

## A CLOSED FORM FOR THE STIRLING POLYNOMIALS IN TERMS OF THE STIRLING NUMBERS

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ABSTRACT. In the paper, by virtue of the Faà di Bruno formula and two identities for the Bell polynomial of the second kind, the authors find a closed form for the Stirling polynomials in terms of the Stirling numbers of the first and second kinds.

### 1. NOTATION AND MAIN RESULT

It is common knowledge [1, p. 48] that the Bernoulli numbers  $B_j$  are defined by

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!} = 1 - \frac{z}{2} + \sum_{j=0}^{\infty} B_{2j} \frac{z^{2j}}{(2j)!}, \quad |z| < 2\pi.$$

For some new developments in recent years about this topic, please refer to [2, 3, 6, 7, 10] and closely-related references therein.

The Stirling numbers of the first and second kinds  $s(n, k)$  and  $S(n, k)$  are important in combinatorial analysis, theory of special functions, and number theory. They can be generated by the rising factorial

$$(x)_n = \prod_{k=0}^{n-1} (x+k) = \sum_{k=0}^n s(n, k) x^k \quad (1)$$

and the exponential function

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!},$$

see [1, p. 213, Theorem A] and [1, p. 206, Theorem A], and can be computed by explicit formulas

$$s(n, k) = (-1)^{n+k} (n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-2}=1}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}} \sum_{\ell_{k-1}=1}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}}$$

for  $1 \leq k \leq n$  and

$$S(n, k) = \frac{1}{k!} \sum_{\ell=1}^k (-1)^{k-\ell} \binom{k}{\ell} \ell^n,$$

see [9, Corollary 2.3] and [1, p. 206]. For recent investigations on the Stirling numbers  $s(n, k)$  and  $S(n, k)$ , please refer to [6, 7, 8, 9] and plenty of references cited therein.

The Stirling polynomials  $S_k(x)$  are a family of polynomials that generalize the Bernoulli numbers  $B_k$  and the Stirling numbers of the second kind  $S(n, k)$ . The

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Stirling polynomials  $S_k(x)$  for nonnegative integers  $k$  are defined by the generating function

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!}.$$

The first six Stirling polynomials  $S_k(x)$  for  $0 \leq k \leq 5$  are

$$1, \quad \frac{x+1}{2}, \quad \frac{3x^2+5x+2}{12}, \quad \frac{x^3+2x^2+x}{8}, \\ \frac{15x^4+30x^3+5x^2-18x-8}{240}, \quad \frac{3x^5+5x^4-5x^3-13x^2-6x}{96}.$$

For more information on  $S_k(x)$ , see the papers [11, 12] and closely-related references therein.

In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.

In this paper, we find a closed form for the Stirling polynomials  $S_k(x)$  in terms of the Stirling numbers  $s(n, k)$  and  $S(n, k)$  of the first and second kinds.

Our main result can be stated as the following theorem.

**Theorem 1.** *For  $k \geq 0$ , the Stirling polynomials  $S_k(x)$  can be computed by the closed form*

$$S_k(x) = (-1)^k k! \sum_{m=0}^k \left[ \sum_{\ell=m}^k \frac{s(\ell+1, m+1)}{(k+\ell)!} \sum_{i=0}^{\ell} (-1)^i \binom{k+\ell}{\ell-i} S(k+i, i) \right] x^m. \quad (2)$$

## 2. PROOF OF THEOREM 1

In combinatorics, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  for  $n \geq k \geq 0$ , are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

See [1, p. 134, Theorem A]. They satisfy

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \quad (3)$$

for complex numbers  $a$  and  $b$ , see [1, p. 135], and

$$B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!} \sum_{i=0}^k (-1)^{k-i} \binom{n+k}{k-i} S(n+i, i), \quad (4)$$

see [3, Theorem 1 and Remark 1], [4, Remark 2.1], [5, p. 30], [8, p. 315], [10, Lemma 2.3], and [13, Example 4.2].

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  by

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)). \quad (5)$$

See [1, p. 139, Theorem C].

