Article

Generalization of the Geometry of Cathelineau Infinitesimal and Grassmannian Chain Complexes

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Abstract: In this article, a generalization of the geometry of Grassmannian chain complex of free abelian groups generated by the projective configuration of points and Cathelineau's infinitesimal complex of polylogarithmic groups is proposed. Firstly, homomorphisms for weight $n = 2$ up to weight $n = 6$ will be introduced to connect sub-complexes of Grassmannian and Cathelineau. Lately, generalization of these morphisms will be shown for weight $n = N$. The associated diagrams will also be proven to be commutative and bi-complex.

Keywords: homomorphism; Grassmannian; generalized geometry; cathelineau's complex

MSC: 19L20, 22E10, 11G55

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1. Introduction

Grassmannian chain complex of free abelian groups generated by the projective configurations of points was first introduced by Suslin [1]. Suslin used two type of differential homomorphisms d and p to connect these free abelian groups. In Grassmannian chain complex each square is commutative and the composition of two same differential morphisms is zero [1]. Classical polylogarithmic functions had studied for many hundred years, first defined by Leibniz. Dilogarithm appear in the work of Spence, Abel, Kummer, Lobachesky, Hill, Roger, and Ramanujan etc but most important was the functional equation known as Abel's five term relations. Trilogarithms and its group $\mathcal{B}_3(F)$ was first introduced by Goncharov using generalized triple cross ratio of six points. Goncharov also generalized polylogarithmic group as $\mathcal{B}_n(F)$ and generalized Bloch-Suslin complex known as Goncharov's complex. Homomorphisms between Grassmannian and Bloch-Suslin complexes for Di-logarithm weight $n = 2$ was defined by Goncharov [2–4]. Goncharov proved that the associated digram is bi-complex and commutative. Goncharov [2] also uses the duality of configurations in order to prove (projected seven-term) functional equation for the trilogarithmic group $\mathcal{B}_3(F)$ and verifies that a Complex forms among Grassmannian and Goncharov's Complexes in weight 3 is commutative. Cathelineau [5–7] defined analogy of Goncharov's complexes in the additive (both infinitesimal and tangential) setting called Cathelineau's complexes.

Cathelineau defined F-vector space as $\beta_2(F)$, generated by four term relation and $\beta_3(F)$, generated by 22 term relations for his generalized chain complex. Siddiqui [8] found projected triple cross ratio and indicated that it should be written as the ratio of two projected cross-ratios. Siddiqui [8,9] also introduced variant of Cathelineau's complexes in both infinitesimal and tangential setting and describe their relations through homomorphisms with Grassmannian chain complexes of the projective configurations for weight $n = 2$. Author also found morphisms between Grassmannian complex and Variant of Cathelineau's infinitesimal complex for weight $n = 3$ and show that the associated diagram is commutative and bi-complex.

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Khalid et al. [10,11] defined generalized morphisms to connect Grassmannian complex with Variant of Cathelineau complex up to weight n=N. Further, the author of [12,13] also generalized higher order differential homomorphisms in Grassmannian complex as n^{th} order differential morphisms.
section 2 presents the basic ideas and background of Grassmanian chain complexes, Polylogarithmic groups, Bloch-Suslin complex, Goncharov's complex and Cathelineau's complex for weight n. In Section 3 geometry through morphisms is defined to connect Grassmannian and Cathelineau's infinitesimal complexes from weight $n = 2$ up to weight $n = 6$ also it is proven that the associated diagrams are bi-complex and commutative. Section 4 produces generalized geometry of Grassmannian and Cathelineau's infinitesimal complexes using generalized morphisms and the main result that the generalized diagram is commutative. Last section is conclusion of the whole work.

2. Preliminary and Background

Detailed background relevant to this research will be discussed in this section. It comprises the Grassmannian complex, Goncharov complex, Cathelineau complex, which is very crucial for this research study.

2.1. Grassmannian Complex

Consider a free abelian group $G_m(n)$ generated by m-vectors of dimension n. Following is the Grassmannian bicomplex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots \xrightarrow{d} & & G_{n+5}(n+2) & \xrightarrow{d} & G_{n+4}(n+2) & \xrightarrow{d} & G_{n+3}(n+2) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots \xrightarrow{d} & & G_{n+4}(n+1) & \xrightarrow{d} & G_{n+3}(n+1) & \xrightarrow{d} & G_{n+2}(n+1) \\
 & & \downarrow p & & \downarrow p & & \downarrow p \\
\cdots \xrightarrow{d} & & G_{n+3}(n) & \xrightarrow{d} & G_{n+2}(n) & \xrightarrow{d} & G_{n+1}(n)
\end{array} \tag{A}$$

d is called differential map given by

$$d : (q_0, \ldots, q_n) \mapsto \sum_{i=0}^{n} (-1)^i (q_0, ..., \hat{q}_i, ..., q_n) \tag{1}$$

and p another differential morphism called projection morphism given by

$$p : (q_0, \ldots, q_n) \mapsto \sum_{i=0}^{n} (-1)^i (q_i | q_0, \ldots, \hat{q}_i, \ldots, q_n) \tag{2}$$

Lemma 1. *The diagram (A) is bi-complex, i.e.* $d \circ d = p \circ p = 0$

Proof. For proof (see [1]) □

Lemma 2. *The diagram (A) is commutative, i.e.* $d \circ p = p \circ d$

Proof. For proof (see [1]) □

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2.2. Polylogarithmic Groups and its Complexes

Let $Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ is $Z-$module called free abelian group generated by $[x] \in \mathbf{P}^1_F$ [2,14], from now F will be used as a field and $F^{\bullet\bullet} = F - \{0,1\}$.

Definition 1. *The group $\mathcal{B}(F)$ is called Scissor congruence group, it is factor group of $Z[F^{\bullet\bullet}]$ and its subgroup generated by Abel's famous five term relation,* $[x] - [y] + [\frac{y}{x}] - [\frac{1-y}{1-x}] + [\frac{1-y^{-1}}{1-x^{-1}}]$ *where* $x \neq y, x, y \neq 0, 1$ ([2])

2.2.1. Weight 1

Let the group $R_1(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ generated by 3 terms relation $[xy] - [x] - [y]$ where $x, y \in F^\times$. Define $\mathcal{B}_1(F)$, it is factor group of $Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ and $R_1(F)$ [2]. The function $\delta : \mathcal{B}_1(F) \to F^\times, [x] \to x$ is an isomorphism, such that $\mathcal{B}_1(F) = F^\times$

2.2.2. Weight 2

The subgroup $R_2(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ [2] generated by the cross ratio of five relations is defined as

$$R_2(F) = \sum_{i=0}^{4} (-1)^i r(q_0, ..., \hat{q}_i, ..., q_4) \tag{3}$$

where

$$r(q_0, q_1, q_2, q_3) = \frac{\triangle(q_0, q_3)\triangle(q_1, q_2)}{\triangle(q_0, q_2)\triangle(q_1, q_3)} \tag{4}$$

It is called cross ratio of four points. Define a map $\delta_2 : Z[\mathbf{P}^1_F/\{0,1,\infty\}] \to \wedge^2 F^\times$, defined as $[x] \to (1-x) \wedge x$, it has been proven that $\delta_2(R_2(F)) = 0$ [2]. Define group $\mathcal{B}_2(F)$ the factor group of $Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_2(F)$. Now introduce Bloch-Suslin complex

$$0 \xrightarrow{\delta} \mathcal{B}_2(F) \xrightarrow{\delta} \wedge^2 F^\times \xrightarrow{\delta} 0$$

where δ is an induced map defined as $\delta : [x]_2 \to (1-x) \wedge x$, this complex is also short exact sequence.

