Simplifying Two Families of Nonlinear Ordinary Differential Equations

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Abstract

In the paper, by virtue of techniques in combinatorial analysis, the authors simplify two families of nonlinear ordinary differential equations in terms of the Stirling numbers of the first kind.

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1 Motivation and main results

In [3, Theorem 1], it was acquired inductively and recursively that the nonlinear differential equations

$$G^{(n)}(t) = \frac{(-1)^n (n-1)!}{(1+t)^n} \sum_{j=2}^{n+1} (j-1)! H_{n-1,j-2} G^j(t) \tag{1}$$

for $n \in \mathbb{N}$ have a solution $G(t) = \frac{1}{\ln(1+t)}$, where

$$H_{n,0} = 1, \quad H_{n,1} = H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad \text{and} \quad H_{n,j} = \sum_{k=j}^{n} \frac{H_{k-1,j-1}}{k} \tag{2}$$

for $2 \le j \le n$ and $n \in \mathbb{N}$. Thereafter, the identity

$$(-1)^m \sum_{j=0}^{\min\{n,m-1\}} (m-j)!(m-1)! H_{m-1,m-j-1} \langle n \rangle_j b_{n-j}^{(m-j+1)}$$

$$= \begin{cases} (-1)^m m! \langle m \rangle_n, & 0 \le n \le m; \\ \sum_{\ell=0}^{n-m-1} \binom{m}{\ell} \dfrac{\langle n \rangle_{\ell+m+1}}{n-\ell} b_{n-\ell}, & n \ge m+1, \end{cases}$$

for $m \in \mathbb{N}$ and $n \ge 0$ was derived in [3, Theorem 2], where the higher order Bernoulli numbers of the second kind $b_n^{(k)}$ can be generated by

$$\left[\frac{t}{\ln(1+t)} \right]^k = \sum_{n=0}^{\infty} b_n^{(k)} \frac{t^n}{n!},$$

with $b_n = b_n^{(1)}$, and the falling factorial $\langle x \rangle_n$ of $x \in \mathbb{R}$ is defined by

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1; \\ 1, & n = 0. \end{cases}$$

In [4, Theorem 2.1], it was inductively and recursively obtained that the family of differential equations

$$F^{(n)}(t) = \frac{(-1)^{n-1}}{(1+\lambda t)^n} \sum_{k=1}^{n} (k-1)! \lambda^{n-k} (n-1)! H_{n-1,k-1} e^{-kF(t)}, \quad n \in \mathbb{N} \tag{3}$$

have a solution

$$F(t) = \ln \left[1 + \frac{\ln(1+\lambda t)}{\lambda} \right].$$

As an application of the above result, the identity

$$(-1)^{m+n+1} \frac{(m+n)!}{(m-1)!} \sum_{\ell=0}^{m+n+1} \frac{D_{\ell,\lambda}}{\ell!} \frac{(-\lambda)^{m+n-\ell-1}}{m+n-\ell} = \sum_{k=1}^{m} \sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-1)^{m_1+m_3} \binom{n}{m_3}$$

$$\times \langle m+n-m_3-1 \rangle_{n-m_3} (k-1)! k^{m_1} \lambda^{m+n-k-m_2} s(m_2,m_1) s(m_3,m_2) H_{m-1,k-1}$$

for $m \in \mathbb{N}$ and $n \ge 0$ was derived in [4, Theorem 3.1], where $D_{n,\lambda}$ is generated by

$$\frac{\lambda \ln\left[1 + \frac{\ln(1+\lambda t)}{\lambda}\right]}{\ln(1+\lambda t)} = \sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^n}{n!}$$

and the Stirling numbers of the first kind $s(n,k)$ can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1.$$

In [6, Corollary 2.3], it was inductively and recursively procured that the Stirling numbers of the first kind $s(n,k)$ for $1 \le k \le n$ can be expressed as

$$s(n,k) = (-1)^{n+k}(n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-2}=1}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}} \sum_{\ell_{k-1}=1}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}}. \tag{4}$$

See also [5, p. 27, Remark 3.3]. Comparing (2) with 4, we observe that

$$(-1)^{n+k} s(n,k) = (n-1)! H_{n-1,k-1}, \quad n \ge k \ge 1. \tag{5}$$

See also [12, pp. 7–8, Section 2.4] and [13, p. 9, Section 2.5].

