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SEVERAL SERIES IDENTITIES INVOLVING THE CATALAN NUMBERS

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Abstract. In the paper, the authors discover several series identities involving the Catalan numbers, the Catalan function, the Riemanian zeta function, and the alternative Hurwitz zeta function.

1. Introduction and main results

It is well known that the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately? Whose solution is the Catalan number C_{n-2}". The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + \dotsb.$$

Three of explicit formulas of C_n for $n \ge 0$ read that

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{4^n \Gamma\bigl(n+\frac{1}{2}\bigr)}{\sqrt{\pi}\,\Gamma(n+2)} = {}_2F_1(1-n,-n;2;1),$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t, \quad \Re(z) > 0$$

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is the classical Euler gamma function,

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^\infty\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_p)_n}\frac{z^n}{n!} $$

is the generalized hypergeometric series defined for $a_i\in\mathbb{C}$, $b_i\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, and $p,q\in\mathbb{N}$, and

$$ (x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge1\\ 1, & n\ge0\end{cases} $$

and

$$ (-x)_n=(-1)^n(x-n+1)_n. $$

In 2014, Beckwith and Harbor [7] proposed a problem: show that

$$ \sum_{n=0}^\infty\frac{2^n}{C_n}=5+\frac32\pi\quad\text{and}\quad\sum_{n=0}^\infty\frac{3^n}{C_n}=22+8\sqrt3\,\pi. $$

This problem was answered in [1, 7] by

$$ \sum_{n=0}^\infty\frac{x^n}{C_n}=1-\frac{x(x-10)}{(4-x)^2}+\frac{24\sqrt{x}}{(4-x)^{5/2}}\arctan\sqrt{\frac{x}{4-x}}\,,\quad 0\le x<4. \tag{1.1} $$

The editorial comment in [1] listed the formulas

$$ \sum_{n=0}^\infty\frac{1}{C_n}=2+\frac{4\pi}{9\sqrt3}, \qquad \sum_{n=0}^\infty\frac{(-1)^n}{C_n}=\frac{14}{25}-\frac{24\sqrt5}{125}\ln\frac{1+\sqrt5}{2}, $$

$$ \sum_{n=0}^\infty\frac{(-2)^n}{C_n}=\frac13-\frac{1}{3\sqrt3}\ln\bigl(2+\sqrt3\bigr), \qquad \sum_{n=0}^\infty\frac{(-3)^n}{C_n}=\frac{10}{49}-\frac{36}{49\sqrt{21}}\ln\frac{5+\sqrt{21}}{2}. $$

The editorial comment in [1] pointed out that the result

$$ \sum_{n=0}^\infty\frac{x^n}{C_n}=2\frac{\sqrt{4-x}\,(8+x)+12\sqrt{x}\,\arctan\frac{\sqrt{x}}{\sqrt{4-x}}}{\sqrt{(4-x)^5}} \tag{1.2} $$

can be found on the website `http://planetmath.org/` and that the problem can be solved easily from

$$ \sum_{n=1}^\infty\frac{2^n}{\binom{2n}{n}}=\frac{\pi}{2}+1,\quad \sum_{n=1}^\infty\frac{n2^n}{\binom{2n}{n}}=\pi+3,\quad \sum_{n=1}^\infty\frac{3^n}{\binom{2n}{n}}=\frac{4\pi\sqrt3}{3}+3,\quad \sum_{n=1}^\infty\frac{n3^n}{\binom{2n}{n}}=\frac{20\pi\sqrt3}{3}+18 $$

which are special cases of the general formula

$$ \sum_{m=1}^\infty\frac{(2x)^{2m}}{m\binom{2m}{m}}=\sum_{m=1}^\infty\frac{(2x)^{2m}}{m(m+1)C_m}=\frac{2x\arcsin x}{\sqrt{1-x^2}},\quad |x|<1 $$