2.2.3. **Weight 3**

As defined in [2]

$$r_3(q_0, ..., q_6) = \frac{\triangle(q_0, q_1, q_3)\triangle(q_1, q_2, q_4)\triangle(q_2, q_0, q_5)}{\triangle(q_0, q_1, q_4)\triangle(q_1, q_2, q_5)\triangle(q_2, q_0, q_3)}$$

it is a triple cross ratio 6 points. Take $R_3(F) \subset Z[\mathbf{P}^1_F/\{0,1,\infty\}]$ [2], defined as

$$R_3(F) = \sum_{i=0}^{6} (-1)^i Alt_6 r_3(q_0, ..., \hat{q}_i, ..., q_6) \tag{5}$$

which is a seven term relation of triple ratio. Goncharov defines $\mathcal{B}_3(F)$, which is quotient subgroup $Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_3(F)$, the Goncharov's complex in weight $n = 3$ is given by

$$\mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^\times \xrightarrow{\delta} \wedge^3 F^\times$$

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2.2.4. Weight n

Goncharov [2] generalized the group $\mathcal{B}_n(F) = Z[\mathbf{P}^1_F/\{0,1,\infty\}]/R_n(F)$, where $R_n(F)$ is a kernel of the map $\delta_n : Z[\mathbf{P}^1_F] \to \mathcal{B}_{n-1}(F) \otimes F^\times$, so generalized Goncharov's complex is given as

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta} \mathcal{B}_{n-2}(F) \otimes \wedge^2(F) \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta} \frac{\wedge^n F^\times}{2 - torsion} \tag{B}$$

2.3. Cathelineau's Complexes

Cathlelineau [6] has defined the F- Vector space which is an infinitesimal form of Bloch groups $\mathcal{B}_n(F)$ as follows

1. $\beta_1(F) = F$
2. $\beta_2(F) = \frac{F[F^{\bullet\bullet}]}{r_2(F)}$. where $r_2(F)$ is the kernel of $\partial_2 : F[F^{\bullet\bullet}] \to F \otimes F^\times$ defined by $[x] \to x \otimes x + (1-x) \otimes (1-x)$. Cathelineau showed that $r_2(F)$ is a sub-vector space generated by four elements $[x] - [y] + x[\frac{y}{x}] + (1-x)[\frac{1-y}{1-x}]$ therefore obtain a complex

$$\beta_2(F) \xrightarrow{\partial} F \otimes_F F^\times$$

where ∂ is an induced map defined as

$$\partial : \langle x \rangle_2 \mapsto x \otimes (x) + (1-x) \otimes (1-x) \tag{6}$$

using tensor properties this map can be written as

$$\partial : \langle x \rangle_2 \mapsto x \otimes (x-1) - (1-x) \otimes (x) \tag{7}$$

The functional equation in $\beta_2(F)$

1. A two term relation $\langle a \rangle_2 = \langle 1-a \rangle_2$
2. An inversion relation $\langle a \rangle_2 = -a\langle \frac{1}{a} \rangle_2$
3. A four term relation $\langle a \rangle_2 - \langle b \rangle_2 + a\langle \frac{b}{a} \rangle_2 + (1-a)\langle \frac{1-b}{1-a} \rangle_2 = 0$
4. A distribution relation $\langle a \rangle_2^m = \sum_{\zeta^m=1} \frac{1-a^m}{1-a\zeta} \langle \zeta a \rangle_2$

If $r_n(F)$ is a kernel of the map defined as $\delta_n : F[F] \to \beta_{n-1} \otimes F^\times \oplus F \otimes \mathcal{B}_{n-1}(F)$ [6]. Now by taking $\beta_n(F)$ the factor group as

$$\beta_n(F) = \frac{F[F^{\bullet\bullet}]}{r_n(F)} \tag{8}$$

The Cathelineau chain complex [6] for groups $\beta_n(F)$ and $\mathcal{B}_n(F)$ is given as

$$\beta_n(F) \xrightarrow{\partial_n} \begin{array}{c} \beta_{n-1}(F) \otimes F^\times \\ \oplus \\ F \otimes \mathcal{B}_{n-1}(F) \end{array} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \begin{array}{c} \beta_2(F) \otimes \wedge^{n-2} F^\times \\ \oplus \\ F \otimes \mathcal{B}_2(F) \otimes \wedge^{n-3} F^\times \end{array} \xrightarrow{\partial_0} F \otimes \wedge^{n-1} F^\times \tag{C}$$

where ∂_n is given by

$$\partial_n : [r] \longmapsto \langle r \rangle_{n-1} \otimes r + (-1)^{n-1}(1-r) \otimes [r]_{n-1} \tag{9}$$

Lemma 3. $\partial_{n-1} \circ \partial_n = 0$ ([6])

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3. Geometry of Cathelineau and Grassmannian Complexes

3.1. Weight 2

Construct the diagram of Grassmannian and Cathelineau infinitesimal complexes for weight $n = 2$.

$$\begin{array}{ccccc} G_6(3) & \xrightarrow{d} & G_5(3) & \xrightarrow{d} & G_4(3) \\ \downarrow p & & \downarrow p & & \downarrow p \\ G_5(2) & \xrightarrow{d} & G_4(2) & \xrightarrow{d} & G_3(2) \\ & & \downarrow f_1^2 & & \downarrow f_0^2 \\ & & \beta_2(F) & \xrightarrow{\partial} & F \otimes F^{\times} \end{array} \tag{D}$$

where

$$f_0^2 : (q_0, q_1, q_2) \to \sum_{i=0}^{2} (-1)^i \triangle(q_0, ..., \hat{q}_i, ..., q_2) \otimes \frac{\triangle(q_0, ..., \hat{q}_{i+1}, ..., q_2)}{\triangle(q_0, ..., \hat{q}_{i+2}, ..., q_2)} \quad (mod\ 3) \tag{10}$$

and

$$f_1^2(q_0, q_1, q_2, q_3) = \langle r(q_0, ..., q_3) \rangle_2 \tag{11}$$

Lemma 4. *f_1^2 is independent of volume formation by vectors in V_2.*

Proof. Let $f_1^2(q_0, q_1, q_2, q_3)$ can be written as

$$f_1^2(q_0, q_1, q_2, q_3) = \left\langle \frac{\triangle(q_0, q_3)\triangle(q_1, q_2)}{\triangle(q_0, q_2)\triangle(q_1, q_3)} \right\rangle_2 \tag{12}$$

so by changing volume $V = \alpha V$ where $\alpha \in$ field F then due to frictions the right side will remain unchanged, therefore f_1^2 is independent of volume form by vectors in V_2. □

Lemma 5. *$f_1^2 \circ p$ is independent of length of vectors in V_2.*

Proof. Let $f_1^2 \circ p(q_0, q_1, q_2, q_3, q_4)$ can be written as

$$f_1^2 \circ p(q_0, q_1, q_2, q_3, q_4) = \sum_{i=0}^{4} \left\langle r(q_i | q_0, ..., \hat{q}_i, ..., q_4) \right\rangle_2 \tag{13}$$

so changing the length of vector like $(q_0, q_1, q_2, q_3, q_4) = \alpha(q_0, q_1, q_2, q_3, q_4)$ where $\alpha \in$ field F then due to ratios the difference will be zero. Therefore f_1^2 is independent of length of vectors in V_2. □

Lemma 6. *f_0^2 is independent of volume form by vectors in V_2.*

Proof. Let $f_0^2(q_0, q_1, q_2)$ can be written as

$$f_0^2(q_0, q_1, q_2) = \frac{\triangle(q_1, q_2)}{\triangle(q_0, q_2)} \otimes \frac{\triangle(q_0, q_2)}{\triangle(q_0, q_1)} - \frac{\triangle(q_0, q_1)}{\triangle(q_0, q_2)} \otimes \frac{\triangle(q_0, q_2)}{\triangle(q_1, q_2)} \tag{14}$$

so if volume $V = \alpha V$ where $\alpha \in$ field F then the right side will remain unchanged so f_0^2 is independent of volume form by vectors in V_2. □

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Lemma 7. *$f_0^2 \circ p$ is independent of length of vectors in V_2.*

Proof.

$$\begin{aligned} f_0^2 \circ p(q_0,q_1,q_2,q_3) =& \frac{\triangle(q_0,q_2,q_3)}{\triangle(q_0,q_1,q_3)} \otimes \frac{\triangle(q_0,q_1,q_3)}{\triangle(q_0,q_1,q_2)} - \frac{\triangle(q_0,q_1,q_2)}{\triangle(q_0,q_1,q_3)} \otimes \frac{\triangle(q_0,q_1,q_3)}{\triangle(q_0,q_2,q_3)} - \\ & \frac{\triangle(q_1,q_2,q_3)}{\triangle(q_1,q_0,q_3)} \otimes \frac{\triangle(q_1,q_0,q_3)}{\triangle(q_1,q_0,q_2)} + \frac{\triangle(q_1,q_0,q_2)}{\triangle(q_1,q_0,q_3)} \otimes \frac{\triangle(q_1,q_0,q_3)}{\triangle(q_1,q_2,q_3)} + \\ & \frac{\triangle(q_2,q_1,q_3)}{\triangle(q_2,q_0,q_3)} \otimes \frac{\triangle(q_2,q_0,q_3)}{\triangle(q_2,q_0,q_1)} - \frac{\triangle(q_2,q_0,q_1)}{\triangle(q_2,q_0,q_3)} \otimes \frac{\triangle(q_2,q_0,q_3)}{\triangle(q_2,q_1,q_3)} - \\ & \frac{\triangle(q_3,q_1,q_2)}{\triangle(q_3,q_0,q_2)} \otimes \frac{\triangle(q_3,q_0,q_2)}{\triangle(q_3,q_0,q_1)} + \frac{\triangle(q_3,q_0,q_1)}{\triangle(q_3,q_0,q_2)} \otimes \frac{\triangle(q_3,q_0,q_2)}{\triangle(q_3,q_1,q_2)} \end{aligned} \tag{15}$$

so by changing the length of vector like $(q_0,q_1,q_2,q_3) = \alpha(q_0,q_1,q_2,q_3)$ where $\alpha \in$ field F then the difference will be zero. Therefore f_0^2 is independent of length of vectors in V_2. □

Lemma 8. $f_0^2 \circ p = 0$.