By virtue of the identity (5), the above-mentioned Theorems 1 and 2 in [3] can be alternatively restated as the following theorem.

Theorem 1. *The nonlinear differential equations*

$$G^{(n)}(t) = \frac{1}{(1+t)^n} \sum_{k=1}^{n} (-1)^k k! s(n,k) G^{k+1}(t), \quad n \in \mathbb{N} \tag{6}$$

have a solution $G(t) = \frac{1}{\ln(1+t)}$ *and the identity*

$$(-1)^m \sum_{j=0}^{\min\{n,m-1\}} (-1)^j (m-j)! s(m,m-j) \langle n \rangle_j b_{n-j}^{(m-j+1)}$$

$$= \begin{cases} (-1)^m m! \langle m \rangle_n, & 0 \le n \le m \\ \displaystyle\sum_{\ell=0}^{n-m-1} \binom{m}{\ell} \frac{\langle n \rangle_{\ell+m+1}}{n-\ell} b_{n-\ell}, & n \ge m+1 \end{cases} \tag{7}$$

is valid for $m \in \mathbb{N}$ *and* $n \ge 0$.

By virtue of the identity (5) once again, the above-mentioned Theorems 2.1 and 3.1 in [4] can be simply reformulated as Theorem 2 below.

Theorem 2. *The family of differential equations*

$$F^{(n)}(t) = \frac{1}{(1+\lambda t)^n} \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \lambda^{n-k} s(n,k) e^{-kF(t)}, \quad n \in \mathbb{N} \tag{8}$$

have a solution

$$F(t) = \ln\left[1 + \frac{\ln(1+\lambda t)}{\lambda}\right].$$

and the identity

$$\sum_{\ell=0}^{m+n+1} \frac{(-1)^\ell}{\ell! \lambda^{\ell+1} (m+n-\ell)} D_{\ell,\lambda} = \frac{(-1)^m}{(m+n)!} \sum_{k=1}^{m} \sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \frac{(-1)^{m_1+m_3+k}}{\lambda^{k+m_2}}$$

$$\times \binom{n}{m_3} \langle m+n-m_3-1 \rangle_{n-m_3} (k-1)! k^{m_1} s(m_2,m_1) s(m_3,m_2) s(m,k) \tag{9}$$

is valid for $m \in \mathbb{N}$ *and* $n \ge 0$.

Although the method used in proofs of [3, Theorem 1] and [4, Theorem 2.1] is effectual, however, in our eyes, proofs of their main results in [3, Theorem 1] and [4, Theorem 2.1] are a little long and tedious (they spent six pages and five pages, respectively), and formulations of their main results are less meaningful and significant (the Stirling numbers of the first kind $s(n,k)$ are more meaningful and significant than the quantities $H_{n,k}$).

The aim of this paper is to simply and concisely verify Theorems 1 and 2 by virtue of techniques in combinatorial analysis.

2   Simple proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1. In combinatorial analysis, the Faà di Bruno formula plays an important role and can be described in terms of the Bell polynomials of the second kind

$$
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}
$$

for $n \ge k \ge 0$, see [1, p. 134, Theorem A], by

$$
\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}[f \circ h(t)] = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}\big(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)\big) \tag{10}
$$

for $n \ge 0$, see [1, p. 139, Theorem C]. Setting $f(u) = \frac{1}{u}$ and $u = h(t) = \ln(1+t)$ in (10) gives

$$
G^{(n)}(t) = \sum_{k=0}^{n} \left(\frac{1}{u}\right)^{(k)} B_{n,k}\left(\frac{1}{1+t}, -\frac{1}{(1+t)^2}, \ldots, (-1)^{n-k}\frac{(n-k)!}{(1+t)^{n-k+1}}\right)
$$

