\sqrt{b})^{n+2}}{2b} \sum_{\ell=0}^{n+2} \frac{(2\ell-3)!!}{(2\ell)!} \frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!} \left(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\right)^{\ell}.$$  

Consequently, we can compute the Motzkin numbers $M_n$ and the restricted hexagonal numbers $H_n$ respectively by

$$M_n = \frac{(-1)^{n+1}}{2} \sum_{\ell=0}^{n+2} (-1)^\ell \frac{(2\ell-3)!!}{(2\ell)!} \frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}$$

and

$$H_n = -\frac{1}{2} \sum_{\ell=0}^{n+2} 5^\ell \frac{(2\ell-3)!!}{(2\ell)!} \frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}.$$  

**Theorem 3.** For $n \geq 0$, the generalized Motzkin numbers $M_n(a, b)$ satisfy

$$M_0(a, b) = 1, \quad M_1(a, b) = a.$$  

(14)
and the recursive formula (9). Consequently, for \( n \geq 0 \), the Motzkin numbers \( M_n \), the Catalan numbers \( C_n \), and the restricted hexagonal numbers \( H_n \) meet the recursive formulas

\[
M_{n+2} = M_{n+1} + \sum_{\ell=0}^{n} M_{\ell} M_{n-\ell},
\]

(15)

\[
C_{n+2} = 2C_{n+1} + \sum_{\ell=0}^{n} C_{\ell} C_{n-\ell},
\]

(16)

\[
H_{n+2} = 3H_{n+1} + \sum_{\ell=0}^{n} H_{\ell} H_{n-\ell}
\]

(17)

respectively.

2. LEMMAS

In order to prove the explicit formula (10), we need the following lemmas.

**Lemma 1** ([1, p. 40, Exercise 5], [11, Section 2.2, p. 849], [13, p. 94], [17, Lemma 3], and [23, Lemma 2.1]). Let \( u(x) \) and \( v(x) \neq 0 \) be two differentiable functions. Let \( U_{(n+1)\times 1}(x) \) be an \((n+1)\times 1\) matrix whose elements \( u_{k,1}(x) = u^{(k-1)}(x) \) for \( 1 \leq k \leq n+1 \), let \( V_{(n+1)\times n}(x) \) be an \((n+1)\times n\) matrix whose elements

\[
v_{i,j}(x) = \begin{cases} (i-1) v^{(i-j)}(x), & i-j \geq 0 \\ 0, & i-j < 0 \end{cases}
\]

for \( 1 \leq i \leq n+1 \) and \( 1 \leq j \leq n \), and let \( |W_{(n+1)\times(n+1)}(x)| \) denote the determinant of the \((n+1)\times(n+1)\) matrix

\[
W_{(n+1)\times(n+1)}(x) = (U_{(n+1)\times 1}(x) \ V_{(n+1)\times n}(x)).
\]

Then the \( n \)th derivative of the ratio \( \frac{u(x)}{v(x)} \) can be computed by

\[
\frac{d^n}{dx^n} \left[ \frac{u(x)}{v(x)} \right] = (-1)^n \frac{|W_{(n+1)\times(n+1)}(x)|}{v^{n+1}(x)}.
\]

**Lemma 2** ([2, p. 134, Theorem A and p. 139, Theorem C]). The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind

\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\sum_{i=1}^{n-k+1} \ell_i = k} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{\ell_i} \right)^{\ell_i}
\]

for \( n \geq k \geq 0 \).
Lemma 3 ([2] p. 135). The Bell polynomials of the second kind $B_{n,k}$ satisfy

$$B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$$

(19)

for $n \geq k \geq 0$.

Lemma 4 ([4] Theorem 4.1], [16, Remark 1], [18, p. 7], [19, Section 3], and [23, Lemma 2.5]). For $n \geq k \geq 0$, we have

$$B_{n,k}(x, 0, 0, \ldots, 0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} (2k-2n)x^{2k-n}.$$  

(20)

More generally, for $n \geq k \geq 0$ and $\lambda \in \mathbb{R}$, we have

$$B_{n,k}(1, 1-\lambda, 1-2\lambda, \ldots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell-q\lambda).$$  

(21)

3. Proofs of Theorems 1 and 3

We are now in a position to prove our main results.

