

## Article

# A Combined Entropy/Phase-Field Approach to Gravity

**Georg J. Schmitz**ACCESS e.V., Intzestr.5, D- 52072 Aachen, Germany; G.J.Schmitz@access.rwth-aachen.de;  
Tel.: +49-241-80-98014

**Abstract:** Terms related to gradients of scalar fields are introduced as scalar products into the formulation of entropy. A Lagrange density is then formulated by adding constraints based on known conservation laws. Applying the Lagrange formalism to the resulting Lagrange density leads to the Poisson equation of gravitation and also includes terms being related to curvature of space. The formalism further leads to terms possibly explaining nonlinear extensions known from modified Newtonian dynamics approaches. The article concludes with a short discussion of the presented methodology and provides an outlook on other phenomena, which might be tackled using this new approach.

**Keywords:** entropic gravity; Lagrange formalism; phase-field models; gradient-entropy; constraints; maximum entropy principle; curvature of space; conservation laws; Modified Newtonian Dynamics; MOND

## 1. Introduction

Recent work on entropic gravity by Erik Verlinde [1] [2] has initiated a recovery of earlier work by the author on thermodynamics of diffuse interfaces [3] and stimulated a generalization of this approach. The approach being depicted in the present article draws on a combination of the very strong and fundamental concepts of entropy, of the phase-field method, and of the Lagrange formalism. The capability of this approach to describe various aspects of gravity will eventually be demonstrated.

The description of entropy by now has been based on the use of scalars for the probabilities  $\Phi_i$  of the individual states  $i$ :

$$s = s(\Phi_i) \quad (1)$$

Scope of the present article is to generalize the entropy formulation and to include terms related to vectors (with options for future extension to tensors) towards a formulation:

$$s = s(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i) \quad (2)$$

Known conservation laws for energy  $E$ , mass  $M$ , charge  $Q$ , momentum  $\vec{P}$ , and spin  $\vec{L}$ , each law being formulated as function of  $(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i)$  are then added as constraints to the variational problem. Each constraint is associated with its own Lagrange multiplier  $(\lambda, g, \varepsilon, \dots)$  in total yielding a free energy density  $f$ :

$$\begin{aligned} f &= s(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i) \\ &\quad - \lambda E(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i) - g M(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i) - \varepsilon Q(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i) \\ &\quad - \vec{c}\vec{P}(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i) - \vec{h}\vec{L}(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i) - \dots \text{ further conservation laws.} \end{aligned} \quad (3)$$

The Lagrange multipliers  $\vec{c}$  and  $\vec{h}$  for momentum  $\vec{P}$  and spin  $\vec{L}$ , respectively, obviously have to be vector valued in order to recover a scalar contribution to the overall scalar equation. Units and scales are implicitly introduced by the Lagrange multipliers. The Lagrange multiplier  $\lambda$  for example

corresponds to  $\lambda = \frac{1}{kT}$  making the energy dimensionless (or giving the dimension of  $kT$  to entropy/probability). The names of the Lagrange multipliers have been arbitrarily selected to be similar to the names of known fundamental constants and have been assigned to the conservation law to which they are most probably related. Relations between these constants are expected to emerge from the Lagrange formalism being applied to above free energy density.

The Lagrange formalism is based on a free energy functional  $F$  being the volume integral of a free energy density  $f$ . Its variational derivative  $\frac{\delta F}{\delta \Phi_i(\vec{r}, t)} = 0$  corresponds to

$$0 = \left\{ \frac{\partial}{\partial \Phi_i(\vec{r}, t)} - \vec{V} \frac{\partial}{\partial \vec{V} \Phi_i(\vec{r}, t)} - \frac{\partial}{\partial t} \frac{\partial}{\partial \dot{\Phi}_i(\vec{r}, t)} \right\} f \quad (4)$$

Executing these derivatives provides the equations of motion of the system. The present paper will demonstrate first results obtained when applying above scheme for mass conservation. It will also not yet address any time dependent phenomena meaning that there will be no dependencies on  $\dot{\Phi}_i(\vec{r}, t)$  considered in the Lagrange density.

## 2. Entropy

The proposed concept requires a formulation of entropy which comprises terms related to vectors (and in future also time derivatives). In order to specify vectors, a reference frame respectively a coordinate system has to be defined. In spite of any vectors (or tensors) being introduced into the terms of the equations, the overall entropy formulation and all the probabilities entering into the typical logarithmic terms have to remain scalars.

### 2.1. Scalar Entropy

Entropy has unveiled its importance in numerous fields. Some most important discoveries are based on entropy like (i) the Boltzmann factor in energy levels of systems (ii) the Gibbs energies of thermodynamic phases, (iii) the Shannon entropy in information systems (iv) the Hawking entropy of black holes, (v) the Flory-Huggins polymerization entropy in polymers [4],[5], and (vi) the crystallization entropy in metals [4],[6] to name only some of the major highlights.

All these approaches for entropy are typically based on logarithmic terms like (see e.g. [7]):

$$s = - \sum_{i=0}^N \Phi_i \ln \Phi_i \quad (5)$$

It seems important to note that entropy can be interpreted as tending to smear out any contrast between different states respectively different objects. Contrast between states/objects is reflected in gradients of an observable property allowing distinguishing between these states/objects. Any object can be defined as a coherent region of space, which can be distinguished from other regions due to a contrast in at least one attribute/property. This attribute may even be the „name“ of the object indicating that somebody – based on some criterion - has identified the region to belong to the same object [8].

Gradients and interfaces, i.e. measures of contrast, are thus most important to describe physical objects and thus should be reflected in the formulation for the entropy. For example, a sphere is an object having the maximum entropy for a given volume. The next section thus describes an approach to include gradients into the entropy formulation i.e. making the step from the “scalar” entropy  $s(\Phi_i)$  being based on scalar values  $\Phi_i$  to the “gradient” entropy also comprising gradients  $\vec{V} \Phi_i$ .

### 2.2 Entropy formulations comprising gradient terms

The name “gradient entropy” is misleading because the value of the entropy and also the probabilities entering into its formulation remain scalars even if gradients are present in its

formulation. The choice of another name might thus be meaningful in future. The “gradient entropy” being described in prior work [3] aims at describing the growth of crystals using a phase-field approach (see section 3). Its relevant content is shortly summarized in the following.

