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SEVERAL EXPLICIT AND RECURSIVE FORMULAS FOR THE GENERALIZED MOTZKIN NUMBERS

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Abstract. In the paper, the authors find two explicit formulas and recover a recursive formula for the generalized Motzkin numbers. Consequently, the authors deduce two explicit formulas and a recursive formula for the Motzkin numbers, the Catalan numbers, and the restricted hexagonal numbers respectively.

1. Introduction

The Motzkin numbers M_n enumerate various combinatorial objects. In 1977, Donaghey and Shapiro [3] gave fourteen different manifestations of the Motzkin numbers M_n. In particular, the Motzkin numbers M_n give the numbers of paths from $(0,0)$ to $(n,0)$ which never dip below the x-axis $y=0$ and are made up only of the steps $(1,0)$, $(1,1)$, and $(1,-1)$.

The first seven Motzkin numbers M_n for $0\le n\le 6$ are $1,1,2,4,9,21,51$. All the Motzkin numbers M_n can be generated by

$$M(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}=\frac{1}{1-x+\sqrt{1-2x-3x^2}}=\sum_{k=0}^{\infty}M_kx^k.$$

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In 2007, Mansour *et al* [10] introduced the (u,l,d)-Motzkin numbers $m_n^{(u,l,d)}$ and obtained [10, Theorem 2.1] that $m_n^{(u,l,d)}=m_n^{(1,l,ud)}$,

$$M_{u,l,d}(x)=\frac{1-lx-\sqrt{(1-lx)^2-4udx^2}}{2udx^2}=\sum_{n=0}^{\infty}m_n^{(u,l,d)}x^n, \tag{1}$$

and

$$m_n^{(u,l,d)}=l^n\sum_{j=0}^{n/2}\frac{1}{j+1}\binom{2j}{j}\binom{n}{2j}\left(\frac{ud}{l^2}\right)^j. \tag{2}$$

From (1) and (2), it is easy to see that $m_n^{(u,l,d)}=m_n^{(d,l,u)}$.

In 2014, Sun [21] generalized the Motzkin numbers M_n to

$$M_n(a,b)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k a^{n-2k}b^k \tag{3}$$

for $a,b\in\mathbb{N}$ in terms of the Catalan numbers

$$C_n=\frac{1}{n+1}\binom{2n}{n} \tag{4}$$

and established the generating function

$$\begin{aligned}M_{a,b}(x)&=\frac{1-ax-\sqrt{(1-ax)^2-4bx^2}}{2bx^2}\\&=\frac{1}{1-ax+\sqrt{(1-ax)^2-4bx^2}}=\sum_{k=0}^{\infty}M_k(a,b)x^k,\end{aligned} \tag{5}$$

where $\lfloor\lambda\rfloor$ denotes the floor function defined by the largest integer less than or equal to $\lambda\in\mathbb{R}$. Wang and Zhang pointed out [22] that

$$M_n(1,1)=M_n,\quad M_n(2,1)=C_{n+1},\quad\text{and}\quad M_n(3,1)=H_n, \tag{6}$$

where H_n denote the restricted hexagonal numbers described by Harary and Read [5].

For more information on many results, applications, and generalizations of the Motzkin numbers M_n, please refer to the papers [3, 7, 8, 21, 22] and closely related references therein. For more information on many results, applications, and generalizations of the Catalan numbers C_n, please refer to the monograph [6], the papers [9, 14, 15, 20], the survey article [12], and closely related references therein.

Comparing (1) with (5) reveals that $M_k(a,b)$ and $m_k^{(u,l,d)}$ are equivalent to each other and satisfy

$$M_k(a,b)=m_n^{(1,a,b)}=m_k^{(b,a,1)}\quad\text{and}\quad m_k^{(u,l,d)}=M_k(l,ud). \tag{7}$$

Therefore, it suffices to consider the generalized Motzkin numbers $M_k(a,b)$, rather than the (u,l,d)-Motzkin numbers $m_n^{(u,l,d)}$, in this paper.

By the second relation in (7), one can reformulated the formula (2) as

$$M_n(a,b)=a^n\sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{j+1}\binom{2j}{j}\binom{n}{2j}\left(\frac{b}{a^2}\right)^j. \tag{8}$$

Substituting (4) into (3) recovers (8) once again.