Proof. From the above diagram take

$$G_4(3) \xrightarrow{p} G_3(2) \xrightarrow{f_0^2} F \otimes F^\times$$

Assume that the four points $(q_0,q_1,q_2,q_3) \in G_4(3)$, apply map p then $p(q_0,q_1,q_2,q_3) = (q_0/q_1,q_2,q_3) - (q_1/q_0,q_2,q_3) + (q_2/q_0,q_1,q_3) - (q_3/q_0,q_1,q_2)$ now apply f_0^2, then

$$\begin{aligned} f_0^2 \circ p(q_0,q_1,q_2,q_3) =& \triangle(q_0,q_2,q_3) \otimes \frac{\triangle(q_0,q_1,q_3)}{\triangle(q_0,q_1,q_2)} - \triangle(q_0,q_1,q_3) \otimes \frac{\triangle(q_0,q_2,q_3)}{\triangle(q_0,q_2,q_1)} + \\ & \triangle(q_0,q_1,q_2) \otimes \frac{\triangle(q_0,q_3,q_2)}{\triangle(q_0,q_3,q_1)} - \triangle(q_1,q_2,q_3) \otimes \frac{\triangle(q_1,q_0,q_3)}{\triangle(q_1,q_0,q_2)} + \\ & \triangle(q_1,q_0,q_3) \otimes \frac{\triangle(q_1,q_2,q_3)}{\triangle(q_1,q_2,q_0)} - \triangle(q_1,q_0,q_2) \otimes \frac{\triangle(q_1,q_3,q_2)}{\triangle(q_1,q_3,q_0)} + \\ & \triangle(q_2,q_1,q_3) \otimes \frac{\triangle(q_2,q_0,q_3)}{\triangle(q_2,q_0,q_1)} - \triangle(q_2,q_0,q_3) \otimes \frac{\triangle(q_2,q_1,q_3)}{\triangle(q_2,q_1,q_0)} + \\ & \triangle(q_2,q_0,q_1) \otimes \frac{\triangle(q_2,q_3,q_1)}{\triangle(q_2,q_3,q_0)} - \triangle(q_3,q_1,q_2) \otimes \frac{\triangle(q_3,q_0,q_2)}{\triangle(q_3,q_0,q_1)} + \\ & \triangle(q_3,q_0,q_2) \otimes \frac{\triangle(q_3,q_1,q_2)}{\triangle(q_3,q_1,q_0)} - \triangle(q_3,q_0,q_1) \otimes \frac{\triangle(q_3,q_2,q_1)}{\triangle(q_3,q_2,q_0)} \\ =& 0 \end{aligned}$$

□

Lemma 9. *The lower square of the diagram D is commutative.*

Proof. Let $(q_0,q_1,q_2,q_3) \in G_4(2)$. Apply morphism d, then

$$d(q_0,q_1,q_2,q_3) = (q_1,q_2,q_3) - (q_0,q_2,q_3) + (q_0,q_1,q_3) - (q_0,q_1,q_2)$$

now apply f_0^2, and get 24 terms , write them in the form of 12 terms as given below.

$$f_0^2 \circ d(q_0,...,q_3) = \triangle(q_2,q_3) \otimes \frac{\triangle(q_1,q_3)}{\triangle(q_1,q_2)} - \triangle(q_1,q_3) \otimes \frac{\triangle(q_2,q_3)}{\triangle(q_2,q_1)} + \triangle(q_1,q_2) \otimes \frac{\triangle(q_3,q_2)}{\triangle(q_3,q_1)} -$$

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$$\begin{aligned}
&\triangle(q_2,q_3)\otimes\frac{\triangle(q_0,q_3)}{\triangle(q_0,q_2)}+\triangle(q_0,q_3)\otimes\frac{\triangle(q_2,q_3)}{\triangle(q_2,q_0)}-\triangle(q_0,q_2)\otimes\frac{\triangle(q_3,q_2)}{\triangle(q_3,q_0)}+\\
&\triangle(q_1,q_3)\otimes\frac{\triangle(q_0,q_3)}{\triangle(q_0,q_1)}-\triangle(q_0,q_3)\otimes\frac{\triangle(q_1,q_3)}{\triangle(q_1,q_0)}+\triangle(q_0,q_1)\otimes\frac{\triangle(q_3,q_1)}{\triangle(q_3,q_0)}-\\
&\triangle(q_1,q_2)\otimes\frac{\triangle(q_0,q_2)}{\triangle(q_0,q_1)}+\triangle(q_0,q_2)\otimes\frac{\triangle(q_1,q_2)}{\triangle(q_1,q_0)}-\triangle(q_0,q_1)\otimes\frac{\triangle(q_2,q_1)}{\triangle(q_2,q_0)}
\end{aligned}\tag{16}$$

Take $(q_0,q_1,q_2,q_3)\in G_4(2)$ again, apply map f_1^2, then

$$f_1^2(q_0,q_1,q_2,q_3)=\left\langle\frac{\triangle(q_0,q_3)\triangle(q_1,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}\right\rangle_2\tag{17}$$

now apply ∂

$$\begin{aligned}
\partial\circ f_1^2(q_0,q_1,q_2,q_3)=&\frac{\triangle(q_0,q_3)\triangle(q_1,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}\otimes\left(\frac{\triangle(q_0,q_3)\triangle(q_1,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}-1\right)-\\
&\left(1-\frac{\triangle(q_0,q_3)\triangle(q_1,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}\right)\otimes\frac{\triangle(q_0,q_3)\triangle(q_1,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}
\end{aligned}\tag{18}$$

using Siegel cross ratio properties [15] then

$$\begin{aligned}
\partial\circ f_1^2(q_0,q_1,q_2,q_3)=&\frac{\triangle(q_0,q_3)\triangle(q_1,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}\otimes\frac{\triangle(q_0,q_1)\triangle(q_2,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}-\\
&\frac{\triangle(q_0,q_1)\triangle(q_2,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}\otimes\frac{\triangle(q_0,q_3)\triangle(q_1,q_3)}{\triangle(q_0,q_2)\triangle(q_1,q_3)}
\end{aligned}\tag{19}$$

after simplifications

$$\begin{aligned}
\partial\circ f_1^2(q_0,...,q_3)=&\triangle(q_2,q_3)\otimes\frac{\triangle(q_1,q_3)}{\triangle(q_1,q_2)}-\triangle(q_1,q_3)\otimes\frac{\triangle(q_2,q_3)}{\triangle(q_2,q_1)}+\triangle(q_1,q_2)\otimes\frac{\triangle(q_3,q_2)}{\triangle(q_3,q_1)}-\\
&\triangle(q_2,q_3)\otimes\frac{\triangle(q_0,q_3)}{\triangle(q_0,q_2)}+\triangle(q_0,q_3)\otimes\frac{\triangle(q_2,q_3)}{\triangle(q_2,q_0)}-\triangle(q_0,q_2)\otimes\frac{\triangle(q_3,q_2)}{\triangle(q_3,q_0)}+\\
&\triangle(q_1,q_3)\otimes\frac{\triangle(q_0,q_3)}{\triangle(q_0,q_1)}-\triangle(q_0,q_3)\otimes\frac{\triangle(q_1,q_3)}{\triangle(q_1,q_0)}+\triangle(q_0,q_1)\otimes\frac{\triangle(q_3,q_1)}{\triangle(q_3,q_0)}-\\
&\triangle(q_1,q_2)\otimes\frac{\triangle(q_0,q_2)}{\triangle(q_0,q_1)}+\triangle(q_0,q_2)\otimes\frac{\triangle(q_1,q_2)}{\triangle(q_1,q_0)}-\triangle(q_0,q_1)\otimes\frac{\triangle(q_2,q_1)}{\triangle(q_2,q_0)}
\end{aligned}\tag{20}$$

