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LÉVY–KHINTCHINE REPRESENTATION OF TOADER–QI MEAN

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Abstract. In the paper, by virtue of a Lévy–Khintchine representation and an alternative integral representation for the weighted geometric mean, the authors establish a Lévy–Khintchine representation and an alternative integral representation for the Toader–Qi mean. Moreover, the authors also collect an probabilistic interpretation and applications in engineering of the Toader–Qi mean.

1. Preliminaries and main results

In this section, we prepare some necessary knowledge and state our main results. We organize this section as three subsections.

1.1. **Toader–Qi mean.** For $a,b>0$ and $q\ne0$, denote

$$M_q(a,b)=\left(\frac{1}{2\pi}\int_0^{2\pi}a^{q\cos^2\theta}b^{q\sin^2\theta}\,\mathrm{d}\,\theta\right)^{1/q}.$$

It is easy to see that

$$M_q(a,b)=\left[\frac{2}{\pi}\int_0^{\pi/2}\left(a^{\cos^2\theta}b^{\sin^2\theta}\right)^q\,\mathrm{d}\,\theta\right]^{1/q}=\left[\frac{2}{\pi}\int_0^{\pi/2}(a^q)^{\cos^2\theta}(b^q)^{\sin^2\theta}\,\mathrm{d}\,\theta\right]^{1/q}.$$

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In [37], it was remarked that

$$M_0(a,b)=\lim_{q\to0}M_q(a,b)=G(a,b)=\sqrt{ab}\,,$$

but it was not known how to determine any mean M_q for $q\ne0$. In [38, p. 382, Section 5], the connection

$$M_q(a,b)=G(a,b)\biggl[J_0\biggl(-i\frac{q}{2}\ln\frac{a}{b}\biggr)\biggr]^{1/q} \tag{1.1}$$

was underlined, where $i=\sqrt{-1}$ is the imaginary unit, the Bessel function of the first kind $J_\nu(z)$ can be defined [9, p. 217, 10.2.2] by

$$J_\nu=\sum_{k=0}^\infty\frac{(-1)^k}{k!\Gamma(\nu+k+1)}\biggl(\frac{z}{2}\biggr)^{2k+\nu},$$

and the classical Euler gamma function $\Gamma(z)$ can be defined [12] by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^n(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

In [23, Lemma 2.1] and [24, Lemma 2.1], the relation

$$M_q(a,b)=G(a,b)\biggl[I_0\biggl(\frac{q}{2}\ln\frac{a}{b}\biggr)\biggr]^{1/q} \tag{1.2}$$

was established, where the modified Bessel functions of the first kind $I_\nu(z)$ can be defined by

$$I_\nu(z)=\sum_{k=0}^\infty\frac{1}{k!\Gamma(\nu+k+1)}\biggl(\frac{z}{2}\biggr)^{2k+\nu},\quad |\arg z|<\pi,\quad \nu\in\mathbb{C}.$$

Since

$$J_0(-iz)=I_0(z)=\sum_{k=0}^\infty\biggl(\frac{z}{2k!}\biggr)^{2k},$$

the formulas (1.1) and (1.2) are essentially the same one.

With the help of (1.2), the mean $M_q(a,b)$, the modified Bessel function of the first kind I_0, and the arithmetic-geometric mean $M(a,b)$, which can be defined [5, 25, 26] by

$$\frac{1}{M(a,b)}=\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}\,\mathrm{d}\,\theta,$$

were bounded in [23, 24, 40, 41, 42] successfully.

We remark that the mean $M_q(a,b)$ can be traced back to [5, 6, 37, 38] and the closely related references therein. See also [2, pp. 401–403].

In the papers [40, 41, 42], the mean $M_1(a,b)$ was denoted by $TQ(a,b)$ and was named the Toader–Qi mean after a Romanian mathematician, Gheorghe Toader, and the first author of this paper. For the sake of convenience, we still adopt in this paper the notation and terminology.

In Section 4 of this paper, we will collect a probabilistic interpretation and applications in engineering of the Toader–Qi mean $TQ(a,b)$.

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1.2. **Lévy–Khintchine representation.** For stating our main results, we need to recall the following notion and conclusions.

Definition 1.1 ([39, Chapter IV]). An infinitely differentiable function f on an interval I is said to be completely monotonic on I if it satisfies

$$(-1)^{n-1} f^{(n-1)}(t) \ge 0$$

for $x \in I$ and $n \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of all positive integers.