In 2015, Wang and Zhang [22, Theorem 1] combinatorially obtained, among other things, the recursive formula

$$M_{n+2}(a,b)=aM_{n+1}(a,b)+b\sum_{\ell=0}^{n}M_\ell(a,b)M_{n-\ell}(a,b),\quad n\ge0. \tag{9}$$

In this paper, we will find two explicit formulas, different from (8), and recover the recursive formula (9) for the generalized Motzkin numbers $M_n(a,b)$. Consequently, we will derive two explicit formula and a recursive formula for the Motzkin numbers M_n, the Catalan numbers C_n, and the restricted hexagonal numbers H_n respectively.

We can state our main results as the following three theorems.

Theorem 1. *For $n\ge0$, we can compute the generalized Motzkin numbers $M_n(a,b)$ by*

$$M_n(a,b)=\frac{1}{2b}\biggl(\frac{4b-a^2}{2a}\biggr)^{n+2}\sum_{\ell=0}^{n+2}\biggl(\frac{2a^2}{4b-a^2}\biggr)^{\ell}\frac{(2\ell-3)!!}{\ell!}\binom{\ell}{n-\ell+2}, \tag{10}$$

where $\binom{p}{q}=0$ for $q>p\ge0$ and the double factorial of negative odd integers $-(2n+1)$ is

$$[-(2n+1)]!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^nn!}{(2n)!},\quad n=0,1,\dotsc.$$

Consequently, we can compute the Motzkin numbers M_n and the restricted hexagonal numbers H_n respectively by

$$M_n=\frac{9}{8}\biggl(\frac{3}{2}\biggr)^{n}\sum_{\ell=0}^{n+2}\biggl(\frac{2}{3}\biggr)^{\ell}\frac{(2\ell-3)!!}{\ell!}\binom{\ell}{n-\ell+2} \tag{11}$$

and

$$H_n=(-1)^n\frac{25}{72}\biggl(\frac{5}{6}\biggr)^{n}\sum_{\ell=0}^{n+2}(-1)^\ell\biggl(\frac{18}{5}\biggr)^{\ell}\frac{(2\ell-3)!!}{\ell!}\binom{\ell}{n-\ell+2}. \tag{12}$$

Theorem 2. *For $n\ge0$, we can compute the generalized Motzkin numbers $M_n(a,b)$ by*

$$M_n(a,b)=-\frac{\bigl(a-2\sqrt{b}\,\bigr)^{n+2}}{2b}\sum_{\ell=0}^{n+2}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}\biggl(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\biggr)^{\ell}. \tag{13}$$

Consequently, we can compute the Motzkin numbers M_n and the restricted hexagonal numbers H_n respectively by

$$M_n=\frac{(-1)^{n+1}}{2}\sum_{\ell=0}^{n+2}(-1)^\ell3^\ell\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}$$

and

$$H_n=-\frac{1}{2}\sum_{\ell=0}^{n+2}5^\ell\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(n-\ell+2)-3]!!}{[2(n-\ell+2)]!!}.$$

Theorem 3. *For $n\ge0$, the generalized Motzkin numbers $M_n(a,b)$ satisfy*

$$M_0(a,b)=1,\quad M_1(a,b)=a, \tag{14}$$

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and the recursive formula (9)*. Consequently, for $n \ge 0$, the Motzkin numbers M_n, the Catalan numbers C_n, and the restricted hexagonal numbers H_n meet the recursive formulas*

$$M_{n+2} = M_{n+1} + \sum_{\ell=0}^{n} M_\ell M_{n-\ell}, \tag{15}$$

$$C_{n+2} = 2C_{n+1} + \sum_{\ell=0}^{n} C_\ell C_{n-\ell}, \tag{16}$$

and

$$H_{n+2} = 3H_{n+1} + \sum_{\ell=0}^{n} H_\ell H_{n-\ell} \tag{17}$$

respectively.

2. Lemmas

In order to prove the explicit formula (10), we need the following lemmas.

Lemma 1 ([1, p. 40, Exercise 5)], [11, Section 2.2, p. 849], [13, p. 94], [17, Lemma 3], and [23, Lemma 2.1])**.** *Let $u(x)$ and $v(x) \ne 0$ be two differentiable functions. Let $U_{(n+1)\times 1}(x)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(x)$ be an $(n+1)\times n$ matrix whose elements*

$$v_{i,j}(x) = \begin{cases} \dbinom{i-1}{j-1} v^{(i-j)}(x), & i-j \ge 0 \\ 0, & i-j < 0 \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(x)|$ denote the determinant of the $(n+1)\times(n+1)$ matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{pmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{pmatrix}.$$

Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{|W_{(n+1)\times(n+1)}(x)|}{v^{n+1}(x)}.$$

Lemma 2 ([2, p. 134, Theorem A and p. 139, Theorem C])**.** *The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind*

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \{0\}\cup\mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \ge k \ge 0$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[f \circ h(t)] = \sum_{k=0}^{n} f^{(k)}(h(t)) \mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big) \tag{18}$$

for $n \ge 0$.