From Eq.(16) and Eq.(20), it is proved that the diagram D is commutative. □

3.2. Weight 3 (Trilogarithm)

For this weight, connect the subcomplex of Cathelineau complex in weight 3 with the subcomplex of Grassmannian given as

$$\begin{array}{ccccc}
G_7(4) & \xrightarrow{d} & G_6(4) & \xrightarrow{d} & G_5(4)\\
\downarrow p & & \downarrow p & & \downarrow p\\
G_6(3) & \xrightarrow{d} & G_5(3) & \xrightarrow{d} & G_4(3)\\
 & & \downarrow f_1^3 & & \downarrow f_0^3\\
 & & \beta_2(F)\otimes F^\times\oplus F\otimes\mathcal{B}_2(F) & \xrightarrow{\partial} & F\otimes\wedge^2F^\times
\end{array}\tag{E}$$

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where

$$f_0^3(q_0,...,q_3) = \sum_{i=0}^{3}(-1)^i \triangle(q_0,...,\hat{q}_i,...,q_3) \otimes \frac{\triangle(q_0,...,\hat{q}_{i+1},...,q_3)}{\triangle(q_0,...,\hat{q}_{i+2},...,q_3)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+2},...,q_3)}{\triangle(q_0,...,\hat{q}_{i+3},...,q_3)} \pmod 4 \quad (21)$$

and

$$f_1^3(q_0,...,q_4) = -\frac{1}{3}\Big(\sum_{i=0}^{4}(-1)^i (\langle r(q_i|q_0,...,\hat{q}_i,...,q_4)\rangle_2 \otimes \prod_{\substack{j\neq i\\ j=0}}^{4} \triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_4)$$
$$- \prod_{\substack{j\neq i\\ j=0}}^{4} \triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_4) \otimes [q_i|q_0,...,\hat{q}_i,...,q_4]_2\Big) \pmod 5 \quad (22)$$

94 **Lemma 10.** $f_0^3 \circ p = 0$

Proof. Let $(q_0,q_1,q_2,q_3,q_4) \in G_5(4)$, apply map p

$$p(q_0,q_1,q_2,q_3,q_4) = \sum_{i=0}^{4}(-1)^i (q_i|q_0,...,\hat{q}_i,...,q_4) \quad (23)$$

On applying map f_0^3 on $p(q_0,q_1,q_2,q_3,q_4)$, then

$$\begin{aligned} f_0^3 \circ p(q_0,...,q_4) &= \frac{\triangle(q_0,q_2,q_3,q_4)}{\triangle(q_0,q_1,q_3,q_4)} \otimes \frac{\triangle(q_0,q_1,q_3,q_4)}{\triangle(q_0,q_1,q_2,q_4)} \wedge \frac{\triangle(q_0,q_1,q_2,q_4)}{\triangle(q_0,q_1,q_2,q_3)} \\ &- \frac{\triangle(q_0,q_1,q_2,q_4)}{\triangle(q_0,q_1,q_2,q_3)} \otimes \frac{\triangle(q_0,q_2,q_3,q_4)}{\triangle(q_0,q_1,q_3,q_4)} \wedge \frac{\triangle(q_0,q_1,q_3,q_4)}{\triangle(q_0,q_1,q_2,q_3)} \\ &. \\ &. \\ &. \\ &- \frac{\triangle(q_4,q_2,q_3,q_0)}{\triangle(q_4,q_1,q_3,q_0)} \otimes \frac{\triangle(q_4,q_1,q_3,q_0)}{\triangle(q_4,q_1,q_2,q_0)} \wedge \frac{\triangle(q_4,q_1,q_2,q_0)}{\triangle(q_4,q_1,q_2,q_3)} \\ &+ \frac{\triangle(q_4,q_1,q_2,q_0)}{\triangle(q_4,q_1,q_2,q_3)} \otimes \frac{\triangle(q_4,q_2,q_3,q_0)}{\triangle(q_4,q_1,q_3,q_0)} \wedge \frac{\triangle(q_4,q_1,q_3,q_0)}{\triangle(q_4,q_1,q_2,q_3)} \\ &= 0 \end{aligned} \quad (24)$$

95 □

96 **Lemma 11.** *The lower square of the diagram E is commutative.*

Proof. Let $(q_0,q_1,q_2,q_3,q_4) \in G_5(3)$ by applying map d it becomes

$$d(q_0,...,q_4) = \sum_{i=0}^{4}(-1)^i (q_0,...,\hat{q}_i,...,q_4) \quad (25)$$

apply map f_0^3, then

$$f_0^3 \circ d(q_0,...,q_4) = \sum_{\substack{j\neq i\\ j=i+1}}^{4} (-1)^j \sum_{i=0}^{4}(-1)^i \triangle(q_0,...,\hat{q}_i,\hat{q}_j...,q_4) \otimes \frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{j+1},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_4)} \wedge$$

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$$\frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_4)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{j+4},...,q_4)} \tag{26}$$

Applying morphism f_1^3 on $(q_0,...,q_4) \in G_5(3)$. then

$$f_1^3(q_0,...,q_4) = -\frac{1}{3}\Big(\sum_{i=0}^{4}(-1)^i(\langle r(q_i|q_0,...,\hat{q}_i,...,q_4)\rangle_2 \otimes \prod_{\substack{j\neq i\\ j=0}}^{4}\triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_4)$$
$$-\prod_{\substack{j\neq i\\ j=0}}^{4}\triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_4)\otimes[q_i|q_0,...,\hat{q}_i,...,q_4]_2\Big) \tag{27}$$

now apply map ∂

$$\partial\circ f_1^3 = -\frac{1}{3}\Big(\sum_{i=0}^{4}(-1)^i(r(q_i|q_0,...,\hat{q}_i,...,q_4)\otimes(1-r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,...,q_4))\wedge$$
$$\prod_{\substack{j\neq i\\ j=0}}^{4}\triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_4)-(1-r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,...,q_4))\otimes r(q_i|q_0,...,\hat{q}_i,\hat{q}_j,...,q_4)\wedge$$
$$\prod_{\substack{j\neq i\\ j=0}}^{4}\triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_4)-\prod_{\substack{j\neq i\\ j=0}}^{4}\triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_4)\otimes(1-r(q_i|q_0,...,\hat{q}_i,...,q_4))\wedge$$
$$r(q_i|q_0,...,\hat{q}_i,...,q_4)\Big) \tag{28}$$

after using tensor, wedge and Siegel cross ratio properties [15], it becomes

$$\partial\circ f_1^3(q_0,...,q_4) = \sum_{\substack{j\neq i\\ j=i+1}}^{4}(-1)^j\sum_{i=0}^{4}(-1)^i\triangle(q_0,...,\hat{q}_i,\hat{q}_j...,q_4)\otimes\frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{j+1},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_4)}\wedge$$
$$\frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_4)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{j+4},...,q_4)} \tag{29}$$

from Eq.(26) and Eq.(29) it is observed that, $f_0^3\circ d = \partial\circ f_1^3$ □

3.3. Weight $n = 4$

In this weight connect the sub-complexes of Cathelineau's infinitesimal and Grassmannian

$$\begin{array}{ccccc}
G_8(5) & \xrightarrow{d} & G_7(5) & \xrightarrow{d} & G_6(5) \\
\downarrow p & & \downarrow p & & \downarrow p \\
G_7(4) & \xrightarrow{d} & G_6(4) & \xrightarrow{d} & G_5(4) \\
 & & \downarrow f_1^4 & & \downarrow f_0^4 \\
 & & \beta_2(F)\otimes\wedge^2F^\times\oplus F\otimes\mathcal{B}_2(F)\wedge F^\times & \xrightarrow{\partial} & F\otimes\wedge^3F^\times
\end{array} \tag{F}$$

where

$$f_0^4(q_0,...,q_4) = \sum_{i=0}^{4}(-1)^i\triangle(q_0,...,\hat{q}_i,...,q_4)\otimes\frac{\triangle(q_0,...,\hat{q}_{i+1},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+2},...,q_4)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+2},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+3},...,q_4)}\wedge$$

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$$\frac{\triangle(q_0,...,\hat{q}_{i+3},...,q_4)}{\triangle(q_0,...,\hat{q}_{i+4},...,q_4)} \pmod 5 \tag{30}$$

and

$$f_1^4(q_0,...,q_5) = \frac{1}{6}\Big(\sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{5} (-1)^i(\langle r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\rangle_2 \otimes \prod_{\substack{k\neq i\\ k=i+1}}^{5} \triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\wedge \prod_{\substack{k\neq j\\ k=j+1}}^{5} \triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5) - \prod_{\substack{k\neq i\\ k=i+1}}^{5} \triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\otimes[r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)]_2\otimes \prod_{\substack{k\neq j\\ k=j+1}}^{5} \triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5) + \prod_{\substack{k\neq j\\ k=j+1}}^{5} \triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5)\otimes[r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)]_2\otimes \prod_{\substack{k\neq i\\ k=i+1}}^{5} \triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\Big) \pmod 6 \tag{31}$$

99 **Lemma 12.** $f_0^4 \circ d = \partial \circ f_1^4$.