Classical thermodynamics are based on Gibbs formulation of the following variational problem [7]:

$$\delta \left\{ - \sum_{i=0}^N \Phi_i \ln \Phi_i - \lambda \left( \sum_{i=0}^N \Phi_i E_i - E_{tot} \right) - \mu \sum_{i=0}^N (\Phi_i - 1) \right\} = 0 \quad (6)$$

with  $\lambda, \mu$  being the Lagrange multipliers accounting for the constraints of energy and probability conservation. The solution of this variational problem yields the well-known relation for the probability of specific state  $\Phi_i$ :

$$\Phi_i = e^{-\lambda E_i / Z} \quad Z = \sum_{i=0}^N e^{-\lambda E_i} \quad (7)$$

with  $Z$  being the partition function being necessary to normalize the probabilities and  $\lambda = \frac{1}{kT}$ . It is interesting to note that instead of the simple variation  $\delta = \frac{\partial}{\partial \Phi_i}$  also the full variational derivative  $\left\{ \frac{\partial}{\partial \Phi_i} - \vec{\nabla} \frac{\partial}{\partial \vec{\nabla} \Phi_i} \right\}$  leads to the same result as there are no explicit dependencies on gradients in the - equivalent - formulation:

$$\left\{ \frac{\partial}{\partial \Phi_i} - \vec{\nabla} \frac{\partial}{\partial \vec{\nabla} \Phi_i} \right\} \left\{ - \sum_{i=0}^N \Phi_i \ln \Phi_i - \lambda \left( \sum_{i=0}^N \Phi_i E_i - E_{tot} \right) - \mu \sum_{i=0}^N (\Phi_i - 1) \right\} = 0 \quad (8)$$

Replacing the logarithmic terms  $\Phi_i \ln \Phi_i$  by scalar products of type  $\vec{a} \vec{\nabla} \Phi_i \ln(\vec{a} \vec{\nabla} \Phi_i)$  drastically changes that situation. Under some simplifying assumptions - especially a Taylor expansion of the logarithmic terms - scalar products comprising gradients occur [3]:

$$\left\{ \frac{\partial}{\partial \Phi_i} - \vec{\nabla} \frac{\partial}{\partial \vec{\nabla} \Phi_i} \right\} \left\{ - \sum_{i=0}^N |\vec{a}|^2 |\vec{\nabla} \Phi_i|^2 - \lambda \left( \sum_{i=0}^N \Phi_i E_i - E_{tot} \right) - \mu \sum_{i=0}^N (\Phi_i - 1) \right\} = 0 \quad (9)$$

The presence of such gradient terms has major consequences for the variational procedure as the  $\vec{\nabla} \frac{\partial}{\partial \vec{\nabla} \Phi_i}$  operator now finds a target. To physically motivate this replacement of the logarithmic terms by scalar products it is helpful to consider models of crystal growth as depicted in the following.

Interfaces in physical systems often have a finite extension of several monolayers and therefore the extension of thermodynamics including sharp interfaces towards a more general thermodynamic description involving diffuse interfaces respectively gradients seems meaningful. Two discrete models describing the growth of a so called Kossel crystal, Fig.1a, will be described in more detail: (i) the Jackson model of faceted growth [6], Fig.1b, and (ii) the Temkin model for the growth of a diffuse interface [9], Fig.1c.

The Schmitz model [3], Fig.1d, extrapolates the discrete approach of Temkin into a continuum description, where the gradient may be identified by

$$\Delta \Phi_n = \Phi_{n-1} - \Phi_n = \frac{d\Phi}{dz} * a_z \quad (10)$$

with “ $a_z$ ” denoting the lattice spacing in z-dimension. Generalization to three dimensions then yields  $\vec{\nabla} \Phi * \vec{a}$  with  $\vec{a}$  representing a vector characterizing the metric imposed by the underlying crystal structure of the solid phase. The inclusion of such gradient type contributions due to diffuse interfaces -- revealing a characteristic length scale being defined by the vector  $\vec{a}$  -- into the classical free energy/entropy formulation of thermodynamics is highly interesting for the description of evolving structures and introduces a length scale into thermodynamics.

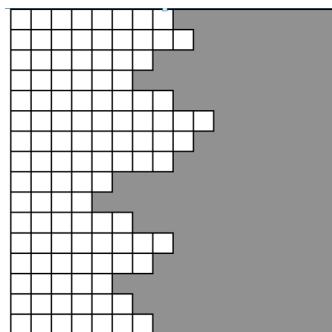


Fig. 1a): Kossel's model of a growing crystal (figure adapted from [4]). Atoms attach to the interface in layers. The model assumes that the atoms may only adhere on the top of already solid atoms ('solid on solid'). A smooth transition with a finite interface thickness can be identified when averaging the fraction of solid atoms parallel to the interface (see fig. 1d).

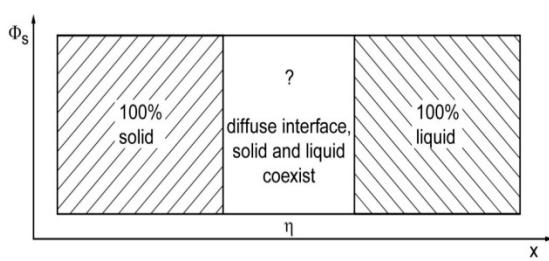


Fig. 1b): The Jackson model [6] assumes an interface being restricted to a single layer separating the two bulk regions. The entropic term in the Jackson model reads:

$$\phi \ln \phi + (1-\phi) \ln (1-\phi)$$

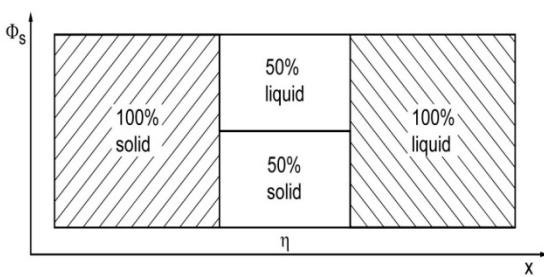


Fig. 1c): The Temkin model [9] assumes multiple steps separating the two bulk regions. The entropy of an intermediate layer  $n$  in this case is also defined by its adjacent layer  $n-1$ . The entropic term in the Temkin model reduces to the Jackson model in case of a single interface layer and reads:

$$\sum_{n=-\infty}^{\infty} (\Phi_{n-1} - \Phi_n) \ln (\Phi_{n-1} - \Phi_n) \\ = \sum_{n=-\infty}^{\infty} (\Delta \Phi_n) \ln (\Delta \Phi_n)$$