Lemma 3 ([2, p. 135])**.** *The Bell polynomials of the second kind* $\mathrm{B}_{n,k}$ *satisfy*

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{19}$$

for $n \ge k \ge 0$.

Lemma 4 ([4, Theorem 4.1], [16, Remark 1], [18, p. 7, (19)], [19, Section 3], and [23, Lemma 2.5])**.** *For* $n \ge k \ge 0$*, we have*

$$\mathrm{B}_{n,k}(x, 1, 0, \dots, 0) = \frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{20}$$

More generally, for $n \ge k \ge 0$ *and* $\lambda \in \mathbb{R}$*, we have*

$$\mathrm{B}_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda). \tag{21}$$

3. Proofs of Theorems 1 and 3

We are now in a position to prove our main results.

Proof of Theorem 1. By virtue of (18), (19), and (20), we obtain for $k \ge 0$ that

$$\begin{aligned}
\left[\sqrt{(1-ax)^2-4bx^2}\,\right]^{(k+2)} &= \sum_{\ell=0}^{k+2}\left\langle\frac12\right\rangle_\ell\big[(1-ax)^2-4bx^2\big]^{1/2-\ell} \\
&\quad\times\mathrm{B}_{k+2,\ell}\big(-2\big[a+\big(4b-a^2\big)x\big], 2\big(a^2-4b\big), 0, \dots, 0\big) \\
&\to \sum_{\ell=0}^{k+2}\left\langle\frac12\right\rangle_\ell \mathrm{B}_{k+2,\ell}\big(-2a, 2\big(a^2-4b\big), 0, \dots, 0\big) \\
&= \sum_{\ell=0}^{k+2}\left\langle\frac12\right\rangle_\ell \big[2\big(a^2-4b\big)\big]^\ell \mathrm{B}_{k+2,\ell}\left(\frac{a}{4b-a^2}, 1, 0, \dots, 0\right) \\
&= \sum_{\ell=0}^{k+2}\left\langle\frac12\right\rangle_\ell \big[2\big(a^2-4b\big)\big]^\ell \frac{(k-\ell+2)!}{2^{k-\ell+2}}\binom{k+2}{\ell}\binom{\ell}{k-\ell+2}\left(\frac{a}{4b-a^2}\right)^{2\ell-k-2}
\end{aligned} \tag{22}$$

as $x \to 0$, where

$$\langle x\rangle_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

denotes the falling factorial of $x \in \mathbb{R}$.

Letting $u(x)=1-ax-\sqrt{(1-ax)^2-4bx^2}$ and $v(x)=x^2$ in Lemma 1 gives

$$
\frac{\mathrm{d}^nM_{a,b}(x)}{\mathrm{d}x^n}=\frac{1}{2b}\frac{(-1)^n}{x^{2(n+1)}}
\begin{vmatrix}
u(x) & \binom{0}{0}x^2 & 0 & \cdots & 0 & 0 & 0\\
u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & \cdots & 0 & 0 & 0\\
u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \cdots & 0 & 0 & 0\\
u^{(3)}(x) & 0 & 2\binom{3}{1} & \cdots & 0 & 0 & 0\\
u^{(4)}(x) & 0 & 0 & \cdots & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots\\
u^{(n-2)}(x) & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2 & 0\\
u^{(n-1)}(x) & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x & \binom{n-1}{n-1}x^2\\
u^{(n)}(x) & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} & 2\binom{n}{n-1}x
\end{vmatrix}
$$

$$
=\frac{1}{2b}\frac{(-1)^n}{x^{2(n+1)}}\left[(-1)^nu^{(n)}(x)
\begin{vmatrix}
\binom{0}{0}x^2 & 0 & 0 & \cdots & 0 & 0 & 0\\
2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & \cdots & 0 & 0 & 0\\
2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & \cdots & 0 & 0 & 0\\
0 & 2\binom{3}{1} & 2\binom{3}{2}x & \cdots & 0 & 0 & 0\\
0 & 0 & 2\binom{4}{2} & \cdots & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2 & 0\\
0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x & \binom{n-1}{n-1}x^2
\end{vmatrix}\right.
$$