Proof. Let the five points be $(q_0,q_1,q_2,q_3,q_4,q_5) \in G_6(4)$, now apply map d, then

$$d(q_0,...,q_5) = \sum_{i=0}^{5}(-1)^i(q_0,...,\hat{q}_i,...,q_5) \tag{32}$$

now apply morphism f_0^4

$$f_0^4 \circ d(q_0,...,q_5) = \sum_{\substack{k\neq i\\ k=i+1}}^{5} (-1)^k \sum_{i=0}^{5}(-1)^i \triangle(q_0,...,\hat{q}_i,\hat{q}_k...,q_5) \otimes \frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{k+1},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{k+2},...,q_5)}\wedge \frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{k+2},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{k+3},...,q_5)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{k+3},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{k+4},...,q_5)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{k+4},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+5},\hat{q}_{k+5},...,q_5)} \tag{33}$$

Apply map f_1^4 on $(q_0,...,q_5) \in G_6(4)$, then

$$f_1^4(q_0,...,q_5) = \frac{1}{6}\Big(\sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{5} (-1)^i(\langle r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\rangle_2 \otimes \prod_{\substack{k\neq i\\ k=i+1}}^{5} \triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\wedge \prod_{\substack{k\neq j\\ k=j+1}}^{5} \triangle(q_0,..., \hat{q}_j,\hat{q}_k,...,q_5) - \prod_{\substack{k\neq i\\ k=i+1}}^{5} \triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\otimes[r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)]_2\otimes \prod_{\substack{k\neq j\\ k=j+1}}^{5} \triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5) + \prod_{\substack{k\neq j\\ k=j+1}}^{5} \triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5)\otimes[r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)]_2\otimes \prod_{\substack{k\neq i\\ k=i+1}}^{5} \triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\Big) \tag{34}$$

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On applying map ∂

$$\begin{aligned}\partial\circ f_1^4(q_0,...,q_5) = \frac{1}{6}\Big(&\sum_{\substack{i\neq j\\ i=0\\ j=i+1}}^{5}(-1)^i\Big[\big(r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\big)\otimes\big(r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)-1\big)\wedge\\
&\prod_{\substack{k\neq i\\ k=i+1}}^{5}\triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\wedge\prod_{\substack{k\neq j\\ k=0}}^{5}\triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5)-\big(1-r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\big)\otimes\\
&\big(r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\big)\wedge\prod_{\substack{k\neq i\\ k=i+1}}^{5}\triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\wedge\prod_{\substack{k\neq j\\ k=j+1}}^{5}\triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5)-\\
&\prod_{\substack{k\neq i\\ k=i+1}}^{5}\triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\otimes\big(1-r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\big)\wedge r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\wedge\\
&\prod_{\substack{k\neq j\\ k=j+1}}^{5}\triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5)+\prod_{\substack{k\neq j\\ k=0}}^{5}\triangle(q_0,...,\hat{q}_j,\hat{q}_k,...,q_5)\otimes\big(1-r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\big)\wedge\\
&r(q_i,q_j|q_0,...,\hat{q}_i,\hat{q}_j,...,q_5)\wedge\prod_{\substack{k\neq i\\ k=i+1}}^{5}\triangle(q_0,...,\hat{q}_i,\hat{q}_k,...,q_5)\Big]\Big)\end{aligned}\tag{35}$$

after using tensor,wedge, Siegel cross ratio properties and dummy indices it becomes

$$\begin{aligned}\partial\circ f_1^4(q_0,...,q_5) = &\sum_{\substack{k\neq i\\ k=i+1}}^{5}(-1)^k\sum_{i=0}^{5}(-1)^i\triangle(q_0,...,\hat{q}_i,\hat{q}_k...,q_5)\otimes\frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{k+1},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{k+2},...,q_5)}\wedge\\
&\frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{k+2},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{k+3},...,q_5)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{k+3},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{k+4},...,q_5)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{k+4},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+5},\hat{q}_{k+5},...,q_5)}\end{aligned}\tag{36}$$

Eq.(33) and Eq.(36) proves $f_0^4\circ d=\partial\circ f_1^4$ □

3.4. Weight $n = 5$

Connect the two sub-complexes given as

$$\begin{array}{ccccc}
G_9(6) & \xrightarrow{d} & G_8(6) & \xrightarrow{d} & G_7(6)\\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p}\\
G_8(5) & \xrightarrow{d} & G_7(5) & \xrightarrow{d} & G_6(5)\\
 & & \downarrow{\scriptstyle f_1^5} & & \downarrow{\scriptstyle f_0^5}\\
 & & \beta_2(F)\otimes\wedge^3F^\times\oplus F\otimes\mathcal{B}_2(F)\wedge^2F^\times & \xrightarrow{\partial} & F\otimes\wedge^4F^\times
\end{array}\tag{G}$$

where

$$\begin{aligned}f_0^5(q_0,...,q_5) = &\sum_{i=0}^{5}(-1)^i\triangle(q_0,...,\hat{q}_i,...,q_5)\otimes\frac{\triangle(q_0,...,\hat{q}_{i+1},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+2},...,q_5)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+2},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+3},...,q_5)}\\
&\frac{\triangle(q_0,...,\hat{q}_{i+3},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+4},...,q_5)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+4},...,q_5)}{\triangle(q_0,...,\hat{q}_{i+5},...,q_5)}\quad(\text{mod } 6)\end{aligned}\tag{37}$$

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and

$$f_1^5(q_0,...,q_6) = \frac{1}{10}\Big(\sum_{\substack{i\neq j\neq k\\ i=0\\ j=i+1\\ k=i+2}}^{6}(-1)^i(\langle r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)\rangle_2 \otimes \prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge$$
$$\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6) - \prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\otimes$$
$$[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)+$$
$$\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\otimes[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)$$
$$\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6) - \prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)\otimes[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2$$
$$\otimes\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\Big) \pmod 7 \tag{38}$$

102 **Lemma 13.** $f_0^5\circ d = \partial\circ f_1^5$.