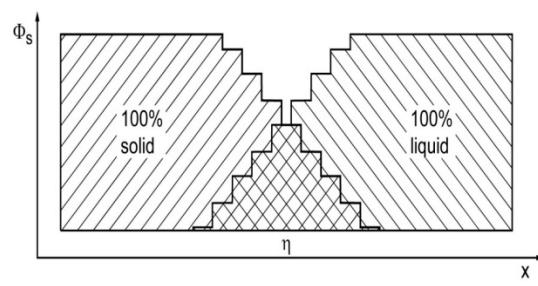
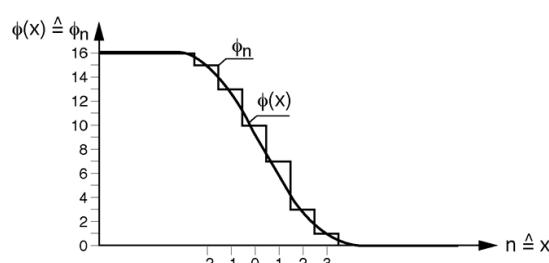


Fig. 1d): The Schmitz model [3] is an extension of the Temkin model for small step widths and approximates the discrete formulation of Temkin by a continuous gradient and by turning the sum of the individual contributions of the discrete layers into an integral (see text):

$$\iiint \vec{a} \nabla \Phi \ln (\vec{a} \nabla \Phi)$$



In case of multiple objects "i", each of these objects will have an own order parameter  $\Phi_i$  and its own interface being characterized by  $\vec{\nabla} \Phi_i$  as depicted in section 3 on phase field models.

Variational minimization of the resulting free energy functional for  $N$  objects comprising entropic terms of type

$$-\sum_{i=0}^N \vec{a} \nabla \Phi_i \ln(\vec{a} \nabla \Phi_i) \quad (11)$$

recovers the phase-field equations of motion under specific, simplifying assumptions. One of these assumptions relates to a Taylor expansion of the  $\vec{a} \nabla \Phi_i \ln(\vec{a} \nabla \Phi_i)$  terms, which for  $\vec{a} \nabla \Phi_i < 1$  provides following approximation:

$$\vec{a} \nabla \Phi_i \ln(\vec{a} \nabla \Phi_i) \sim (\vec{a} \nabla \Phi_i)^2 - \vec{a} \nabla \Phi_i \quad (12)$$

The scalar character of the entropy has been maintained by the use of scalar products when inserting the gradient terms. The vector  $\vec{a}$  in this early model [3] corresponds to the lattice vector of the growing crystal determining the distance between individual, discrete atomic layers of the Temkin model.

No such vector is a priori specified for a more generic formulation being the objective of the present work. Thus there is a need to specify a reference vector  $\vec{n}$  taking the role of the crystal lattice vector  $\vec{a}$  in the early crystallization model [3]. This vector has to be introduced as part of a scalar entity and the use of a scalar product thus seems mandatory. An interesting option for the choice of  $\vec{n}$  is the use of the scalar-triple (or parallelepipedal) product spanning a volume with value  $v$  along with an oriented coordinate system consisting of the three base vectors  $\vec{e}_x, \vec{e}_y, \vec{e}_z$ :

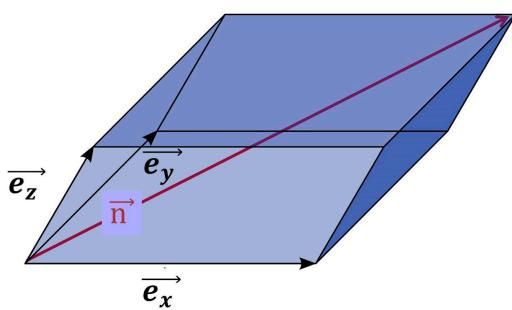


Fig. 2: Scalar triple product:  $\vec{e}_z(\vec{e}_x \times \vec{e}_y) = v$ . The volume being spanned by these vectors is  $v$ . The diagonal vector of this coordinate system reads  $\vec{e}_x + \vec{e}_y + \vec{e}_z = \vec{n}$  with its norm being  $(\vec{e}_x^2 + \vec{e}_y^2 + \vec{e}_z^2)^{0.5} = |\vec{n}|$ . This vector also is the normal vector to the tangential plane to that volume.

A periodic repetition of this elementary finite volume may eventually span the entire space and can be approximated by a continuous field  $\vec{n} = \vec{n}(\vec{r})$ . For the purpose of this paper the norm of the vector  $|\vec{n}|$  and the volume of the parallelepiped  $v$  are considered as constant.

Terms including scalar products between the coordinate system defined by  $\vec{n}$  and gradients of the scalar fields  $\Phi_i$  then - in analogy with equation (12) - read:

$$\vec{n} \nabla \Phi_i \ln(\vec{n} \nabla \Phi_i) \sim (\vec{n} \nabla \Phi_i)^2 - \vec{n} \nabla \Phi_i \quad (13)$$

### 3. Phase-Field Models

The overall strategy of the methodology proposed in this article comprises compiling the free energy density from a combination of entropic terms (see previous section) and from constraints given by conservation laws. The specification of different  $\Phi_i(\vec{r}, t)$  becomes mandatory for describing different objects and also for formulating the constraints of mass, energy and momentum conservation in terms of the variables  $\Phi_i(\vec{r}, t), \vec{\nabla} \Phi_i(\vec{r}, t), \dot{\Phi}_i(\vec{r}, t)$ :

$$constraint = f(\Phi_i(\vec{r}, t), \vec{\nabla} \Phi_i(\vec{r}, t), \dot{\Phi}_i(\vec{r}, t)) \quad (14)$$

The basis to tackle this task is provided by the phase-field method being shortly reviewed in terms of its relevance to the scope of the present methodology.