Definition 1.2 ([4, Definition 1.2]). An infinitely differentiable function $f : I \to [0, \infty)$ is called a Bernstein function on an interval I if $f'(t)$ is completely monotonic on I.

Proposition 1.1 ([36, Theorem 3.2]). *A function $f : (0, \infty) \to [0, \infty)$ is a Bernstein function if and only if it admits the representation*

$$f(x) = \alpha + \beta x + \int_0^\infty \left(1 - e^{-xt}\right) \mathrm{d}\,\mu(t), \tag{1.3}$$

where $\alpha, \beta \ge 0$ and μ is a positive measure satisfying $\int_0^\infty \min\{1, t\} \mathrm{d}\,\mu(t) < \infty$.
In particular, the triplet (α, β, μ) determines f uniquely and vice versa.

The integral representation (1.3) for $f(x)$ is called the Lévy-Khintchine representation of $f(x)$. The representing measure μ and the characteristic triplet (α, β, μ) are often respectively called the Lévy measure and the Lévy triplet of the Bernstein function $f(x)$.

It was pointed out in [36, Chapter 3] that the notion of the Bernstein functions originated from the potential theory and is useful to harmonic analysis, probability, statistics, and integral transforms.

Definition 1.3 ([36, Chapter 2, Definition 2.1]). A Stieltjes function is a function $f : (0, \infty) \to [0, \infty)$ which can be written in the form

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s + x} \mathrm{d}\,\mu(s), \tag{1.4}$$

where $a, b \ge 0$ are nonnegative constants and μ is a nonnegative measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{1}{1 + s} \mathrm{d}\,\mu(s) < \infty.$$

The integral appearing in (1.4) is also called the Stieltjes transform of the measure μ. For more information on this kind of functions (transforms), please refer to the monographs [36, Chapter 2], [39, Chapter VIII] and the closely related references therein.

1.3. **Main results.** The aims of this paper are to establish an integral representation and the Lévy–Khintchine representation of the Toader–Qi mean $TQ(x+a, x+b)$ for $b > a > 0$ and to verify that the Toader–Qi mean $TQ(x, x+b-a)$ for $b > a > 0$ is a Bernstein functions and the divided difference of $TQ(x, x+b-a)$ is a Stieltjes function on $(0, \infty)$.

Our main results can be stated as the following theorems.

Theorem 1.1. *For $b > a > 0$, the Toader–Qi mean $TQ(x+a, x+b)$ for $x > -a$ has the integral representation*

$$TQ(x + a, x + b) = TQ(a, b) + x\left[1 + \frac{1}{\pi} \int_a^b \frac{h(a, b; t)}{t} \frac{1}{t + x} \mathrm{d}\,t\right] \tag{1.5}$$

and the Lévy–Khintchine representation

$$TQ(x+a,x+b)=TQ(a,b)+x+\frac{b-a}{\pi}\int_0^\infty\frac{H(a,b;s)}{s}e^{-as}(1-e^{-xs})\,\mathrm{d}\,s, \tag{1.6}$$

where

$$h(a,b;t)=\frac{2}{\pi}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)(t-a)^{\cos^2\theta}(b-t)^{\sin^2\theta}\,\mathrm{d}\,\theta,\quad t\in[a,b],$$
$$H(a,b;s)=\frac{2}{\pi}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)F\bigl(\cos^2\theta,(b-a)s\bigr)\,\mathrm{d}\,\theta,\quad s\in(0,\infty),$$

and

$$F(\lambda,s)=\int_0^1\biggl(\frac{1}{u}-1\biggr)^\lambda\biggl(1-\frac{\lambda}{1-u}\biggr)e^{-su}\,\mathrm{d}\,u,\quad(\lambda,s)\in(0,1)\times(0,\infty) \tag{1.7}$$

are positive. Consequently,

(1) *the Toader–Qi mean $TQ(x,x+b-a)$ is a Bernstein function of x on $(0,\infty)$,*
(2) *the divided difference*

$$\frac{TQ(x,x+b-a)-TQ(a,b)}{x-a} \tag{1.8}$$

is a Stieltjes function of x on $(0,\infty)$.