$$
+2\binom{n}{n-1}x
\begin{vmatrix}
u(x) & \binom{0}{0}x^2 & 0 & 0 & 0 & \cdots & 0 & 0\\
u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & 0 & \cdots & 0 & 0\\
u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & 0 & \cdots & 0 & 0\\
u^{(3)}(x) & 0 & 2\binom{3}{1} & 2\binom{3}{2}x & \binom{3}{3}x^2 & \cdots & 0 & 0\\
u^{(4)}(x) & 0 & 0 & 2\binom{4}{2} & 2\binom{4}{3}x & \cdots & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots\\
u^{(n-2)}(x) & 0 & 0 & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2\\
u^{(n-1)}(x) & 0 & 0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x
\end{vmatrix}
$$

$$
\left.-2\binom{n}{n-2}\binom{n-1}{n-1}x^2
\begin{vmatrix}
u(x) & \binom{0}{0}x^2 & 0 & 0 & \cdots & 0 & 0\\
u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & \cdots & 0 & 0\\
u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & \cdots & 0 & 0\\
u^{(3)}(x) & 0 & 2\binom{3}{1} & 2\binom{3}{2}x & \cdots & 0 & 0\\
u^{(4)}(x) & 0 & 0 & 2\binom{4}{2} & \cdots & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots\\
u^{(n-3)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-3}{n-4}x & \binom{n-3}{n-3}x^2\\
u^{(n-2)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-2}{n-4} & 2\binom{n-2}{n-3}x
\end{vmatrix}\right]
$$

$$
=\frac{1}{2b}\frac{u^{(n)}(x)}{x^2}-\frac{2n}{x}\frac{1}{2b}\frac{(-1)^{n-1}}{x^{2n}}
\begin{vmatrix}
u(x) & \binom{0}{0}x^2 & 0 & \cdots & 0 & 0\\
u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & \cdots & 0 & 0\\
u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \cdots & 0 & 0\\
u^{(3)}(x) & 0 & 2\binom{3}{1} & \cdots & 0 & 0\\
u^{(4)}(x) & 0 & 0 & \cdots & 0 & 0\\
\cdots & \cdots & \cdots & \ddots & \cdots & \cdots\\
u^{(n-2)}(x) & 0 & 0 & \cdots & 2\binom{n-2}{n-3}x & \binom{n-2}{n-2}x^2\\
u^{(n-1)}(x) & 0 & 0 & \cdots & 2\binom{n-1}{n-3} & 2\binom{n-1}{n-2}x
\end{vmatrix}
$$

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$$-n(n-1)\frac{1}{x^2}\frac{1}{2b}\frac{(-1)^{n-2}}{x^{2(n-1)}}\begin{vmatrix} u(x) & \binom{0}{0}x^2 & 0 & 0 & \cdots & 0 & 0\\ u'(x) & 2\binom{1}{0}x & \binom{1}{1}x^2 & 0 & \cdots & 0 & 0\\ u''(x) & 2\binom{2}{0} & 2\binom{2}{1}x & \binom{2}{2}x^2 & \cdots & 0 & 0\\ u^{(3)}(x) & 0 & 2\binom{3}{1} & 2\binom{3}{2}x & \cdots & 0 & 0\\ u^{(4)}(x) & 0 & 0 & 2\binom{4}{2} & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots\\ u^{(n-3)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-3}{n-4}x & \binom{n-3}{n-3}x^2\\ u^{(n-2)}(x) & 0 & 0 & 0 & \cdots & 2\binom{n-2}{n-4} & 2\binom{n-2}{n-3}x \end{vmatrix}$$

$$= \frac{1}{2b}\frac{u^{(n)}(x)}{x^2} - \frac{2n}{x}\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}} - \frac{n(n-1)}{x^2}\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}}$$

$$= \frac{1}{x^2}\left[\frac{u^{(n)}(x)}{2b} - 2nx\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}} - n(n-1)\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}}\right].$$