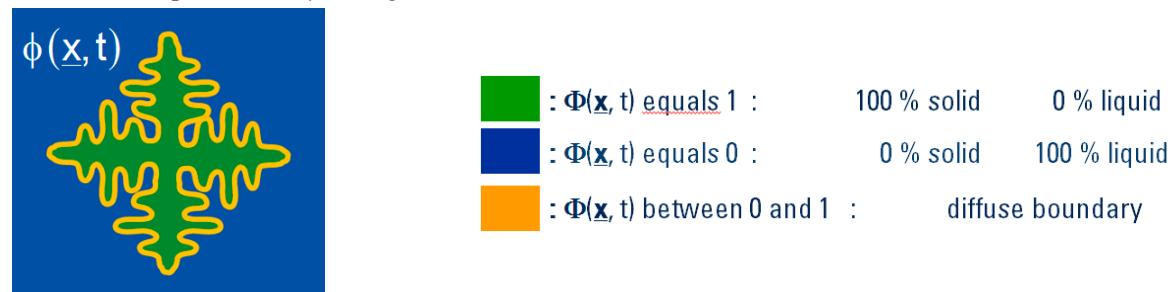
Phase-field models have their origin in the description of phase transitions. Phase transitions play a vital role in many areas of physics at all scales including e.g. magnetism, superconductivity,

solidification, condensation, solid state transformations, very fundamental transitions like the Higgs-Kibble mechanism unifying electroweak interaction, or possibly even nucleation and growth of galaxies.

Theories of phase transitions have their origin in early models of van der Waals (1893), Korteweg (1901), Ginzburg-Landau (1950), Cahn-Hilliard (1958), Allen-Cahn (1960), Halperin, Hohenberg & Ma (1977). While the early models did not include any spatial resolution, especially the Cahn-Hilliard equation for the first time addressed demixing phenomena in a spatially resolved approach. The order parameter entering into the equations was the – conserved - concentration of alloy elements. The Allen-Cahn equation then included the option of non-conserved order parameters for the first time. It seems essential to highlight, that phase transitions are best described by non-conserved order parameters, as for example the fraction liquid of a system will turn from 1 to 0 in a solidification process and thus is not a conserved quantity.

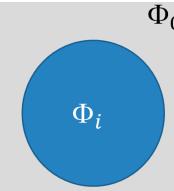
The first phase-field concept has been proposed in unpublished work by Langer [10] and was first publicly documented by Fix [11] and Caginalp [12]. The simulation of the evolution of complex 3D dendritic structures using phase-field models by Kobayashi [13] marked the trigger for an intense use of this methodology in materials sciences. The binary transitions/ equilibria between two states have later been extended to multi-phase equilibria in a multi-phase-field-model [14]. Higher order derivatives of the order parameter eventually lead to atomic resolution of rigid lattices in so called phase-field crystal models [15]. Phase-field models nowadays have reached a high degree of maturity and found applications in describing complex microstructures in technical alloy systems [16]. Reviews on phase-field modelling are found e.g. in [17, 18].

Core of most phase-field models is the description of the evolution of the shape of an object in time. To describe the evolution of this shape it is necessary in a first step to mathematically describe the initial shape of the object, Fig.3:



**Fig. 3:** Basic setting for the description of a complex shaped object by an order parameter. The order parameter field  $\Phi(\vec{r}, t)$  takes the value 1 wherever and whenever the object is present. A diffuse continuous interface marks the transition from the object to the “non-object” as shown here for a solid object in a liquid.

This initial shape thus is expressed as a scalar field of an “order parameter”, which is alternatively named “feature indicator” or most common “phase-field variable”. For the present objective it is sufficient to restrict the further discussion to rigid objects. Neither the evolution of their shape nor their motion in vacuum space will be addressed. This is the most basic use of a phase-field model just drawing on the definition of the “phase-field”-variable. The phase field description of a static object – here a sphere – being placed in vacuum is schematically depicted in Fig. 4:

$\Phi_0$ 	$\Phi_i = 1$ where sphere $i$ is present $\Phi_i = 0$ where sphere $i$ is absent $\Phi_0 = 1$ where sphere $i$ is absent $\Phi_0 = 0$ where sphere $i$ is present	$V_i = \iiint_{-\infty}^{+\infty} \Phi_i(\vec{r}, t)$ <p style="text-align: center;">is the volume of the sphere <math>i</math></p>
---	--	---

$\Phi_0 = (1 - \Phi_1)$ $\dot{\Phi}_1^2 = \dot{\Phi}_0^2$ $\dot{\Phi}_1 = -\dot{\Phi}_0$ $\vec{\nabla}\Phi_1 = -\vec{\nabla}\Phi_0$ $\Delta\Phi_1 = -\Delta\Phi_0$	interface points i.e. the surface of the sphere are characterized by following: $\vec{\nabla}\vec{\Phi}_i \neq 0$ $\Delta\Phi_i \neq 0$ $\Phi_0\Phi_i \neq 0$ $\Phi_i \ln \Phi_i \neq 0$	$J_i = \iiint_{-\infty}^{+\infty} r_i^2 \rho_i \Phi_i(\vec{r}, t)$ <i>is the moment of inertia of the sphere</i> $m_i = \iiint_{-\infty}^{+\infty} \rho_i \Phi_i(\vec{r}, t)$ <i>is the mass of the sphere</i> $R_i = \frac{\iiint_{-\infty}^{+\infty} r_i \Phi_i(\vec{r}, t)}{\iiint_{-\infty}^{+\infty} \Phi_i(\vec{r}, t)}$ <i>is the centroid of the sphere</i>
$\frac{\vec{\nabla}\vec{\Phi}_i}{ \vec{\nabla}\vec{\Phi}_i } = \vec{n}_i$ $\vec{\nabla}\vec{n}_i = \text{curvature}$ of interface		