Proof of Theorem 1 By virtue of (18), (19), and (20), we obtain for $k \geq 0$ that

$$\left[\sqrt{(1-ax)^2-4bx^2}\right]^{k+2} = \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \frac{1}{2} \left[\left(1-ax\right)^2-4bx^2\right]^{1/2-\ell}$$

$$\times B_{k+2,\ell}\left(-2\left[a + (4b - a^2)x\right], 2(a^2 - 4b), 0, \ldots, 0\right)$$

$$\rightarrow \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} B_{k+2,\ell}\left(-2a, 2(a^2 - 4b), 0, \ldots, 0\right)$$

$$= \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left[2(a^2 - 4b)\right]^{\ell/2} B_{k+2,\ell}\left(\frac{a}{4b-a^2}, 1, 0, \ldots, 0\right)$$

$$= \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left[2(a^2 - 4b)\right]^{\ell/2} \left(\frac{a}{4b-a^2}\right)^{2\ell-k-2}$$

(22)

as $x \rightarrow 0$, where

$$\langle x \rangle_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

denotes the falling factorial of $x \in \mathbb{R}$.
Letting $u(x) = 1 - ax - \sqrt{(1 - ax)^2 - 4bx^2}$ and $v(x) = x^2$ in Lemma 1 gives

$$\frac{d^n M_{a,b}(x)}{dx^n} = \frac{1}{2b \cdot x^{2(n+1)}} \begin{vmatrix}
(0)_n x^2 & 0 & \cdots & 0 & 0 & 0 \\
(1)_n x^2 & (1)_n x^2 & \cdots & 0 & 0 & 0 \\
(2)_n x^2 & (2)_n x^2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
(0)_{n-2} x^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\end{vmatrix}$$

$$= \frac{1}{2b \cdot x^{2(n+1)}} (-1)^n u^{(n)}(x)$$

$$+ 2 \left( \frac{n}{n-1} \right) x^n$$

$$- 2 \left( \frac{n}{n-2} \right) \left( \frac{n-1}{n-1} \right) x^{n-2}$$

$$= \frac{1}{2b \cdot x^2} - \frac{2n}{x} \frac{1}{2b} (-1)^{n-1} x^{2n-2}$$

$$\begin{vmatrix}
(0)_n x^2 & 0 & \cdots & 0 & 0 & 0 \\
(1)_n x^2 & (1)_n x^2 & \cdots & 0 & 0 & 0 \\
(2)_n x^2 & (2)_n x^2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
(0)_{n-2} x^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\end{vmatrix}$$

$$\begin{vmatrix}
(0)_n x^2 & 0 & \cdots & 0 & 0 & 0 \\
(1)_n x^2 & (1)_n x^2 & \cdots & 0 & 0 & 0 \\
(2)_n x^2 & (2)_n x^2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
(0)_{n-2} x^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\end{vmatrix}$$

$$\begin{vmatrix}
(0)_n x^2 & 0 & \cdots & 0 & 0 & 0 \\
(1)_n x^2 & (1)_n x^2 & \cdots & 0 & 0 & 0 \\
(2)_n x^2 & (2)_n x^2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
(0)_{n-2} x^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\end{vmatrix}$$

$$\begin{vmatrix}
(0)_n x^2 & 0 & \cdots & 0 & 0 & 0 \\
(1)_n x^2 & (1)_n x^2 & \cdots & 0 & 0 & 0 \\
(2)_n x^2 & (2)_n x^2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
(0)_{n-2} x^2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\end{vmatrix}$$
Therefore, by L’Hôpital’s rule, we have