Therefore, by L'Hôspital's rule, we have

$$\begin{aligned}\lim_{x\to0}\frac{\mathrm{d}^nM_{a,b}(x)}{\mathrm{d}x^n} &= \lim_{x\to0}\left\{\frac{1}{x^2}\left[\frac{u^{(n)}(x)}{2b} - 2nx\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}} - n(n-1)\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}}\right]\right\}\\ &= \lim_{x\to0}\left\{\frac{1}{2x}\left[\frac{u^{(n+1)}(x)}{2b} - 2nx\frac{\mathrm{d}^{n}M_{a,b}(x)}{\mathrm{d}x^{n}} - n(n+1)\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}}\right]\right\}\\ &= \frac{1}{2}\lim_{x\to0}\left[\frac{u^{(n+2)}(x)}{2b} - 2nx\frac{\mathrm{d}^{n+1}M_{a,b}(x)}{\mathrm{d}x^{n+1}} - n(n+3)\frac{\mathrm{d}^{n}M_{a,b}(x)}{\mathrm{d}x^{n}}\right]\\ &= \frac{1}{2}\left[\lim_{x\to0}\frac{u^{(n+2)}(x)}{2b} - n(n+3)\lim_{x\to0}\frac{\mathrm{d}^{n}M_{a,b}(x)}{\mathrm{d}x^{n}}\right]\end{aligned}$$

which is equivalent to

$$\lim_{x\to0}\frac{\mathrm{d}^nM_{a,b}(x)}{\mathrm{d}x^n} = \frac{1}{(n+1)(n+2)}\lim_{x\to0}\frac{u^{(n+2)}(x)}{2b} = \frac{1}{2b(n+1)(n+2)}\lim_{x\to0}u^{(n+2)}(x).$$

Considering

$$\lim_{x\to0}\frac{\mathrm{d}^nM_{a,b}(x)}{\mathrm{d}x^n} = n!M_n(a,b),$$

making use of (22), and simplifying lead to the explicit formula (10).

Letting $(a,b)=(1,1)$ and $(a,b)=(3,1)$ respectively in (10) and considering the three relations in (6) derive (11) and (12) immediately. The proof of Theorem 1 is complete. □

Proof of Theorem 2. From (5), it is derived that

$$\sqrt{(1-ax)^2-4bx^2} = 1-ax-2b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}.$$

This implies that

$$M_k(a,b) = -\frac{1}{2b}\frac{1}{(k+2)!}\lim_{x\to0}\left[\sqrt{(1-ax)^2-4bx^2}\right]^{(k+2)}, \quad k\ge0. \tag{23}$$

It is easy to see that

(1) when $a^2-4b\ge0$ and $x\le\min\bigl\{\frac{1}{a+2\sqrt{b}},\frac{1}{a-2\sqrt{b}}\bigr\}=\frac{1}{a+2\sqrt{b}}$, we have

$$\begin{aligned}
\Bigl[\sqrt{(1-ax)^2-4bx^2}\Bigr]^{(k+2)}&=\Biggl[\sqrt{\bigl(a^2-4b\bigr)\biggl(x-\frac{1}{a+2\sqrt{b}}\biggr)\biggl(x-\frac{1}{a-2\sqrt{b}}\biggr)}\Biggr]^{(k+2)}\\
&=\sqrt{a^2-4b}\,\Biggl(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\,\sqrt{\frac{1}{a-2\sqrt{b}}-x}\Biggr)^{(k+2)}\\
&=\sqrt{a^2-4b}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\Biggl(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\Biggr)^{(\ell)}\Biggl(\sqrt{\frac{1}{a-2\sqrt{b}}-x}\Biggr)^{(k-\ell+2)}\\
&=(-1)^k\sqrt{a^2-4b}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\biggl\langle\frac12\biggr\rangle_\ell\biggl(\frac{1}{a+2\sqrt{b}}-x\biggr)^{1/2-\ell}\biggl\langle\frac12\biggr\rangle_{k-\ell+2}\biggl(\frac{1}{a-2\sqrt{b}}-x\biggr)^{\ell-k-3/2}\\
&\to(-1)^k\sqrt{a^2-4b}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\biggl\langle\frac12\biggr\rangle_\ell\biggl(\frac{1}{a+2\sqrt{b}}\biggr)^{1/2-\ell}\biggl\langle\frac12\biggr\rangle_{k-\ell+2}\biggl(\frac{1}{a-2\sqrt{b}}\biggr)^{\ell-k-3/2}\\
&=(k+2)!\bigl(a-2\sqrt{b}\bigr)^{k+2}\sum_{\ell=0}^{k+2}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(k-\ell+2)-3]!!}{[2(k-\ell+2)]!!}\biggl(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\biggr)^{\ell}
\end{aligned}$$

as $x\to0$;