**Fig. 4:** Some important mathematical relations related to the phase-field description of a massive sphere

The mass of a sphere  $i$  then is given as

$$m_i = \iiint_{-\infty}^{+\infty} \rho_i \Phi_i(\vec{r}, t) \quad (15)$$

It should be noted that the integral extends over the entire space and not only over covers the domain of the sphere. The phase-field variable  $\Phi_i$  – being 0 wherever there is no sphere  $i$  – here acts as a type of stencil selecting the domain of the sphere out of infinite space. The conservation for a total mass of a system of  $N$  objects/spheres then is given as

$$M_{tot} = \sum_{i=0}^N m_i = \iiint_{-\infty}^{+\infty} \sum_{i=0}^N \rho_i \Phi_i(\vec{r}, t) \quad (16)$$

where the index  $i=0$  refers to vacuum. Specifying the total mass as the integral of an average mass density

$$M_{tot} = \iiint_{-\infty}^{+\infty} \rho_{average} \quad (17)$$

allows formulating the constraint for mass conservation as follows

$$-g \left( \sum_{i=0}^N \rho_i \Phi_i(\vec{r}, t) - \rho_{average} \right) = 0 \quad (18a)$$

In case only a maximum of two objects coexist at the same point e.g. at the interface between vacuum and a massive object with index  $i$ , it is possible to rewrite the function  $\Phi_0(\vec{r}, t)$  as  $(1 - \Phi_i(\vec{r}, t))$  yielding:

$$-g \left( \sum_{i=1}^N \rho_i \Phi_i(\vec{r}, t) + \rho_{vac} (1 - \Phi_i(\vec{r}, t)) - \rho_{average} \right) = 0 \quad (18b)$$

This assumption holds everywhere and anytime as long as the massive objects are immersed in vacuum and do not have any mutual contact with each other. The mass density of the vacuum  $\rho_{vac}$  will become important in view of the cosmological constant entering into the scheme in this way (see chapter 5).

#### 4. Lagrange Formalism

Similar to entropy also the Lagrange formalism takes a significant role in many areas of physics. Besides the derivation of the Boltzmann factor being depicted above, the Lagrange formalism is a major basis for quantum mechanics and has especially been used to derive relations between symmetries and conservation laws. The Noether theorems being derived using the Lagrange formalism showed that invariance of physics laws under a translation implies the conservation of momentum or invariance under translation in time implies the conservation of energy. A further striking observation is that major physics laws all contain a Laplacian operator (resp. a Poisson type equation) somehow suggesting a common ground of all these models, which comprise all different length scales like gravitation, electrostatics, thermal conductivity, diffusion, flow, phase-field, Schrödinger equations, density functional equations and many others. Some operators being present in the Lagrange scheme have the property of generating Laplacian operators.

The basic concept of the Lagrange formalism is based on a functional being a scalar function  $F$  of a variable  $\Phi_i$  which itself is a function of space and time  $\Phi_i = \Phi_i(\vec{r}, t)$  and being an integral of a density function  $f$ .

$$F[\Phi_i(\vec{r}, t)] = \int f(\text{variables}) \, dr \, dt \quad (19)$$

The density function  $f$  is a function of a number of variables especially comprising  $\Phi_i(\vec{r}, t)$ ,  $\vec{\nabla}\Phi_i(\vec{r}, t)$ ,  $\dot{\Phi}_i(\vec{r}, t)$ :

$$f = f(\Phi_i(\vec{r}, t), \vec{\nabla}\Phi_i(\vec{r}, t), \dot{\Phi}_i(\vec{r}, t), \dots) \quad (20)$$

The fundamental Lagrange formalism allows for extensions to further variables, to higher order derivatives, and also to tensor fields. The present article will only consider first order derivatives. Setting the variation of  $F$  with respect to one of the variables  $\Phi_i(\vec{r}, t)$  to zero provides the corresponding Euler equation by setting the variational derivative of the function  $f$  to zero:

$$\frac{\delta F}{\delta \Phi_i(\vec{r}, t)} = \left\{ \frac{\partial}{\partial \Phi_i(\vec{r}, t)} - \vec{\nabla} \frac{\partial}{\partial \vec{\nabla}\Phi_i(\vec{r}, t)} - \frac{\partial}{\partial t} \frac{\partial}{\partial \dot{\Phi}_i(\vec{r}, t)} \right\} f = 0 \quad (21)$$

The results are the desired equations of motion for the different  $\Phi_i(\vec{r}, t)$ .

#### 5. Derivation of the Gravitational Law

The entropy for the static field as formulated in chapter 2 in Taylor approximation reads:

$$s = - \sum_{i=0}^N ((\vec{n} \vec{\nabla} \Phi_i)^2 - \vec{n} \vec{\nabla} \Phi_i) = - \sum_{i=0}^N \vec{n}^2 \vec{\nabla} \Phi_i^2 \cos^2(\vec{n}, \vec{\nabla} \Phi_i) - \vec{n} \vec{\nabla} \Phi_i \quad (22)$$

The scalar product for the square term here has been formally executed introducing a  $\cos^2$  function of the angle between the vectors  $\vec{n}$  and  $\vec{\nabla} \Phi_i$ . When applying the Lagrange formalism, this  $\cos^2$  function eventually will lead to a non-linear generalization of the Newton–Poisson equation as e.g. used in modified Newtonian dynamic (MOND) approaches [19] - [22]. Further aspects of this generalization are detailed and discussed in section 6. In the following, the function is first considered as a constant making the product  $\vec{n}^2 \cos^2(\vec{n}, \vec{\nabla} \Phi_i) = 1$ . The overall entropy term only comprises terms related to  $\vec{\nabla} \Phi_i(\vec{r}, t)$  and no terms related to  $\Phi_i(\vec{r}, t)$  and  $\dot{\Phi}_i(\vec{r}, t)$ . Only the gradient related terms of the functional derivative in the Lagrange formalism thus become active:

$$-\vec{\nabla} \frac{\partial}{\partial \vec{\nabla} \Phi_i} \left\{ - \sum_{i=0}^N (\vec{\nabla} \Phi_i^2 - \vec{n} \vec{\nabla} \Phi_i) \right\} = 0 \quad (23)$$

Executing this derivative yields:

$$\vec{\nabla} \{ 2\vec{\nabla} \Phi_i - \vec{n} \} = 2\Delta \Phi_i - \vec{\nabla} \vec{n} = 0 \quad (24)$$

The Poisson equation can be directly identified. The divergence of the normal vector of a surface is the measure of the curvature of that surface. The vector  $\vec{n}$  is the diagonal of the volume and perpendicular to the tangential plane to that volume. The divergence of this vector  $\vec{\nabla} \vec{n}$  thus is a measure for the curvature  $\kappa$  of space.