\[
\begin{align*}
\lim_{x \to 0} \frac{d^n M_{a,b}(x)}{dx^n} &= \lim_{x \to 0} \left\{ \frac{u(n)(x)}{2b} - \frac{2nx}{x^2} \frac{d^n M_{a,b}(x)}{dx^n} - \frac{n(n-1)}{x^2} \frac{d^{n-2} M_{a,b}(x)}{dx^{n-2}} \right\} \\
&= \lim_{x \to 0} \left\{ \frac{1}{2x} \left[ u^{(n+1)}(x) - 2nx \frac{d^n M_{a,b}(x)}{dx^n} - n(n+1) \frac{d^{n-1} M_{a,b}(x)}{dx^{n-1}} \right] \right\} \\
&= \frac{1}{2} \lim_{x \to 0} \left[ \frac{u^{(n+2)}(x)}{2b} - 2nx \frac{d^{n+1} M_{a,b}(x)}{dx^{n+1}} - n(n+3) \frac{d^n M_{a,b}(x)}{dx^n} \right] \\
&= \frac{1}{2} \lim_{x \to 0} \frac{u^{(n+2)}(x)}{2b} - n(n+3) \lim_{x \to 0} \frac{d^n M_{a,b}(x)}{dx^n}
\end{align*}
\]

which is equivalent to

\[
\lim_{x \to 0} \frac{d^n M_{a,b}(x)}{dx^n} = \frac{1}{(n+1)(n+2)} \lim_{x \to 0} \frac{u^{(n+2)}(x)}{2b} = \frac{1}{2b(n+1)(n+2)} \lim_{x \to 0} u^{(n+2)}(x).
\]

Considering

\[
\lim_{x \to 0} \frac{d^n M_{a,b}(x)}{dx^n} = n!M_n(a, b),
\]

making use of (22), and simplifying lead to the explicit formula (10).

Letting \((a, b) = (1, 1)\) and \((a, b) = (3, 1)\) respectively in (10) and considering the three relations in (6) derive (11) and (12) immediately. The proof of Theorem 1 is complete.

**Proof of Theorem 2** From (5), it is derived that

\[
\sqrt{(1 - ax)^2 - 4bx^2} = 1 - ax - 2b \sum_{k=0}^{\infty} M_k(a, b)x^{k+2}.
\]

This implies that

\[
M_k(a, b) = -\frac{1}{2b} \frac{1}{(k+2)!} \lim_{x \to 0} \left[ \sqrt{(1 - ax)^2 - 4bx^2} \right]^{(k+2)}, \quad k \geq 0.
\]
(1) when \( a^2 - 4b \geq 0 \) and \( x \leq \min \{ \frac{1}{a+2\sqrt{b}}, \frac{1}{a-2\sqrt{b}} \} = \frac{1}{a+2\sqrt{b}} \), we have

\[
\left(\sqrt{(1-ax)^2 - 4bx^2}\right)^{(k+2)} = \left[ \sqrt{(a^2 - 4b) \left( x - \frac{1}{a+2\sqrt{b}} \right) \left( x - \frac{1}{a-2\sqrt{b}} \right)} \right]^{(k+2)}
\]

\[
= \sqrt{a^2 - 4b} \left( \sqrt{\frac{1}{a+2\sqrt{b}} - x} \sqrt{\frac{1}{a-2\sqrt{b}} - x} \right)^{(k+2)}
\]

\[
= \sqrt{a^2 - 4b} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \sqrt{\frac{1}{a+2\sqrt{b}} - x} \right)^{(\ell)} \left( \sqrt{\frac{1}{a-2\sqrt{b}} - x} \right)^{(k-\ell+2)}
\]

\[
= (-1)^k \sqrt{a^2 - 4b} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{1}{2} \right)^{\ell} \left( \frac{1}{a+2\sqrt{b}} - x \right)^{(1/2-\ell)} \left( \frac{1}{2} \right)^{(k-\ell+2)} \left( \frac{1}{a-2\sqrt{b}} - x \right)^{(k-3/2)}
\]

\[
\rightarrow (-1)^k \sqrt{a^2 - 4b} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{1}{2} \right)^{\ell} \left( \frac{1}{a+2\sqrt{b}} - x \right)^{(1/2-\ell)} \left( \frac{1}{2} \right)^{(k-\ell+2)} \left( \frac{1}{a-2\sqrt{b}} - x \right)^{(k-3/2)}
\]

\[
= (k+2)! (a - 2\sqrt{b})^{k+2} \sum_{\ell=0}^{k+2} \frac{(2\ell-3)!!}{(2\ell)!!} \frac{[2(k-\ell+2)-3]!!}{[2(k-\ell+2)]!!} \left( \frac{a + 2\sqrt{b}}{a-2\sqrt{b}} \right)^{\ell}
\]

as \( x \to 0 \);