(2) when $a^2-4b<0$ and

$$\begin{aligned}
\frac{1}{a+2\sqrt{b}}=\max\biggl\{\frac{1}{a+2\sqrt{b}},\frac{1}{a-2\sqrt{b}}\biggr\}&>x\\
&>\min\biggl\{\frac{1}{a+2\sqrt{b}},\frac{1}{a-2\sqrt{b}}\biggr\}=\frac{1}{a-2\sqrt{b}},
\end{aligned}$$

we have

$$\begin{aligned}
\Bigl[\sqrt{(1-ax)^2-4bx^2}\Bigr]^{(k+2)}&=\Biggl[\sqrt{\bigl(4b-a^2\bigr)\biggl(\frac{1}{a+2\sqrt{b}}-x\biggr)\biggl(x-\frac{1}{a-2\sqrt{b}}\biggr)}\Biggr]^{(k+2)}\\
&=\sqrt{4b-a^2}\,\Biggl(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\,\sqrt{x-\frac{1}{a-2\sqrt{b}}}\Biggr)^{(k+2)}\\
&=\sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\Biggl(\sqrt{\frac{1}{a+2\sqrt{b}}-x}\Biggr)^{(\ell)}\Biggl(\sqrt{x-\frac{1}{a-2\sqrt{b}}}\Biggr)^{(k-\ell+2)}\\
&=\sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}(-1)^\ell\biggl\langle\frac12\biggr\rangle_\ell\biggl(\frac{1}{a+2\sqrt{b}}-x\biggr)^{1/2-\ell}\biggl\langle\frac12\biggr\rangle_{k-\ell+2}\biggl(x-\frac{1}{a-2\sqrt{b}}\biggr)^{1/2-(k-\ell+2)}\\
&\to\sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}(-1)^\ell\biggl\langle\frac12\biggr\rangle_\ell\biggl(\frac{1}{a+2\sqrt{b}}\biggr)^{1/2-\ell}\biggl\langle\frac12\biggr\rangle_{k-\ell+2}\biggl(\frac{1}{2\sqrt{b}-a}\biggr)^{1/2-(k-\ell+2)}\\
&=\bigl(2\sqrt{b}-a\bigr)^{k+1}\sqrt{4b-a^2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\frac{(2\ell-3)!!}{2^\ell}\biggl(\frac{a+2\sqrt{b}}{2\sqrt{b}-a}\biggr)^{\ell-1/2}(-1)^{k-\ell}\frac{[2(k-\ell+2)-3]!!}{2^{k-\ell+2}}\\
&=\bigl(2\sqrt{b}-a\bigr)^{k+2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\frac{(2\ell-3)!!}{2^\ell}\biggl(\frac{a+2\sqrt{b}}{2\sqrt{b}-a}\biggr)^{\ell}(-1)^{k-\ell}\frac{[2(k-\ell+2)-3]!!}{2^{k-\ell+2}}
\end{aligned}$$

$$
\begin{aligned}
&= \big(a-2\sqrt{b}\big)^{k+2}\sum_{\ell=0}^{k+2}\binom{k+2}{\ell}\frac{(2\ell-3)!!}{2^{\ell}}\frac{[2(k-\ell+2)-3]!!}{2^{k-\ell+2}}\biggl(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\biggr)^{\ell}\\
&= (k+2)!\big(a-2\sqrt{b}\big)^{k+2}\sum_{\ell=0}^{k+2}\frac{(2\ell-3)!!}{(2\ell)!!}\frac{[2(k-\ell+2)-3]!!}{[2(k-\ell+2)]!!}\biggl(\frac{a+2\sqrt{b}}{a-2\sqrt{b}}\biggr)^{\ell}
\end{aligned}
$$

as $x\to0$.

By virtue of (23), we obtain the formula (13) readily.