Proof. Let $(q_0,...,q_6)\in G_7(5)$ on apply map d

$$d(q_0,...,q_6) = \sum_{i=0}^{6}(-1)^i(q_0,...,\hat{q}_i,...,q_6) \tag{39}$$

By applying morphism f_0^5

$$f_0^5\circ d(q_0,...,q_6) = \sum_{\substack{l\neq i\\ l=i+1}}^{6}(-1)^l\sum_{i=0}^{6}(-1)^i\triangle(q_0,...,\hat{q}_i,\hat{q}_l...,q_6)\otimes\frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{l+1},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{jl+2},...,q_6)}\wedge$$
$$\frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{l+2},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{l+3},...,q_6)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{l+3},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{l+4},...,q_6)}\wedge$$
$$\frac{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{l+4},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+5},\hat{q}_{l+5},...,q_6)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+5},\hat{q}_{l+5},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+6},\hat{q}_{l+6},...,q_6)} \tag{40}$$

Apply f_1^5 on $(q_0,...,q_6)\in G_7(5)$

$$f_1^5(q_0,...,q_6) = \frac{1}{10}\Big(\sum_{\substack{i\neq j\neq k\\ i=0\\ j=i+1\\ k=i+2}}^{6}(-1)^i(\langle r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)\rangle_2 \otimes \prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge$$
$$\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6) - \prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\otimes$$

$$[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)$$
$$+\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\otimes[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge$$
$$\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)-\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)\otimes[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes$$
$$\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\Big) \tag{41}$$

Apply map ∂

$$\partial\circ f_1^5(q_0,...,q_6)=\frac{1}{10}\Big(\sum_{\substack{i\neq j\neq k\\ i=0\\ j=i+1\\ k=i+2}}^{6}(-1)^i(r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)\otimes(1-r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,$$
$$...,q_6))\wedge\prod_{\substack{l\neq i\\ l=0i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)$$
$$-(1-r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6))\otimes(r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6))\wedge$$
$$\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)$$
$$-\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\otimes[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\wedge$$
$$\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)+\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\otimes[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes$$
$$\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)-\prod_{\substack{l\neq k\\ l=k+1}}^{6}\triangle(q_0,...,\hat{q}_k,\hat{q}_l,...,q_6)\otimes$$
$$[r(q_i,q_j,q_k|q_0,...,\hat{q}_i,\hat{q}_j,\hat{q}_k,...,q_6)]_2\otimes\prod_{\substack{l\neq i\\ l=i+1}}^{6}\triangle(q_0,...,\hat{q}_i,\hat{q}_l,...,q_6)\wedge\prod_{\substack{l\neq j\\ l=j+1}}^{6}\triangle(q_0,...,\hat{q}_j,\hat{q}_l,...,q_6)\Big) \tag{42}$$

after using tensor,wedge, Siegel cross ratio properties and dummy indices it becomes

$$\partial\circ f_1^5(q_0,...,q_6)=\sum_{\substack{l\neq i\\ l=i+1}}^{6}(-1)^l\sum_{i=0}^{6}(-1)^i\triangle(q_0,...,\hat{q}_i,\hat{q}_l...,q_6)\otimes\frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{l+1},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{jl+2},...,q_6)}\wedge$$
$$\frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{l+2},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{l+3},...,q_6)}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{l+3},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{l+4},...,q_6)}\wedge$$

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$$\frac{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{l+4},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+5},\hat{q}_{l+5},...,q_6)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+5},\hat{q}_{l+5},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+6},\hat{q}_{l+6},...,q_6)} \tag{43}$$

Hence Eq.(40) and Eq.(43) proves $f_0^5 \circ d = \partial \circ f_1^5$ □

3.5. Weight n = 6

Connect the two simplicial complexes Grassmannian and Cathelineau as

$$\begin{array}{ccccc}
G_{10}(7) & \xrightarrow{d} & G_9(7) & \xrightarrow{d} & G_8(7) \\
\downarrow p & & \downarrow p & & \downarrow p \\
G_9(6) & \xrightarrow{d} & G_8(6) & \xrightarrow{d} & G_7(6) \\
 & & \downarrow f_1^6 & & \downarrow f_0^6 \\
 & & \beta_2(F)\otimes\wedge^4 F^\times \oplus F\otimes\mathcal{B}_2(F)\wedge^3 F^\times & \xrightarrow{\partial} & F\otimes\wedge^5 F^\times
\end{array} \tag{H}$$

where

$$f_0^6(q_0,...,q_6) = \sum_{i=0}^{6}(-1)^i \triangle(q_0,...,\hat{q}_i,...,q_6)\otimes \frac{\triangle(q_0,...,\hat{q}_{i+1},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+2},...,q_6)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+2},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+3},...,q_6)}$$
$$\frac{\triangle(q_0,...,\hat{q}_{i+3},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+4},...,q_6)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+4},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+5},...,q_6)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+5},...,q_6)}{\triangle(q_0,...,\hat{q}_{i+6},...,q_6)} \pmod 7 \tag{44}$$

and

$$f_1^6(q_0,...,q_7) = -\frac{1}{15}\Big(\sum_{\substack{i_0=0\\ i_1=i_0+1\\ i_2=i_0+2\\ i_3=i_0+1}}^{7} (-1)^{i_0}(\langle r(q_{i_0},q_{i_1},q_{i_2},q_{i_3}|q_0,...,\hat{q}_{i_0},\hat{q}_{i_1},\hat{q}_{i_2},\hat{q}_{i_3},...,q_7)\rangle_2 \otimes$$
$$\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{7} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{7} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_2\\ j=i_2+1}}^{7} \triangle(q_0,...,\hat{q}_{i_2},\hat{q}_j,...,q_7)\wedge$$
$$\prod_{\substack{j\neq i_3\\ j=i_3+1}}^{7} \triangle(q_0,...,\hat{q}_{i_3},\hat{q}_j,...,q_7) - \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{7} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_7)\otimes$$
$$[r(q_{i_0},q_{i_1},q_{i_2},q_{i_3}|q_0,...,\hat{q}_{i_0},\hat{q}_{i_1},\hat{q}_{i_2},\hat{q}_{i_3},...,q_7)]_2 \otimes \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{7} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_7)\wedge$$
$$\prod_{\substack{j\neq i_2\\ j=i_2+1}}^{7} \triangle(q_0,...,\hat{q}_{i_2},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_3\\ j=li_3+1}}^{7} \triangle(q_0,...,\hat{q}_{i_3},\hat{q}_j,...,q_7) + \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{7} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_7)\otimes$$
$$[r(q_{i_0},q_{i_1},q_{i_2},q_{i_3}|q_0,...,\hat{q}_{i_0},\hat{q}_{i_1},\hat{q}_{i_2},\hat{q}_{i_3},...,q_7)]_2 \otimes \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{7} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_7)\wedge$$
$$\prod_{\substack{j\neq i_2\\ j=i_2+1}}^{7} \triangle(q_0,...,\hat{q}_{i_2},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_3\\ j=i_3+1}}^{7} \triangle(q_0,...,\hat{q}_{i_3},\hat{q}_j,...,q_7)$$

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$$- \prod_{\substack{j \neq i_2 \\ j=i_2+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_2}, \hat{q}_j, ..., q_7) \otimes [r(q_{i_0}, q_{i_1}, q_{i_2}, q_{i_3} | q_0, ..., \hat{q}_{i_0}, \hat{q}_{i_1}, \hat{q}_{i_2}, \hat{q}_{i_3}, ..., q_7)]_2 \otimes$$
$$\prod_{\substack{j \neq i_0 \\ j=i_0+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_0}, \hat{q}_j, ..., q_7) \wedge \prod_{\substack{j \neq i_1 \\ j=i_1+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_2}, \hat{q}_2, ..., q_7) \wedge \prod_{\substack{j \neq i_3 \\ j=i_3+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_3}, \hat{q}_j, ..., q_7)$$
$$+ \prod_{\substack{j \neq i_3 \\ j=i_3+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_3}, \hat{q}_j, ..., q_7) \otimes [r(q_{i_0}, q_{i_1}, q_{i_2}, q_{i_3} | q_0, ..., \hat{q}_{i_0}, \hat{q}_{i_1}, \hat{q}_{i_2}, \hat{q}_{i_3}, ..., q_7)]_2 \otimes$$
$$\prod_{\substack{j \neq i_0 \\ j=i_0+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_0}, \hat{q}_j, ..., q_7) \wedge \prod_{\substack{j \neq i_1 \\ j=i_1+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_1}, \hat{q}_j, ..., q_7) \wedge \prod_{\substack{j \neq i_2 \\ j=i_2+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_2}, \hat{q}_j, ..., q_7) \Big)$$
$$\pmod 8 \tag{45}$$

105 **Lemma 14.** $f_0^6 \circ d = \partial \circ f_1^6$.