(2) when \( a^2 - 4b < 0 \) and

\[
\frac{1}{a+2\sqrt{b}} = \max \{ \frac{1}{a+2\sqrt{b}}, \frac{1}{a-2\sqrt{b}} \} > x
\]

we have

\[
\left(\sqrt{(1-ax)^2 - 4bx^2}\right)^{(k+2)} = \left[ \sqrt{(4b-a^2) \left( \frac{1}{a+2\sqrt{b}} - x \right) \left( x - \frac{1}{a-2\sqrt{b}} \right)} \right]^{(k+2)}
\]

\[
= \sqrt{4b-a^2} \left( \sqrt{\frac{1}{a+2\sqrt{b}} - x} \sqrt{x - \frac{1}{a-2\sqrt{b}}} \right)^{(k+2)}
\]

\[
= \sqrt{4b-a^2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \sqrt{\frac{1}{a+2\sqrt{b}} - x} \right)^{(\ell)} \left( \sqrt{x - \frac{1}{a-2\sqrt{b}}} \right)^{(k-\ell+2)}
\]

\[
= \sqrt{4b-a^2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} (-1)^{\ell} \left( \frac{1}{2} \right)^{\ell} \left( \frac{1}{a+2\sqrt{b}} - x \right)^{(1/2-\ell)} \left( \frac{1}{2} \right)^{(k-\ell+2)} \left( \frac{1}{a-2\sqrt{b}} - x \right)^{(1/2-(k-\ell+2))}
\]

\[
\rightarrow \sqrt{4b-a^2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} (-1)^{\ell} \left( \frac{1}{2} \right)^{\ell} \left( \frac{1}{a+2\sqrt{b}} - x \right)^{(1/2-\ell)} \left( \frac{1}{2} \right)^{(k-\ell+2)} \left( \frac{1}{a-2\sqrt{b}} - x \right)^{(1/2-(k-\ell+2))}
\]

\[
= (2\sqrt{b} - a)^{k+1} \sqrt{4b-a^2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \frac{(2\ell-3)!!}{(2\ell)!!} \frac{[2(k-\ell+2)-3]!!}{[2(k-\ell+2)]!!} \left( \frac{a + 2\sqrt{b}}{2\sqrt{b} - a} \right)^{\ell}
\]

\[
= (2\sqrt{b} - a)^{k+2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \frac{(2\ell-3)!!}{(2\ell)!!} \frac{[2(k-\ell+2)-3]!!}{[2(k-\ell+2)]!!} (2\sqrt{b} - a)^{\ell}
\]
Squaring on both sides of the above equation gives

\[
(a - 2\sqrt{b})^{k+2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{(2\ell - 3)!!}{2^\ell} \right) \left( \frac{[2(k - \ell + 2) - 3]!!}{a + 2\sqrt{b}} \right)^{\ell} \left( \frac{a + 2\sqrt{b}}{a - 2\sqrt{b}} \right)^{2k-\ell+2} = (k + 2)!(a - 2\sqrt{b})^{k+2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{(2\ell - 3)!!}{2^\ell} \right) \left( \frac{[2(k - \ell + 2) - 3]!!}{a + 2\sqrt{b}} \right)^{\ell} \left( \frac{a + 2\sqrt{b}}{a - 2\sqrt{b}} \right)^{2k-\ell+2}
\]

as \( x \to 0 \).

By virtue of (23), we obtain the formula (13) readily.