Letting $(a,b)=(1,1)$ and $(a,b)=(3,1)$ respectively in (13) and making use of the first and third relations in (6) lead to (11) and (12) immediately. The proof of Theorem 2 is complete. □

Proof of Theorem 3. From (5), it is derived that

$$
\sqrt{(1-ax)^2-4bx^2}\,=1-ax-2b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}.
$$

Squaring on both sides of the above equation gives

$$
\begin{aligned}
(1-ax)^2-4bx^2 &= 1-2ax+\big(a^2-4b\big)x^2\\
&=\Biggl[1-ax-2b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}\Biggr]^2\\
&=1+a^2x^2+4b^2\Biggl[\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}\Biggr]^2\\
&\quad-2ax-4b\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}+4abx\sum_{k=0}^{\infty}M_k(a,b)x^{k+2}\\
&=1-2ax+a^2x^2+4b^2x^4\sum_{k=0}^{\infty}\Biggl[\sum_{\ell=0}^{k}M_\ell(a,b)M_{k-\ell}\Biggr]x^k\\
&\quad-4b\sum_{k=2}^{\infty}M_{k-2}(a,b)x^k+4ab\sum_{k=3}^{\infty}M_{k-3}(a,b)x^k\\
&=1-2ax+a^2x^2-4b\sum_{k=2}^{\infty}M_{k-2}(a,b)x^k\\
&\quad+4ab\sum_{k=3}^{\infty}M_{k-3}(a,b)x^k+4b^2\sum_{k=4}^{\infty}\Biggl[\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b)\Biggr]x^k\\
&=1-2ax+a^2x^2-4b\big[M_0(a,b)x^2+M_1(a,b)x^3\big]+4abM_0(a,b)x^3\\
&\quad-4b\sum_{k=4}^{\infty}M_{k-2}(a,b)x^k+4ab\sum_{k=4}^{\infty}M_{k-3}(a,b)x^k\\
&\quad+4b^2\sum_{k=4}^{\infty}\Biggl[\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b)\Biggr]x^k
\end{aligned}
$$

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$$
\begin{aligned}
&= 1 - 2ax + \bigl[a^2 - 4bM_0(a,b)\bigr]x^2 + 4b[aM_0(a,b) - M_1(a,b)]x^3 \\
&\quad - 4b\sum_{k=4}^{\infty}\Biggl[M_{k-2}(a,b) - aM_{k-3}(a,b) - b\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b)\Biggr]x^k
\end{aligned}
$$

which means that

$$
a^2 - 4b = a^2 - 4bM_0(a,b), \quad 4b[aM_0(a,b) - M_1(a,b)] = 0,
$$

and

$$
M_{k-2}(a,b) - aM_{k-3}(a,b) - b\sum_{\ell=0}^{k-4}M_\ell(a,b)M_{k-\ell-4}(a,b) = 0, \quad k \ge 4.
$$

Consequently, the identities in (14) and the recursive formula (9) follow.

Taking $(a,b) = (1,1)$, $(a,b) = (2,1)$, and $(a,b) = (3,1)$ respectively in (9) and considering the three relations in (6) lead to (15), (16), and (17) immediately. The proof of Theorem 3 is complete. □

4. Two remarks

Remark 1. From the proof of Theorem 1, we can conclude that

$$
x^2\frac{\mathrm{d}^n M_{a,b}(x)}{\mathrm{d}x^n} + 2nx\frac{\mathrm{d}^{n-1}M_{a,b}(x)}{\mathrm{d}x^{n-1}} + n(n-1)\frac{\mathrm{d}^{n-2}M_{a,b}(x)}{\mathrm{d}x^{n-2}} = \frac{u^{(n)}(x)}{2b}, \quad n \ge 2.
$$

This implies that the generating function $M_{a,b}(x)$ expressed in (5) is an explicit solution of the linear ordinary differential equations

$$
x^2 f^{(n)}(x) + 2nxf^{(n-1)}(x) + n(n-1)f^{(n-2)}(x) = F_{n;a,b}(x)
$$

for all $n \ge 2$, where, by (19) and (20) or (21),

$$
\begin{aligned}
F_{n;a,b}(x) &= \frac{n!\bigl(4b-a^2\bigr)^n}{2^{n+1}b}\frac{\sqrt{(1-ax)^2-4bx^2}}{[a+(4b-a^2)x]^n} \\
&\quad \times \sum_{\ell=1}^{n}\frac{2^\ell(2\ell-3)!!}{\ell!(4b-a^2)^\ell}\binom{\ell}{n-\ell}\frac{\bigl[a+\bigl(4b-a^2\bigr)x\bigr]^{2\ell}}{[(1-ax)^2-4bx^2]^\ell}.
\end{aligned}
$$

Remark 2. This paper is a company and continuation of the article [24].

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