in [15, p. 452, Theorem]. Koshy and Gao [12] obtained

$$ \sum_{n=0}^\infty\frac{x^n}{C_n}=\begin{cases}1+\dfrac{x(4-x)^{3/2}+6x(4-x)^{1/2}+24\sqrt{x}\,\arcsin\frac{\sqrt{x}}{2}}{(4-x)^{5/2}}, & 0\le x<4;\\[2ex] 1-\dfrac{|x|(4-x)^{3/2}+6\sqrt{|x|(4-x)}+24\sqrt{|x|}\,\ln\frac{\sqrt{-x}+\sqrt{4-x}}{2}}{(4-x)^{5/2}}, & -4<x\le0.\end{cases} \tag{1.3} $$

In 2016, motivated by Problem 11765 in [7] mentioned above, Amdeberhan and his four coauthors [6] proposed a general problem: find a closed-form formula for the series $\sum_{n=0}^\infty \frac{z^n}{C_n}$. They obtained in [6] that

$$\sum_{n=0}^\infty \frac{z^n}{C_n} = {}_2F_1\left(1,2;\frac12;\frac z4\right) = \frac{2(z+8)}{(4-z)^2} + \frac{24\sqrt z}{(4-z)^{5/2}}\arcsin\frac{\sqrt z}{2} \tag{1.4}$$

for $|z|<4$ by several methods.

It is clear that the formulas (1.1) and (1.2) are the same one and that the formulas (1.3) and (1.4) are also the same one. Since

$$\arctan\sqrt{\frac{x}{4-x}} = \arcsin\frac{\sqrt x}{2}$$

for $0\le x<4$, the four formulas (1.1) to (1.4) are essentially the same one. It seems that there are close and similar ideas in [6, 7] and that the paper [6] is almost an expanded version of [7]. Great minds think alike!

The Catalan numbers C_n have a long history [11, 13, 14, 16, 33] and have been generalized and developed [2, 3, 19, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 35] in recent years.

In this paper, we will discover several series identities involving the Catalan numbers C_n, the Catalan function

$$C_x = \frac{4^x\Gamma\left(x+\frac12\right)}{\sqrt\pi\,\Gamma(x+2)}$$

for $x\ge0$, the Riemanian zeta function

$$\zeta(s) = \sum_{k=1}^\infty \frac{1}{k^s}$$

for $\Re(s)>1$, and the alternative Hurwitz zeta function

$$\zeta_a(s,q) = \sum_{n=0}^\infty \frac{(-1)^n}{(q+n)^s}$$

for $\Re(s)>1$ and $\Re(q)>0$.

Our main results can be stated as the following theorems.

Theorem 1.1. *For $\lambda>0$, we have*

$$\int_0^1 x^\lambda \ln C_x \,\mathrm{d}\,x = \frac{1}{\lambda+1}\left[\zeta_a(1,\lambda+2) - \sum_{n=1}^\infty \frac{1}{n^2}\sum_{k=0}^\infty \frac{(-1)^k}{n^k}\frac{2^{k+2}-2}{\lambda+k+3}\right]. \tag{1.5}$$

If $\lambda\in\mathbb N$, then

$$\int_0^1 x^\lambda \ln C_x \,\mathrm{d}\,x = \frac{1}{\lambda+1}\left[\sum_{n=1}^\infty \frac{(-1)^{n+1}\left(2^{n+1}-2\right)}{n+\lambda+2}\zeta(n+1) - \zeta_a(1,\lambda+2)\right]. \tag{1.6}$$

Theorem 1.2. *For $\lambda<\frac12$, we have*

$$\sum_{n=0}^\infty \frac{C_n}{4^n} = \sum_{n=0}^\infty \frac{1}{n!(1-\lambda)^{n+1/2}}\left(\frac12\right)_n \sum_{j=0}^n \frac{(-\lambda)^{n-j}}{j+1}\binom{n}{j} = 2. \tag{1.7}$$