Corollary 1.1. *For $b>a>0$ and $x>-a^q$, we have*

$$TQ(x+a^q,x+b^q)=[M_q(a,b)]^q+x\biggl[1+\frac{1}{\pi}\int_{a^q}^{b^q}\frac{h(a^q,b^q;t)}{t}\frac{1}{t+x}\,\mathrm{d}\,t\biggr]$$

and

$$TQ(x+a^q,x+b^q)=[M_q(a,b)]^q+x+\frac{b^q-a^q}{\pi}\int_0^\infty\frac{H(a^q,b^q;s)}{s}e^{-a^qs}(1-e^{-xs})\,\mathrm{d}\,s.$$

Consequently, the divided difference

$$\frac{TQ(x,x+b^q-a^q)-[M_q(a,b)]^q}{x-a^q},\quad q\ne0$$

is a Stieltjes function of x on $(0,\infty)$.

2. Lemmas

For proving our main results, we need the following lemmas.

Lemma 2.1. *Let $\lambda\in(0,1)$ and $b>a>0$. The principal branch of the weighted geometric mean*

$$G_{a,b;\lambda}(z)=(a+z)^\lambda(b+z)^{1-\lambda}$$

for $z\in\mathbb{C}\setminus[-b,-a]$ can be represented by

$$\frac{G_{a,b;\lambda}(z)-a^\lambda b^{1-\lambda}}{z}=\begin{cases}1+\dfrac{\sin(\lambda\pi)}{\pi}\displaystyle\int_a^b\frac{(t-a)^\lambda(b-t)^{1-\lambda}}{t}\frac{1}{t+z}\,\mathrm{d}\,t, & z\ne0;\\ \dfrac{(1-\lambda)a+\lambda b}{a^{1-\lambda}b^\lambda}, & z=0.\end{cases} \tag{2.1}$$

Proof. This follows from letting $n=2$ in [29, Theorems 3.1 and 4.6], rearranging, and taking the limit $z\to0$. See also [13, Remark 4.3], [17, eq. (2.1)], [18, eq. (2.1)], [21, p. 728, Section 4, eq. (4.1)], [22, Section 4, eq. (4.1)] and the closely related references therein. □

Lemma 2.2. *Let $\lambda \in (0,1)$ and $b > a > 0$. The principal branch of the weighted geometric mean $G_{a,b;\lambda}(z)$ for $z \in \mathbb{C} \setminus [-b,-a]$ has the Lévy–Khintchine representation*

$$G_{a,b;\lambda}(z) = a^\lambda b^{1-\lambda} + z + \frac{\sin(\lambda\pi)}{\pi}(b-a)\int_0^\infty \frac{F(\lambda,(b-a)s)}{s}e^{-as}(1-e^{-zs})\,\mathrm{d}\,s, \tag{2.2}$$

where $F(\lambda,s)$ is defined by (1.7).

Proof. This is a slight reformulation of those results in [15, p. 131, Lemma 3.8], [20, Lemma 2.2], [30, Theorem 3.2], and [32, Theorem 1.1] and the closely related references therein. □

3. Proofs of main results

Proof of Theorem 1.1. Applying (2.1) gives

$$TQ(x+a,x+b) = \frac{2}{\pi}\int_0^{\pi/2}(x+a)^{\cos^2\theta}(x+b)^{\sin^2\theta}\,\mathrm{d}\,\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\biggl[a^{\cos^2\theta}b^{\sin^2\theta} + x + \frac{\sin\bigl(\pi\cos^2\theta\bigr)}{\pi}x\int_a^b \frac{(t-a)^{\cos^2\theta}(b-t)^{\sin^2\theta}}{t}\frac{1}{t+x}\,\mathrm{d}\,t\biggr]\,\mathrm{d}\,\theta$$

$$= TQ(a,b) + x + \frac{2}{\pi}\frac{x}{\pi}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)\int_a^b \frac{(t-a)^{\cos^2\theta}(b-t)^{\sin^2\theta}}{t}\frac{1}{t+x}\,\mathrm{d}\,t\biggr]\,\mathrm{d}\,\theta$$

$$= TQ(a,b) + x + \frac{x}{\pi}\int_a^b \frac{1}{t}\biggl[\frac{2}{\pi}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)(t-a)^{\cos^2\theta}(b-t)^{\sin^2\theta}\,\mathrm{d}\,\theta\biggr]\frac{1}{t+x}\,\mathrm{d}\,t.$$

The integral representation (1.5) is thus proved.