The constraint for mass conservation - its specification being fully detailed in equations 18 a&b - is associated with the Lagrange Multiplier  $g$  and only depends on  $\Phi_i(\vec{r}, t)$ . This constraint does not contain any terms related to  $\vec{\nabla} \Phi_i(\vec{r}, t)$  and  $\dot{\Phi}_i(\vec{r}, t)$ :

$$\frac{\partial}{\partial \Phi_i} \left\{ -g \left( \sum_1^N \rho_i \Phi_i - \rho_{average} \right) \right\} = -g \rho_i \quad (25)$$

Overall, the following equation for the field  $\Phi_i$  arises when adding this constraint term (25) to the term for the entropy equation (24):

$$2\Delta \Phi_i - \vec{\nabla} \vec{n} - g \rho_i = 0 \quad (26a)$$

Using the special formulation for the constraint of mass conservation comprising also the density of the vacuum (equation 18b) generates an additional term:

$$2\Delta \Phi_i - \vec{\nabla} \vec{n} + g \rho_{vac} - g \rho_i = 0 \quad (26b)$$

Calibrating the Lagrange multiplier  $g$  to  $8\pi G$  with  $G$  being the gravitational constant yields:

$$2\Delta \Phi_i - \vec{\nabla} \vec{n} + 8\pi G \rho_{vac} = 8\pi G \rho_i \quad (27)$$

Comparing with the cosmological constant  $\Lambda$  [23]

$$\Lambda = \frac{8\pi G \rho_{vac}}{c^2} \quad (28)$$

allows rewriting to

$$\Delta \Phi_i - \frac{1}{2} \vec{\nabla} \vec{n} + \frac{c^2 \Lambda}{2} = 4\pi G \rho_i \quad (29)$$

Neglecting curvature  $\kappa$  of space (i.e. setting  $\vec{\nabla} \vec{n} = 0$ ) and setting  $\Lambda = 0$  this equation becomes identical with the Poisson equation of gravitation i.e. with classical Newton's law

$$\Delta \Phi_0 = 4\pi G \rho_0 \quad (30)$$

In a first summary the application of the Lagrange scheme to the gradient-entropy terms as depicted in the present paper obeying the constraint of mass conservation in a strikingly direct derivation has led to

- the Poisson equation of gravity (Newton's law)
- a term related to curvature of space (which probably can be related to Einstein's general theory of relativity)
- a term introducing the mass density of vacuum (which seems related to the cosmological constant)
- terms related to a nonlinear generalization of the Newton-Poisson equation as used in modified Newtonian dynamic (MOND) approaches [19]-[22].

These MOND terms will be discussed in more detail in the following chapter.

## 6. Modified Newtonian Dynamics

Formulations and equations in Modified Newtonian Dynamics (MOND) approaches [19 -22] are constructed based on experimental findings and are made to fit the experimental observations especially on velocity distributions in galaxies. These approaches do not draw on dark matter as the

basis for the description of the observed behavior. In spite of successfully describing the experimental observations, the MOND formulations are under controversial discussion as they still lack a deeper theoretical framework for their derivation from fundamental principles. The present work indicates a possible approach towards a deeper understanding of the background of the empirical MOND equations.

This chapter aims to elucidate relations of the proposed approach to the modified Newtonian dynamics (MOND) being already shortly indicated in chapter 2. Recovering the equation for the gradient entropy from chapter 2 including also the curvature related term reads:

$$(\vec{n}\vec{\nabla}\Phi)^2 - \vec{n}\vec{\nabla}\Phi = \vec{n}^2\vec{\nabla}\Phi^2 \cos^2(\varphi) - \vec{n}\vec{\nabla}\Phi \quad (31a)$$

with the angle between space diagonal  $\vec{n}$  and the gradient of the scalar field  $\vec{\nabla}\Phi$  being denoted as  $\varphi$ . The variational derivative then looks as follows:

$$\vec{\nabla} \frac{\partial}{\partial \vec{\nabla}\Phi} \left\{ \vec{n}^2\vec{\nabla}\Phi^2 \cos^2(\varphi) - \vec{n}\vec{\nabla}\Phi \right\} = 0 \quad (32a)$$

$$\vec{\nabla} \left\{ 2\vec{n}^2 \cos^2(\varphi) \vec{\nabla}\Phi + 2\vec{n}^2\vec{\nabla}\Phi^2 \cos(\varphi)\sin(\varphi) \frac{\partial\varphi}{\partial \vec{\nabla}\Phi} - \vec{n} \right\} = 0 \quad (32b)$$

which, with

$$\begin{aligned} \vec{n}^2 &= \text{constant} = 1, & \cos^2(\varphi) &= 1 - \sin^2(\varphi), \\ 2\cos(\varphi)\sin(\varphi) &= \sin(2\varphi), & \text{and} & \sin^2(\varphi) = \frac{\tan^2(\varphi)}{1 + \tan^2(\varphi)} \end{aligned}$$

leads to

$$2\Delta\Phi - 2\vec{\nabla} \left\{ \frac{\tan^2(\varphi)}{1 + \tan^2(\varphi)} \vec{\nabla}\Phi \right\} + \vec{\nabla} \left\{ \vec{\nabla}\Phi^2 \sin(2\varphi) \frac{\partial\varphi}{\partial \vec{\nabla}\Phi} \right\} - \vec{\nabla}\vec{n} = 0 \quad (32c)$$

"Newton" – "MOND\_I" + "MOND\_II" – "Einstein"