Taking  $f(u) = u^{-(x+1)}$  and  $u = h(t) = \frac{1-e^{-t}}{t}$  in the formula (5) and using the limit

$$\lim_{t \rightarrow 0} u = \lim_{t \rightarrow 0} h(t) = \lim_{t \rightarrow 0} \frac{1-e^{-t}}{t} = 1$$

yield

$$\begin{aligned} \frac{d^k}{dt^k} \left[ \left( \frac{t}{1-e^{-t}} \right)^{x+1} \right] &= \sum_{\ell=0}^k \langle -(x+1) \rangle_{\ell} B_{k,\ell}(h'(t), h''(t), \dots, h^{(k-\ell+1)}(t)) \\ &\rightarrow \sum_{\ell=0}^k \langle -(x+1) \rangle_{\ell} B_{k,\ell} \left( \lim_{t \rightarrow 0} h'(t), \lim_{t \rightarrow 0} h''(t), \dots, \lim_{t \rightarrow 0} h^{(k-\ell+1)}(t) \right) \end{aligned}$$

as  $t \rightarrow 0$ , where

$$\langle x \rangle_n = \prod_{\ell=0}^{n-1} (x - \ell) = \begin{cases} x(x-1) \cdots (x-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

is the falling factorial of  $x \in \mathbb{R}$  for  $n \in \{0\} \cup \mathbb{N}$ . It is not difficult to see that

$$\begin{aligned} \langle -(x+1) \rangle_{\ell} &= \prod_{m=0}^{\ell-1} [-(x+1) - m] = (-1)^{\ell} \prod_{m=0}^{\ell-1} (x+m+1) \\ &= (-1)^{\ell} \prod_{m=1}^{\ell} (x+m) = \frac{(-1)^{\ell}}{x} \prod_{m=0}^{\ell} (x+m) = \frac{(-1)^{\ell}}{x} (x)_{\ell+1}, \end{aligned}$$

where  $(x)_n$  is defined by (1). Hence, it follows that

$$\langle -(x+1) \rangle_{\ell} = \frac{(-1)^{\ell}}{x} \sum_{m=0}^{\ell+1} s(\ell+1, m) x^m, \quad \ell \geq 0.$$

Since

$$h^{(\ell)}(t) = \int_{1/e}^1 s^{t-1} (\ln s)^{\ell} ds \rightarrow \frac{(-1)^{\ell}}{1+\ell}, \quad t \rightarrow 0, \quad \ell \geq 0,$$

by virtue of (3) and (4), we have

$$\begin{aligned} B_{k,\ell} \left( \lim_{t \rightarrow 0} h'(t), \lim_{t \rightarrow 0} h''(t), \dots, \lim_{t \rightarrow 0} h^{(k-\ell+1)}(t) \right) &= B_{k,\ell} \left( -\frac{1}{2}, \frac{1}{3}, \dots, \frac{(-1)^{k-\ell+1}}{k-\ell+2} \right) \\ &= (-1)^k B_{k,\ell} \left( \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k-\ell+2} \right) = \frac{(-1)^{k+\ell} k!}{(k+\ell)!} \sum_{i=0}^{\ell} (-1)^i \binom{k+\ell}{\ell-i} S(k+i, i). \end{aligned}$$

Accordingly, we obtain

$$\begin{aligned} &\lim_{t \rightarrow 0} \frac{d^k}{dt^k} \left[ \left( \frac{t}{1-e^{-t}} \right)^{x+1} \right] \\ &= \sum_{\ell=0}^k \left[ \frac{(-1)^{\ell}}{x} \sum_{m=0}^{\ell+1} s(\ell+1, m) x^m \right] \left[ \frac{(-1)^{k+\ell} k!}{(k+\ell)!} \sum_{i=0}^{\ell} (-1)^i \binom{k+\ell}{\ell-i} S(k+i, i) \right] \\ &= \frac{(-1)^k k!}{x} \sum_{\ell=0}^k \frac{1}{(k+\ell)!} \left[ \sum_{i=0}^{\ell} (-1)^i \binom{k+\ell}{\ell-i} S(k+i, i) \right] \left[ \sum_{m=0}^{\ell+1} s(\ell+1, m) x^m \right] \\ &= \frac{(-1)^k k!}{x} \sum_{m=0}^{k+1} \left[ \sum_{\ell=m-1}^k \frac{s(\ell+1, m)}{(k+\ell)!} \sum_{i=0}^{\ell} (-1)^i \binom{k+\ell}{\ell-i} S(k+i, i) \right] x^m \end{aligned}$$

The explicit formula (2) is thus proved. The proof of Theorem 1 is complete.

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