Proof. Let $(q_0, ..., q_7) \in G_8(6)$ on applying map d

$$d(q_0, ..., q_7) = \sum_{i=0}^{7} (-1)^i (q_0, ..., \hat{q}_i, ..., q_7) \tag{46}$$

Now apply map f_0^6

$$f_0^6 \circ d(q_0, ..., q_7) = \sum_{\substack{j \neq i \\ j=i+1}}^{7} (-1)^j \sum_{i=0}^{7} (-1)^i \triangle(q_0, ..., \hat{q}_i, \hat{q}_j ..., q_7) \otimes \frac{\triangle(q_0, ..., \hat{q}_{i+1}, \hat{q}_{j+1}, ..., q_7)}{\triangle(q_0, ..., \hat{q}_{i+2}, \hat{q}_{j+2}, ..., q_7)} \wedge$$
$$\frac{\triangle(q_0, ..., \hat{q}_{i+2}, \hat{q}_{j+2}, ..., q_7)}{\triangle(q_0, ..., \hat{q}_{i+3}, \hat{q}_{j+3}, ..., q_7)} \wedge \frac{\triangle(q_0, ..., \hat{q}_{i+3}, \hat{q}_{j+3}, ..., q_7)}{\triangle(q_0, ..., \hat{q}_{i+4}, \hat{q}_{j+4}, ..., q_7)} \frac{\triangle(q_0, ..., \hat{q}_{i+4}, \hat{q}_{j+4}, ..., q_7)}{\triangle(q_0, ..., \hat{q}_{i+5}, \hat{q}_{j+5}, ..., q_7)} \wedge$$
$$\frac{\triangle(q_0, ..., \hat{q}_{i+4}, \hat{q}_{j+5}, ..., q_7)}{\triangle(q_0, ..., \hat{q}_{i+6}, \hat{q}_{j+6}, ..., q_7)} \wedge \frac{\triangle(q_0, ..., \hat{q}_{i+6}, \hat{q}_{j+6}, ..., q_7)}{\triangle(q_0, ..., \hat{q}_{i+7}, \hat{q}_{j+7}, ..., q_7)} \tag{47}$$

Apply f_1^6 on $(q_0, ..., q_7) \in G_8(6)$, then

$$f_1^6(q_0, ..., q_7) = -\frac{1}{15} \Big(\sum_{\substack{i_0=0 \\ i_1=i_0+1 \\ i_2=i_0+2 \\ i_3=i_0+1}}^{7} (-1)^{i_0} (\langle r(q_{i_0}, q_{i_1}, q_{i_2}, q_{i_3} | q_0, ..., \hat{q}_{i_0}, \hat{q}_{i_1}, \hat{q}_{i_2}, \hat{q}_{i_3}, ..., q_7) \rangle_2 \otimes$$
$$\prod_{\substack{j \neq i_0 \\ j=i_0+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_0}, \hat{q}_j, ..., q_7) \wedge \prod_{\substack{j \neq i_1 \\ j=i_1+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_1}, \hat{q}_j, ..., q_7) \wedge \prod_{\substack{j \neq i_2 \\ j=i_2+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_2}, \hat{q}_j, ..., q_7) \wedge$$
$$\prod_{\substack{j \neq i_3 \\ j=i_3+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_3}, \hat{q}_j, ..., q_7) - \prod_{\substack{j \neq i_0 \\ j=i_0+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_0}, \hat{q}_j, ..., q_7) \otimes$$
$$[r(q_{i_0}, q_{i_1}, q_{i_2}, q_{i_3} | q_0, ..., \hat{q}_{i_0}, \hat{q}_{i_1}, \hat{q}_{i_2}, \hat{q}_{i_3}, ..., q_7)]_2 \otimes \prod_{\substack{j \neq i_1 \\ j=i_1+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_1}, \hat{q}_j, ..., q_7) \wedge$$
$$\prod_{\substack{j \neq i_2 \\ j=i_2+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_2}, \hat{q}_j, ..., q_7) \wedge \prod_{\substack{j \neq i_3 \\ j=li_3+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_3}, \hat{q}_j, ..., q_7) + \prod_{\substack{j \neq i_1 \\ j=i_1+1}}^{7} \triangle(q_0, ..., \hat{q}_{i_1}, \hat{q}_j, ..., q_7) \otimes$$

$$[r(q_{i_0},q_{i_1},q_{i_2},q_{i_3}|q_0,...,\hat{q}_{i_0},\hat{q}_{i_1},\hat{q}_{i_2},\hat{q}_{i_3},...,q_7)]_2 \otimes \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{7} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_7)\wedge$$
$$\prod_{\substack{j\neq i_2\\ j=i_2+1}}^{7} \triangle(q_0,...,\hat{q}_{i_2},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_3\\ j=i_3+1}}^{7} \triangle(q_0,...,\hat{q}_{i_3},\hat{q}_j,...,q_7)$$
$$-\prod_{\substack{j\neq i_2\\ j=i_2+1}}^{7} \triangle(q_0,...,\hat{q}_{i_2},\hat{q}_j,...,q_7) \otimes [r(q_{i_0},q_{i_1},q_{i_2},q_{i_3}|q_0,...,\hat{q}_{i_0},\hat{q}_{i_1},\hat{q}_{i_2},\hat{q}_{i_3},...,q_7)]_2\otimes$$
$$\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{7} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{7} \triangle(q_0,...,\hat{q}_{i_2},\hat{q}_2,...,q_7) \wedge \prod_{\substack{j\neq i_3\\ j=i_3+1}}^{7} \triangle(q_0,...,\hat{q}_{i_3},\hat{q}_j,...,q_7)$$
$$+\prod_{\substack{j\neq i_3\\ j=i_3+1}}^{7} \triangle(q_0,...,\hat{q}_{i_3},\hat{q}_j,...,q_7) \otimes [r(q_{i_0},q_{i_1},q_{i_2},q_{i_3}|q_0,...,\hat{q}_{i_0},\hat{q}_{i_1},\hat{q}_{i_2},\hat{q}_{i_3},...,q_7)]_2\otimes$$
$$\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{7} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{7} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_7) \wedge \prod_{\substack{j\neq i_2\\ j=i_2+1}}^{7} \triangle(q_0,...,\hat{q}_{i_2},\hat{q}_j,...,q_7)\Big) \tag{48}$$

Apply map ∂ and all properties, then

$$\partial \circ f_1^6(q_0,...,q_7) = \sum_{\substack{j\neq i\\ j=i+1}}^{7} (-1)^j \sum_{i=0}^{7} (-1)^i \triangle(q_0,...,\hat{q}_i,\hat{q}_j ...,q_7) \otimes \frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{j+1},...,q_7)}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_7)} \wedge$$
$$\frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_7)}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_7)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_7)}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{j+4},...,q_7)} \frac{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{j+4},...,q_7)}{\triangle(q_0,...,\hat{q}_{i+5},\hat{q}_{j+5},...,q_7)} \wedge$$
$$\frac{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{j+5},...,q_7)}{\triangle(q_0,...,\hat{q}_{i+6},\hat{q}_{j+6},...,q_7)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+6},\hat{q}_{j+6},...,q_7)}{\triangle(q_0,...,\hat{q}_{i+7},\hat{q}_{j+7},...,q_7)} \tag{49}$$

Hence Eq.(47) and Eq.(49) proves $\quad f_0^6 \circ p = \partial \circ f_1^6 \quad \square$

4. Generalized Geometry (Weight n = N)

For generalization, construct the generalized diagram by connecting the two sub-complexes using generalized morphisms.

$$\begin{array}{ccccc}
G_{n+3}(n+1) & \xrightarrow{d} & G_{n+3}(n+1) & \xrightarrow{d} & G_{n+2}(n+1) \quad (N \geq 2) \\
\downarrow p & & \downarrow p & & \downarrow p \\
G_{n+3}(n) & \xrightarrow{d} & G_{n+2}(n) & \xrightarrow{d} & G_{n+1}(n) \\
 & & \downarrow f_1^6 & & \downarrow f_0^n \\
 & & \beta_2(F) \otimes \wedge^{n-2} F^\times \oplus F \otimes \mathcal{B}_2(F) \wedge^{n-3} F^\times & \xrightarrow{\partial} & F \otimes \wedge^{n-1} F^\times
\end{array} \tag{I}$$

where

$$f_0^n(q_0,...,q_n) = \sum_{i=0}^{n} (-1)^i \triangle(q_0,...,\hat{q}_i,...,q_n) \otimes \frac{\triangle(q_0,...,\hat{q}_{i+1},...,q_n)}{\triangle(q_0,...,\hat{q}_{i+2},...,q_n)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+2},...,q_n)}{\triangle(q_0,...,\hat{q}_{i+3},...,q_n)} \wedge$$

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$$\frac{\triangle(q_0,...,\hat{q}_{i+3},...,q_n)}{\triangle(q_0,...,\hat{q}_{i+4},...,q_n)} \wedge ... \wedge \frac{\triangle(q_0,...,\hat{q}_{i+n-2},...,q_n)}{\triangle(q_0,...,\hat{q}_{i+n-1},...,q_n)} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+n-1},...,q_n)}{\triangle(q_0,...,\hat{q}_{i+n},...,q_n)} \pmod{n+1} \tag{50}$$

and

$$f_1^n(q_0,...,q_{n+1}) = (-1)^N \frac{1}{{}^nC_2}\Big(\sum_{\substack{i_0=0\\ i_1=i_0+1\\ i_2=i_0+2\\ \vdots \\ i_{n-3}=i_0+n-3}}^{n+1} (-1)^{i_0}(\langle r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1})\rangle_2 \otimes$$

$$\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1}) \wedge \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_{n+1}) \wedge ... \wedge$$

$$\prod_{\substack{j\neq i_{n-3}\\ mj=i_{n-3}+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...,q_{n+1}) - \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1}) \otimes$$

$$[r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1})]_2 \otimes \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_{n+1})$$

$$\wedge ... \wedge \prod_{\substack{j\neq i_{n-3}\\ j=1+i_{n-3}}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...,q_{n+1}) +$$

.