Applying (2.2) yields

$$TQ(x+a,x+b) = \frac{2}{\pi}\int_0^{\pi/2}(x+a)^{\cos^2\theta}(x+b)^{\sin^2\theta}\,\mathrm{d}\,\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\biggl[a^{\cos^2\theta}b^{\sin^2\theta} + x + \frac{\sin\bigl(\pi\cos^2\theta\bigr)}{\pi}(b-a)\int_0^\infty \frac{F\bigl(\cos^2\theta,(b-a)s\bigr)}{s}e^{-as}(1-e^{-xs})\,\mathrm{d}\,s\biggr]\,\mathrm{d}\,\theta$$

$$= TQ(a,b) + x + \frac{2(b-a)}{\pi^2}\int_0^{\pi/2}\biggl[\sin\bigl(\pi\cos^2\theta\bigr)\int_0^\infty \frac{F\bigl(\cos^2\theta,(b-a)s\bigr)}{s}e^{-as}(1-e^{-xs})\,\mathrm{d}\,s\biggr]\,\mathrm{d}\,\theta$$

$$= TQ(a,b) + x + \frac{b-a}{\pi}\int_0^\infty \frac{1}{s}\biggl[\frac{2}{\pi}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)F\bigl(\cos^2\theta,(b-a)s\bigr)\,\mathrm{d}\,\theta\biggr]e^{-as}(1-e^{-xs})\,\mathrm{d}\,s.$$

The Lévy–Khintchine representation (1.6) is thus proved.

Since $F(\lambda,s)$ is positive on $(0,\infty)$, comparing the representation (1.6) with (1.3) readily reveals that the Toader–Qi mean $TQ(x,x+b-a)$ is a Bernstein function of x on $(0,\infty)$.

Reformulating (1.5) as

$$\frac{TQ(x+a,x+b)-TQ(a,b)}{x} = 1 + \frac{1}{\pi}\int_a^b \frac{h(a,b;t)}{t}\frac{1}{t+x}\,\mathrm{d}\,t$$

and comparing with (1.4) immediately shows that the divided difference (1.8) is a Stieltjes function. The proof of Theorem 1.1 is complete. □

Proof of Corollary 1.1. This follows from replacing a and b in Theorem 1.1 respectively by a^q and b^q for $q \neq 0$. □

4. An interpretation and an application

In this section, we collect a probabilistic interpretation and applications in engineering of the Toader–Qi mean $TQ(a,b)$. For more detailed information, please refer to the closely related posts at the ResearchGate and the paper [7, 8].

4.1. **An probabilistic interpretation.** It is not difficult to see that

$$TQ\bigl(1,e^{-4z}\bigr)=\frac{2}{\pi}\int_0^{\pi/2}e^{-4z\sin^2x}\operatorname{d}x=\frac{2}{\pi}\int_0^{\pi/2}e^{-2z}e^{2z\cos(2x)}\operatorname{d}x$$
$$=e^{-2z}\sum_{n=0}^\infty\frac{(2z)^n}{n!}\frac{2}{\pi}\int_0^{\pi/2}\frac{\bigl(e^{2ix}+e^{-2ix}\bigr)^n}{2^n}\operatorname{d}x=e^{-2z}\sum_{k=0}^\infty\binom{2k}{k}\frac{(2z)^{2k}}{(2k)!}=\sum_{k=0}^\infty\biggl(e^{-z}\frac{z^k}{k!}\biggr)^2.$$

For the probabilistic interpretation, let N_1 and N_2 form a pair of independent identically distributed random variables with Poisson distribution parameter z. Then

$$P\{N_1=N_2\}=\sum_{k=0}^\infty P\{N_1=N_2=k\}$$
$$=\sum_{k=0}^\infty P\{N_1=k\}P\{N_2=k\}=\sum_{k=0}^\infty\biggl(e^{-z}\frac{z^k}{k!}\biggr)^2=TQ\bigl(1,e^{-4z}\bigr).$$

The formulas (1.1) and (1.2) imply directly the following probabilistic interpretation. If N_1 and N_2 are two independent Poisson distributed random variables both with the same parameter $\frac{z}{2}$ for $z>0$, then $I_0(z)$ is the probability that $N_1=N_2$. A simple proof is

$$P\{N_1=N_2\}=\sum_{k=0}^\infty P\{N_1=N_2=k\}=\sum_{k=0}^\infty P\{N_1=k\}P\{N_2=k\}=I_0(z).$$