For small angles of  $\varphi$  with  $\sin(2\varphi) \sim 0$  and  $\tan^2(\varphi) \sim 0$  the classical Newton-Poisson equation is recovered, while for angles of  $\varphi$  approaching  $\pi/2$  the MOND terms generate additional contributions. In both cases the curvature related term persists. The MOND Eulerian [20] for comparison reads

$$\vec{\nabla} \left\{ \mu \left( \frac{|\vec{\nabla}\Phi|}{a_0} \right) \vec{\nabla}\Phi \right\} \quad (33)$$

with  $\mu(x)$  being "....an as-yet unspecified function (known as the "interpolating function"), and  $a_0$  is a new fundamental constant ( $a_0 \sim 10^{-8}$  cm s<sup>-2</sup>) which marks the transition between the Newtonian and deep-MOND regimes. Agreement with Newtonian mechanics requires  $\mu(x) \rightarrow 1$  for  $x \gg 1$ , and consistency with astronomical observations requires  $\mu(x) \rightarrow x$  for  $x \ll 1$ . Beyond these limits, the interpolating function is not specified by the theory, although it is possible to weakly constrain it empirically....." [20]. Examples for the MOND interpolation function are the "standard" [22] and the "simple" [24] interpolation functions:

$$\mu(x) = \frac{x}{\sqrt{1+x^2}} \quad [\text{standard}] \quad \text{or} \quad \mu(x) = \frac{x}{1+x} \quad [\text{simple}] \quad (34)$$

The term  $\frac{\tan^2(\varphi)}{1+\tan^2(\varphi)}$  corresponds to the square of the "standard" interpolation function when setting  $x = \tan^2(\varphi)$ . Setting  $x = \tan^2(\varphi)$  directly yields the "simple" interpolation function.

In spite of these similarities with the MOND type formulations there are also obvious differences like (i) the persisting, original Newton term, (ii) the negative sign of the MOND\_I term, (iii) the new, additional term denoted as MOND\_II, (iv) the  $\vec{\nabla}\vec{n}$  term likely to be related to "Einstein" curvature, or (v) the "Λ term", see equation (29).

It eventually seems most interesting to elaborate a relation and a physical interpretation for the angle  $φ$  between the gradient of the potential  $\vec{\nabla}\Phi$  and the direction of diagonal vector  $\vec{n}$  of the underlying coordinate system. A possible geometric interpretation in 2 dimensions is depicted in figure 5:

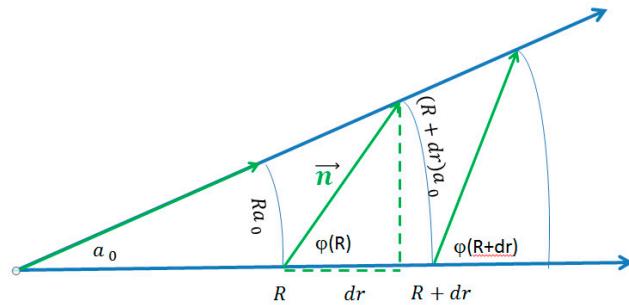


Figure 5: Proposed explanation for a change in direction of the normal vector of space leading to an angle  $φ$  which increases with increasing radius  $R$  (see text)

The area  $A(R)$  of a small segment between  $R$  and  $R+dr$  being bounded by the angle  $α_0$  reads

$$A(R) = α_0 R dr \quad (35)$$

Postulating this area  $A$  of the parallelepiped with diagonal  $\vec{n}$  to be constant when increasing the radius  $R$  by  $dr$  requires

$$dr = σ' \frac{1}{R} \quad (36)$$

with  $σ'$  being a proportionality constant with dimensions  $[L^2]$ . The tangent of the angle  $φ$  between radial direction and the diagonal of the parallelepiped can then be approximated as a function of  $R$  by

$$\tan(φ(R)) = \frac{α_0 R}{dr} = \frac{α_0}{σ'} R^2 = \frac{R^2}{σ} \quad (37)$$

with  $σ = \frac{σ'}{α_0}$  being a yet undefined constant probably related to the  $a_0$  parameter in MOND approaches, which indicates the transition between Newtonian and MOND regimes. In spite of being most interesting, it is beyond the scope of the present article to further elucidate the origin and interpretation of this angle and/or the value of  $σ$ .

In summary following equation for gravity could be obtained when formally performing the scheme being proposed in this article to its full depth:

$$2ΔΦ - 2\vec{\nabla} \left\{ \frac{\tan^2(φ)}{1 + \tan^2(φ)} \vec{\nabla}\Phi \right\} + \vec{\nabla} \left\{ \vec{\nabla}\Phi^2 \sin(2φ) \frac{\partial φ}{\partial \vec{\nabla}\Phi} \right\} - \vec{\nabla}\vec{n} = 8πGρ_i - c^2Λ \quad (38)$$

This equation comprises a combination of terms similarly – at least in a qualitative way – appearing in a number of other theories on gravitation. In its simplest approximation i.e. for

$$φ = 0, \quad \vec{\nabla}\vec{n} = 0, \quad Λ = 0 \quad (39)$$

this equation – being derived from a mere entropic approach – clearly recovers the classical Newtonian law. All other terms remain subject to future discussions.

## 7. Summary and Future Perspectives

The approach for the description of gravity being described in the present article is based on (i) an entropy formulation comprising scalar products of gradients of a scalar field with (ii) the diagonal vector of a volume element, (iii) a field description of objects being based on this scalar

field, and (iv) a formulation of the constraint for mass conservation in terms of this scalar field. Performing (v) the Lagrange formalism onto the resulting formulations in a strikingly direct derivation has led to

- the Poisson equation of gravity (Newton's law)
- a term related to curvature of space (which probably can be related to Einstein's general theory of relativity)
- a term introducing the mass density of vacuum (which seems related to the cosmological constant)
- terms related to a nonlinear generalization of the Newton–Poisson equation as used in modified Newtonian dynamic (MOND) approaches

The resulting formulation suggests a co-existence resp. a superposition of different types of models being currently intensely discussed. A deeper investigation and interpretation of the presented approach by the respective communities thus seems worth some effort. It should be noted that the present paper has only touched a subset of an overall scheme, which may be extended into several directions being depicted in the following table:

	Lagrange derivative	entropy term	Conserved quantity	Lagrange multipliers
established procedures	$\frac{\partial}{\partial \Phi_i}$	$s(\Phi_i)$	energy	scalar ( $\lambda$ )
this work	$\frac{\partial}{\partial \Phi_i}$	$s(\Phi_i, \vec{V}\Phi_i)$	mass	scalar (g)
	$\nabla \frac{\partial}{\partial \nabla \Phi_i}$	$s(\Phi_i, \vec{V}\Phi_i)$		
future topics	$\frac{\partial}{\partial t} \frac{\partial}{\partial \dot{\Phi}_i}$	$s(\Phi_i, \vec{V}\Phi_i, \dot{\Phi}_i)$	mass, energy, charge, momentum, spin	scalars: ( $\lambda, g, \epsilon, \dots$ ) vectors: $\vec{c}$ and $\vec{h}$
	tensor type formulations	....	energy-momentum tensor, charge, spin	...to be continued....