.

.

$$(-1)^{n+1} \prod_{\substack{j\neq i_{n-3}\\ j=1+i_{n-3}}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...q_{n+1}) \otimes [r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,$$

$$q_{n+1})]_2 \otimes \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1}) \wedge ... \wedge \prod_{\substack{j\neq i_{n-2}\\ j=1+i_{n-1}}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-2}},\hat{q}_j,...,q_{n+1}) \pmod{n+2} \tag{51}$$

108 **Theorem 1.** *The lower square of the generalized diagram I is commutative.*

Proof. Let $(q_0,...,q_{n+1}) \in G_{n+2}(n)$ and apply map d, then

$$d(q_0,...,q_{n+1}) = \sum_{i=0}^{n+1}(-1)^i(q_0,...,\hat{q}_i,...,q_{n+1}) \tag{52}$$

Apply map f_0^n on $d(q_0,...,q_{n+1})$, then

$$f_0^n \circ d(q_0,...,q_{n+1}) = \sum_{j=i+1}^{n+1}(-1)^j \sum_{i=0}^{n+1}(-1)^i \triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_{n+1}) \otimes \frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{j+1},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_{n+1})} \wedge$$

$$\frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_{n+1})} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{j+4},...,q_{n+1})} \wedge ... \wedge$$

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$$\frac{\triangle(q_0,...,\hat{q}_{i+n-1},\hat{q}_{j+n-1},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+n},\hat{q}_{j+n},...,q_{n+1})} \wedge \frac{\triangle(q_0,...,\hat{q}_{i+n},\hat{q}_{j+n},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+n+1},\hat{q}_{j+n+1},...,q_{n+1})} \tag{53}$$

Apply morphism f_1^n on $(q_0,...,q_{n+1}) \in G_{n+2}(n)$, then

$$\begin{aligned}
f_1^n(q_0,...,q_{n+1}) = (-1)^N \frac{1}{{}^nC_2}\Big(&\sum_{\substack{i_0=0\\ i_1=i_0+1\\ i_2=i_0+2\\ \vdots \\ i_{n-3}=i_0+n-3}}^{n+1} (-1)^{i_0}(\langle r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1})\rangle_2 \otimes \\
&\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1}) \wedge \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_{n+1}) \wedge ...\wedge \\
&\prod_{\substack{j\neq i_{n-3}\\ mj=i_{n-3}+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...,q_{n+1}) - \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1}) \otimes \\
&[r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1})]_2 \otimes \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_{n+1}) \\
&\wedge ... \wedge \prod_{\substack{j\neq i_{n-3}\\ j=1+i_{n-3}}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...,q_{n+1}) + \\
&. \\
&. \\
&. \\
&(-1)^{n+1} \prod_{\substack{j\neq i_{n-3}\\ j=1+i_{n-3}}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...q_{n+1}) \otimes [r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},..., \\
&q_{n+1})]_2 \otimes \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1}) \wedge ... \wedge \prod_{\substack{j\neq i_{n-2}\\ j=1+i_{n-1}}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-2}},\hat{q}_j, \\
&...,q_{n+1})\Big) \pmod{n+2}
\end{aligned} \tag{54}$$

apply map ∂, it becomes

$$\begin{aligned}
\partial \circ f_1^n(q_0,...,q_{n+1}) = (-1)^N \frac{1}{{}^nC_2}\Big(&\sum_{\substack{i_0=0\\ i_1=i_0+1\\ i_2=i_0+2\\ \vdots \\ i_{n-3}=i_0+n-3}}^{n+1} (-1)^{i_0}(r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1}) \otimes \\
&(1 - r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1}) \wedge \prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1}) \\
&\wedge \prod_{\substack{j\neq i_1\\ j=i_1+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_{n+1}) \wedge ... \wedge \prod_{\substack{j\neq i_{n-3}\\ j=i_{n-3}+1}}^{n+1} \triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...,q_{n+1}) -
\end{aligned}$$

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$$(1-r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1}))\otimes r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,$$
$$q_{n+1})\wedge\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1}\triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1})\wedge\prod_{\substack{j\neq i_1\\ j=i_1+1}}^{n+1}\triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_{n+1})\wedge...\wedge$$
$$\prod_{\substack{j\neq i_{n-3}\\ j=i_{n-3}+1}}^{n+1}\triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...,q_{n+1})-\prod_{\substack{j\neq i_1\\ j=i_1+1}}^{n+1}\triangle(q_0,...,\hat{q}_{i_1},\hat{q}_j,...,q_{n+1})\otimes$$
$$(1-r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1}))\wedge r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,$$
$$q_{n+1})$$
$$\wedge\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1}\triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1})\wedge...\wedge\prod_{\substack{j\neq i_{n-3}\\ j=1+i_{n-3}}}^{n+1}\triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...,q_{n+1})$$
$$+$$
$$.$$
$$.$$
$$.$$
$$(-1)^{n+1}\prod_{\substack{j\neq i_{n-3}\\ j=1+i_{n-3}}}^{n+1}\triangle(q_0,...,\hat{q}_{i_{n-3}},\hat{q}_j,...q_{n+1})\otimes(1-r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...$$
$$,q_{n+1}))\wedge r(q_{i_0},...,q_{i_{n-3}}|q_0,...,\hat{q}_{i_0},...,\hat{q}_{i_{n-3}},...,q_{n+1}))\wedge\prod_{\substack{j\neq i_0\\ j=i_0+1}}^{n+1}\triangle(q_0,...,\hat{q}_{i_0},\hat{q}_j,...,q_{n+1})$$
$$\wedge...\wedge\prod_{\substack{j\neq i_{n-2}\\ j=1+i_{n-2}}}^{n+1}\triangle(q_0,...,\hat{q}_{i_{n-2}},\hat{q}_j,...,q_{n+1})\Big) \tag{55}$$

now apply all properties of wedge, tensor, Siegel and dummy indices, it becomes

$$\begin{aligned}\partial\circ f_1^n(q_0,...,q_{n+1}) = \sum_{j=i+1}^{n+1}(-1)^j\sum_{i=0}^{n+1}(-1)^i\triangle(q_0,...,\hat{q}_i,\hat{q}_j,...,q_{n+1})\otimes\frac{\triangle(q_0,...,\hat{q}_{i+1},\hat{q}_{j+1},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_{n+1})}\wedge\\ \frac{\triangle(q_0,...,\hat{q}_{i+2},\hat{q}_{j+2},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_{n+1})}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+3},\hat{q}_{j+3},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+4},\hat{q}_{j+4},...,q_{n+1})}\wedge...\wedge\\ \frac{\triangle(q_0,...,\hat{q}_{i+n-1},\hat{q}_{j+n-1},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+n},\hat{q}_{j+n},...,q_{n+1})}\wedge\frac{\triangle(q_0,...,\hat{q}_{i+n},\hat{q}_{j+n},...,q_{n+1})}{\triangle(q_0,...,\hat{q}_{i+n+1},\hat{q}_{j+n+1},...,q_{n+1})}\end{aligned} \tag{56}$$

So from Eq.(53) and Eq.(56) and using dummy indices, theorem 1 is hence proved. □

5. Conclusion

In this study the generalization of morphisms f_0^n and f_1^n are presented to connect Cathelineau infinitesimal and Grassmannian chain complexes for generalized geometry. This work will play significant role in the fields of Algebraic K-theory, Chain complexes, Algebraic Topology, Homological Algebra and Polylogarithmic group theory.

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