$$
= \sum_{k=0}^{n} \frac{(-1)^k k!}{u^{k+1}} \left(\frac{1}{1+t}\right)^n (-1)^{n+k} B_{n,k}(0!, 1!, 2!, \ldots, (n-k)!) = \frac{1}{(1+t)^n} \sum_{k=0}^{n} \frac{(-1)^k k!}{\ln^{k+1}(1+t)} s(n,k)
$$

$$
= \frac{1}{(1+t)^n} \sum_{k=0}^{n} (-1)^k k! s(n,k) G^{k+1}(t) = \frac{1}{(1+t)^n} \sum_{k=1}^{n} (-1)^k k! s(n,k) G^{k+1}(t)
$$

for $n \in \mathbb{N}$, where we used in the above lines the formula $s(n,0) = 0$ for $n \in \mathbb{N}$ and the identities

$$
B_{n,k}\big(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \tag{11}
$$

and

$$
B_{n,k}(0!, 1!, 2!, \ldots, (n-k)!) = (-1)^{n-k} s(n,k) \tag{12}
$$

listed in [1, p. 135, Theorem B]. This means that the nonlinear differential equations (6) have a solution $G(t) = \frac{1}{\ln(1+t)}$.

The proof of the identity (7) follows from combining proofs of the equation (6) and [3, Theorem 2] together.   □

Proof of Theorem 2. Setting $f(u) = \ln(1+u)$ and $u = h(t) = \frac{\ln(1+\lambda t)}{\lambda}$ in (10) gives

$$
F^{(n)}(t) = \sum_{k=0}^{n} [\ln(1+u)]^{(k)} B_{n,k}\left(\frac{1}{1+\lambda t}, -\frac{\lambda}{(1+\lambda t)^2}, \ldots, (-1)^{n-k}\frac{(n-k)!\lambda^{n-k}}{(1+\lambda t)^{n-k+1}}\right)
$$

$$
= \sum_{k=0}^{n} [\ln(1+u)]^{(k)} \left(\frac{1}{1+\lambda t}\right)^n (-1)^{n+k} \lambda^{n-k} B_{n,k}(0!, 1!, 2!, \ldots, (n-k)!)
$$

$$= \left(\frac{1}{1+\lambda t}\right)^n \sum_{k=0}^{n} [\ln(1+u)]^{(k)} (-1)^{n+k} \lambda^{n-k} (-1)^{n-k} s(n,k)$$

$$= \frac{1}{(1+\lambda t)^n} \left[\ln(1+u)\lambda^n s(n,0) + \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{(1+u)^k} \lambda^{n-k} s(n,k)\right]$$

$$= \frac{1}{(1+\lambda t)^n} \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{\left[1 + \frac{\ln(1+\lambda t)}{\lambda}\right]^k} \lambda^{n-k} s(n,k)$$

$$= \frac{1}{(1+\lambda t)^n} \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \lambda^{n-k} s(n,k) e^{-kF(t)}$$

for $n \in \mathbb{N}$, where we used in the above lines the formula $s(n,0) = 0$ for $n \in \mathbb{N}$ and the identities (11) and (12). This means that the nonlinear differential equations (8) have a solution $F(t) = \ln\left[1 + \frac{\ln(1+\lambda t)}{\lambda}\right]$.

The proof of the identity (9) follows from combining proofs of the equation (8) and [4, Theorem 3.1] together. □

3   Remarks

Finally we list several remarks on our main and the closely related results.

Remark 1. The relation (5) shows that we should not call the quantities $H_{n,k}$ defined in (2) the generalized harmonic numbers, as did in [4, p. 745].

Remark 2. Till now we can see that formulations and proofs of Theorems 1 and 2 in this paper are simpler, more concise, more meaningful, and more significant than those in the papers [3, 4].

Remark 3. About simplification from the differential equations (1) and (3) to (6) and (8), the idea comes from the papers [2, 7, 8, 9, 10, 11, 12, 13, 14] and the closely related references therein.

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