4.2. **Applications in engineering.** In [7, Section 2] and [8, Section II], the mean and the mean square over one period T of a continuous (and periodic) signal $x(t)$ were defined by

$$\kappa(x)=\frac{1}{T}\int_T x(t)\operatorname{d}t\quad\text{and}\quad\kappa\bigl(x^2\bigr)=\frac{1}{T}\int_T x^2(t)\operatorname{d}t.$$

It is easy to see that

$$TQ(a,b)=\frac{1}{2\pi}\int_0^{2\pi}\bigl(a^{\cos\theta}b^{\sin\theta}\bigr)^2\operatorname{d}\theta=\kappa\Bigl(\bigl(a^{\cos\theta}b^{\sin\theta}\bigr)^2\Bigr),$$

where $T=2\pi$ and $x(t)=a^{\cos t}b^{\sin t}$. In [7], the root of the mean square values of sampled periodic signals was calculated by using non-integer number of samples per period. This implies that the Toader–Qi mean $TQ(a,b)$ can be applied to engineering.

5. Remarks

Finally we give several remarks on some related thing, including several inequalities for bounding the Toader–Qi mean $TQ(a,b)$.

Remark 5.1. Lemmas 2.1 and 2.2 are equivalent to each other. For the outline to prove this equivalence, please refer to [19, Remark 4.1] and [31, Theorem 1.1].

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Remark 5.2. By the arithmetic-geometric-harmonic mean inequality, we have

$$a\cos^2\theta+b\sin^2\theta\ge a^{\cos^2\theta}b^{\sin^2\theta}\ge\frac{1}{\frac{\cos^2\theta}{a}+\frac{\sin^2\theta}{b}}.$$

Integrating on all sides on $\bigl(0,\frac{\pi}{2}\bigr)$ gives

$$A(a,b)=\frac{a+b}{2}>TQ(a,b)>\frac{a+b}{2ab}=H(a,b)$$

and

$$\sqrt[q]{A(a^q,b^q)}>M_q(a,b)>\sqrt[q]{H(a^q,b^q)},$$

where $A(a,b)$ and $H(a,b)$ denote respectively the arithmetic and harmonic means. Comparing the geometric mean $G(a,b)$ and the Toader–Qi mean $TQ(a,b)$, which mean is bigger? The answer is

$$TQ(a,b)<\frac{A(a,b)+G(a,b)}{2}<\frac{2A(a,b)+G(a,b)}{3}$$

did in [23, Remark 4.1] and [24, Remark 1]. Consequently, we have

$$M_q(a,b)<\left[\frac{A(a^q,b^q)+G(a^q,b^q)}{2}\right]^{1/q}<\left[\frac{2A(a^q,b^q)+G(a^q,b^q)}{3}\right]^{1/q},\quad q\ne0.$$

Remark 5.3. In [10, Theorem 1.1], it was derived that

$$[\lambda a+(1-\lambda)b]-a^\lambda b^{1-\lambda}<\frac{\sin(\lambda\pi)}{\pi}\left((2\lambda-1)(b-a)+[(1-\lambda)b-\lambda a]\ln\frac{b}{a}\right)\tag{5.1}$$

for $b>a>0$ and $\lambda\in(0,1)$. Replacing λ by $\cos^2\theta$ and integrating over $\bigl(0,\frac{\pi}{2}\bigr)$ on both sides of (5.1) arrives at

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi/2}&\bigl[a\cos^2\theta+b\sin^2\theta\bigr]\,\mathrm{d}\,\theta-\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}\,\mathrm{d}\,\theta\\&<\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin\bigl(\pi\cos^2\theta\bigr)}{\pi}\left((2\cos^2\theta-1)(b-a)+\bigl[b\sin^2\theta-a\cos^2\theta\bigr]\ln\frac{b}{a}\right)\mathrm{d}\,\theta\end{aligned}$$