Proof of Theorem 2. From (5), it is derived that

\[
\sqrt{(1 - ax)^2 - 4bx^2} = 1 - ax - 2b \sum_{k=0}^{\infty} M_k(a,b)x^{k+2}.
\]

Squaring on both sides of the above equation gives

\[
(1 - ax)^2 - 4bx^2 = 1 - 2ax + (a^2 - 4b)x^2
\]

\[
= \left[ 1 - ax - 2b \sum_{k=0}^{\infty} M_k(a,b)x^{k+2} \right]^2
\]

\[
= 1 + a^2x^2 + 4b^2 \left[ \sum_{k=0}^{\infty} M_k(a,b)x^{k+2} \right]^2
\]

\[
- 2ax - 4b \sum_{k=0}^{\infty} M_k(a,b)x^{k+2} + 4abx \sum_{k=0}^{\infty} M_k(a,b)x^{k+2}
\]

\[
= 1 - 2ax + a^2x^2 + 4b^2x^4 \left[ \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{k} M_\ell(a,b)M_{k-\ell} \right) x^k \right]
\]

\[
- 4b \sum_{k=0}^{\infty} M_{k-2}(a,b)x^k + 4ab \sum_{k=0}^{\infty} M_{k-3}(a,b)x^k
\]

\[
= 1 - 2ax + a^2x^2 - 4b \sum_{k=0}^{\infty} M_{k-2}(a,b)x^k
\]

\[
+ 4ab \sum_{k=0}^{\infty} M_{k-3}(a,b)x^k + 4b^2 \sum_{k=0}^{\infty} \left[ \sum_{\ell=0}^{k-4} M_\ell(a,b)M_{k-\ell-4}(a,b) \right] x^k
\]

\[
= 1 - 2ax + a^2x^2 - 4b \left[ M_0(a,b)x^2 + M_1(a,b)x^3 \right] + 4abM_0(a,b)x^3
\]

\[
- 4b \sum_{k=0}^{\infty} M_{k-2}(a,b)x^k + 4ab \sum_{k=0}^{\infty} M_{k-3}(a,b)x^k
\]

\[
+ 4b^2 \sum_{k=0}^{\infty} \left( \sum_{\ell=0}^{k-4} M_\ell(a,b)M_{k-\ell-4}(a,b) \right) x^k
\]
This implies that the generating function
\[ F = 1 - 2ax + [a^2 - 4bM_0(a, b)]x^2 + 4b[aM_0(a, b) - M_1(a, b)]x^3 \]

which means that
\[ a^2 - 4b = a^2 - 4bM_0(a, b), \quad 4b[aM_0(a, b) - M_1(a, b)] = 0, \]

and
\[ M_{k-2}(a, b) - aM_{k-3}(a, b) - b \sum_{\ell=0}^{k-4} M_\ell(a, b)M_{k-\ell-4}(a, b) = 0, \quad k \geq 4. \]

Consequently, the identities in (14) and the recursive formula (9) follow.

The proof of Theorem 3 is complete. \[ \square \]

Two remarks

1. From the proof of Theorem 1 we can conclude that
\[ x^2 \frac{d^n M_{a,b}(x)}{dx^n} + 2nx \frac{d^{n-1} M_{a,b}(x)}{dx^{n-1}} + n(n-1) \frac{d^{n-2} M_{a,b}(x)}{dx^{n-2}} = \frac{u^{(n)}(x)}{2b}, \quad n \geq 2. \]

This implies that the generating function \( M_{a,b}(x) \) expressed in [5] is an explicit solution of the linear ordinary differential equations

\[ x^2 f^{(n)}(x) + 2nx f^{(n-1)}(x) + n(n-1) f^{(n-2)}(x) = F_{n;a,b}(x) \]

for all \( n \geq 2 \), where, by [19] and [20] or [21],

\[ F_{n;a,b}(x) = \frac{n!(4b-a^2)^n}{2^n a} \frac{\sqrt{(1-ax)^2 - 4bx^2}}{[a + (4b-a^2)x]^n} \times \sum_{\ell=1}^{n} \frac{2\ell(2\ell-3)!!}{\ell!(4b-a^2)^\ell} \frac{[a + (4b-a^2)x]^{2\ell}}{(1-ax)^{2\ell} - 4bx^{2\ell} }. \]

2. This paper is a company and continuation of the article [24].

References

THREE FORMULAS FOR GENERALIZED MOTZKIN NUMBERS