Especially the mass constraint being investigated in the present article can and should be replaced by the energy-momentum tensor in a relativistic formulation. A number of further, open questions remain to be tackled, such as (i) the origin and interpretation of the angle  $\varphi$  in the formulations, (ii) the origin and interpretation of the length scale [ $L^2$ ] being introduced by the scalar products (iii) a careful comparison with other theories, (iv) a comparison with experimental observations and many others.

The appealing simplicity of the approach provides an interesting, new and alternative view on things and perhaps even a better understanding of phenomena in future, which seems to make it worth to be further exploited.

**Acknowledgments:** The ideas being documented in the present article have emerged over several years in parallel to ongoing projects being funded by various institutions. However, the efforts toward compiling these ideas did not receive any dedicated funding. Covering the costs for open source publication by my employer ACCESS e.V. is gratefully acknowledged. Thanks are due to the MICRESS® group at ACCESS e.V. for numerous discussions and constructive criticism. Special thanks are due to Bernd Böttger, Gottfried Laschet, and Markus Apel for commenting on the final manuscript and to Ulrike Hecht for drawing my attention to the MOND just at the right time.

**Conflicts of Interest:** The author declares no conflict of interest. The founding sponsors had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, and in the decision to publish the results.

## References

1. E.P.Verlinde: High Energy Physics - Theory (hep-th) JHEP 1104:029,201,DOI: 10.1007/JHEP04(2011)029 arXiv:1001.0785v1 [hep-th]
2. E. P. Verlinde: "Emergent Gravity and the Dark Universe", 7 Nov 2016. arxiv.org/abs/1611.02269
3. G.J. Schmitz: "Thermodynamics of Diffuse Interfaces" in: H.Emmerich, B.Nestler, M.Schreckenberg (eds): "Interface and Transport Dynamics" Springer Lecture Notes in Computational Science and engineering, ISBN 3-540-40367-1 (2003) p. 47-64
4. D. Woodruff.: The solid liquid interface. Cambridge University Press, 1973
5. P. J. Flory: Principles of Polymer Chemistry; Cornell University Press: New York, 1953.
6. K.A. Jackson,: Liquid Metals and solidification ASM. Cleveland Ohio, (1958) 174
7. E.T Jaynes: Probability Theory: The logic of Science, chapter 11, (internet: <http://omega.albany.edu:8008/JaynesBook.html>) (accessed March 2017)
8. G. J. Schmitz, B. Böttger, M. Apel, J. Eiken, G. Laschet, R. Altenfeld, R. Berger, G. Boussinot & A. Viardin: "Towards a metadata scheme for the description of materials – the description of microstructures" Science and Technology of Advanced Materials 17:1(2016) 410-430, DOI:10.1080/14686996.2016.1194166
9. D.E Temkin: Crystallization Processes (ed. N.N. Sirota, F.K. Gorskii and V.M. Varikash). English translation published by consultants Bureau, New York (1966)
10. J. Langer (1978). Manuscript on a concept of phase field models (unpublished)
11. Fix, G.: Phase field models for free boundary problems. in: Free boundary problems Vol. II, (ed. A.Fasano, M.Primicerio) Piman Boston (1983)
12. G. Caginalp, P.C. Fife: Phys. Rev. B 33 11 (1986) 7792
13. R. Kobayashi: Modeling and numerical simulations of dendritic crystal growth. Physica D 63 (1993) 410-423
14. I. Steinbach, F. Pezzolla, B. Nestler, M. Seefelberg, R. Prieler, G.J. Schmitz, J.L.L. Rezende "A phase field concept for multiphase systems" Physica D 94(1996), pp 135-147
15. K. R. Elder, Mark Katakowski, Mikko Haataja, and Martin Grant: "Modeling Elasticity in Crystal Growth" Phys. Rev. Lett. 88, 245701 – Published 4 June 2002
16. G. J. Schmitz, B. Böttger, J. Eiken, M. Apel , A. Viardin, A. Carré and G. Laschet: "Phase-field based simulation of microstructure evolution in technical alloy grades": Int J Adv Eng Sci Appl Math 2 4 (2010) 126 DOI 10.1007/s12572-011-0026-y
17. N. Provatas, K. Elder: Phase-Field Methods in Materials Science and Engineering, Wiley VCH, Weinheim (2010), ISBN: 978-3-527-40747-7
18. I. Steinbach: Phase-field models in Materials Science –Topical Review: Modelling Simul. Mater. Sci. Eng. 17 (2009)073001
19. J. Bekenstein and M. Milgrom: "Does the missing mass problem signal the breakdown of Newtonian gravity?". Astrophys. J. 286: 7–14. Bibcode:1984ApJ...286....7B. doi:10.1086/162570.
20. [https://en.wikipedia.org/wiki/Modified\\_Newtonian\\_dynamics](https://en.wikipedia.org/wiki/Modified_Newtonian_dynamics) (accessed Feb. 2017)
21. M. Milgrom: "The MOND" <https://arxiv.org/pdf/1404.7661.pdf>
22. M. Milgrom: "A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis": Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 270, July 15, 1983, p. 365-370. DOI: 10.1086/161130
23. [https://de.wikipedia.org/wiki/Kosmologische\\_Konstante](https://de.wikipedia.org/wiki/Kosmologische_Konstante) (accessed March 2017)
24. B. Famaey, J. Binney: "Modified Newtonian Dynamics in the Milky Way" <https://arxiv.org/abs/astro-ph/0506723>