which can be simplified as

$$\begin{aligned}TQ(a,b)-A(a,b)&>-\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin\bigl(\pi\cos^2\theta\bigr)}{\pi}\left((2\cos^2\theta-1)(b-a)+\bigl[b\sin^2\theta-a\cos^2\theta\bigr]\ln\frac{b}{a}\right)\mathrm{d}\,\theta\\&=-\frac{2(b-a)}{\pi^2}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)\cos(2\theta)\,\mathrm{d}\,\theta-\frac{2}{\pi^2}\ln\frac{b}{a}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)\bigl[b\sin^2\theta-a\cos^2\theta\bigr]\,\mathrm{d}\,\theta\\&=-\frac{2}{\pi^2}\ln\frac{b}{a}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)\bigl[b\sin^2\theta-a\cos^2\theta\bigr]\,\mathrm{d}\,\theta=-J_0\Bigl(\frac{\pi}{2}\Bigr)\frac{b-a}{2\pi}\ln\frac{b}{a},\end{aligned}$$

where

$$\begin{aligned}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)\cos(2\theta)\,\mathrm{d}\,\theta&=\int_0^{\pi/2}\sin\left[\pi\frac{1+\cos(2\theta)}{2}\right]\cos(2\theta)\,\mathrm{d}\,\theta\\&=\int_0^{\pi/2}\cos\left[\frac{\pi}{2}\cos(2\theta)\right]\cos(2\theta)\,\mathrm{d}\,\theta=\frac{1}{2}\int_0^{\pi}\cos\Bigl(\frac{\pi}{2}\cos\theta\Bigr)\cos\theta\,\mathrm{d}\,\theta\\&=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos\left[\frac{\pi}{2}\cos\Bigl(\theta+\frac{\pi}{2}\Bigr)\right]\cos\Bigl(\theta+\frac{\pi}{2}\Bigr)\,\mathrm{d}\,\theta=-\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos\Bigl(\frac{\pi}{2}\sin\theta\Bigr)\sin\theta\,\mathrm{d}\,\theta=0,\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)\sin^2\theta\,\mathrm{d}\,\theta&=\int_0^{\pi/2}\sin\biggl[\pi\frac{1+\cos(2\theta)}{2}\biggr]\frac{1-\cos(2\theta)}{2}\,\mathrm{d}\,\theta\\
=\int_0^{\pi/2}\cos\biggl[\frac{\pi}{2}\cos(2\theta)\biggr]\frac{1-\cos(2\theta)}{2}\,\mathrm{d}\,\theta&=\frac12\int_0^{\pi/2}\cos\biggl[\frac{\pi}{2}\cos(2\theta)\biggr]\,\mathrm{d}\,\theta\\
&=\frac14\int_0^{\pi}\cos\Bigl(\frac{\pi}{2}\cos\theta\Bigr)\,\mathrm{d}\,\theta=\frac{\pi}{4}J_0\Bigl(\frac{\pi}{2}\Bigr),\end{aligned}$$

and

$$\int_0^{\pi/2}\sin\bigl(\pi\cos^2\theta\bigr)\cos^2\theta\,\mathrm{d}\,\theta=\int_0^{\pi/2}\sin\biggl[\pi\frac{1+\cos(2\theta)}{2}\biggr]\frac{1+\cos(2\theta)}{2}\,\mathrm{d}\,\theta$$
$$=\int_0^{\pi/2}\cos\biggl[\frac{\pi}{2}\cos(2\theta)\biggr]\frac{1+\cos(2\theta)}{2}\,\mathrm{d}\,\theta=\frac12\int_0^{\pi/2}\cos\biggl[\frac{\pi}{2}\cos(2\theta)\biggr]\,\mathrm{d}\,\theta=\frac{\pi}{4}J_0\Bigl(\frac{\pi}{2}\Bigr)$$

by virtue of the formula

$$\int_0^{\pi}\cos(z\cos x)\cos(nx)\,\mathrm{d}\,x=\pi\cos\frac{n\pi}{2}J_n(z)$$

in [3, p. 425, item 18]. In conclusion, we obtain

$$TQ(a,b)>A(a,b)-J_0\Bigl(\frac{\pi}{2}\Bigr)\frac{b-a}{2\pi}\ln\frac{b}{a},\quad b>a>0.$$

Consequently, it follows that

$$M_q(a,b)>\biggl[A\bigl(a^q,b^q\bigr)-qJ_0\Bigl(\frac{\pi}{2}\Bigr)\frac{b^q-a^q}{2\pi}\ln\frac{b}{a}\biggr]^{1/q},\quad b>a>0,\quad q\ne0.$$

Remark 5.4. The papers [1, 4, 10, 11, 14, 16, 17, 18, 19, 23, 24, 27, 28, 29, 30, 31, 32, 33, 34, 35] belong to the same series for establishing integral representations of several mathematical means.

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