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Theorem 1.3. *For $x \in (-4, 0]$, we have*

$$\sum_{n=0}^{\infty} \frac{x^n}{C_n} = \frac{24\sqrt{-x}}{(4-x)^{5/2}} \ln\left(\frac{\sqrt{-x}+\sqrt{4-x}}{2}\right) + \frac{2x}{(4-x)^2} + 1. \tag{1.8}$$

Theorem 1.4. *For $\lambda < \frac{1}{2}$ and $p > 0$, we have*

$$\sum_{n=1}^{\infty} \frac{n(n+1)C_n}{4^n(n+p)} = 2\sum_{n=0}^{\infty} \frac{1}{n!(1-\lambda)^{n+3/2}} \left(\frac{3}{2}\right)_n \sum_{j=0}^{n} \frac{(-\lambda)^{n-j}}{j+p+1}\binom{n}{j}. \tag{1.9}$$

Theorem 1.5. *For $\lambda < \frac{1}{2}$, we have*

$$C_n = \frac{2^{2n+5}}{\pi} \sum_{m=0}^{\infty} \frac{(-2n)_m}{m!(1-\lambda)^{m-2n}} \sum_{j=0}^{m} (-\lambda)^{m-j} \binom{m}{j} \sum_{k=0}^{\infty} \binom{-2n-3}{k} \frac{1}{j+2k+3}. \tag{1.10}$$

Theorem 1.6. *The zeta function $\zeta(z)$ satisfies*

$$\zeta(3) = \frac{8}{7} \sum_{n=0}^{\infty} \frac{(n+1)(2n)!!C_n}{4^n(2n+1)^2(2n+1)!!}. \tag{1.11}$$

2. Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([4, Lemma 2.1] and [21, p. 138, 5.5.8])**.** *The function $\psi = \frac{\Gamma'}{\Gamma}$ is strictly concave on $(0, \infty)$ and satisfies the duplication formula*

$$\psi(2x) = \frac{1}{2}\psi(x) + \frac{1}{2}\psi\left(x + \frac{1}{2}\right) + \ln 2.$$

Lemma 2.2. *For $x, k \in \mathbb{R}$ and $\lambda > 0$, we have*

$$\frac{x^{\lambda+1}}{x+k} = \sum_{j=0}^{\lambda} (-1)^j k^j x^{\lambda-j} + (-1)^{\lambda+1} \frac{k^{\lambda+1}}{x+k}. \tag{2.1}$$

Proof. A simple computation yields

$$\begin{aligned}\sum_{j=0}^{\lambda} (-1)^j k^{j-1} x^{\lambda-j} + (-1)^{\lambda+1} \frac{k^{\lambda}}{x+k} &= \frac{x^{\lambda}}{k} \sum_{j=0}^{\lambda} (-1)^j \left(\frac{k}{x}\right)^j + (-1)^{\lambda+1} \frac{k^{\lambda}}{x+k} \\ &= \frac{x^{\lambda}}{k} \frac{1-(-1)^{\lambda+1}\left(\frac{k}{x}\right)^{\lambda+1}}{1+\frac{k}{x}} + (-1)^{\lambda+1}\frac{k^{\lambda}}{x+k} = \frac{x^{\lambda+1}}{k(x+k)}.\end{aligned}$$

The required proof is complete. □

Remark 2.1. Taking $\lambda = 1$ in the identity (2.1) leads to

$$\frac{k^2}{x+k} = k - x + \frac{x^2}{x+k}$$

which can be found in [9, p. 96].

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Lemma 2.3 ([5, Theorem (The λ-method)]). *For $0<\alpha\le1$, $\eta>0$, $\lambda<\frac12$, and $\xi\in\mathbb{R}$, suppose that the function G given by $G(x)=\frac{g(x)}{(1-\alpha x^\eta)^\xi}$ satisfies $g,G\in L^1[0,1]$ and that*

$$b_j=b_j(\alpha,\eta)=\alpha^j\int_0^1 t^{j\eta}g(t)\,\mathrm{d}\,t,$$

then

$$\int_0^1\frac{g(x)}{(1-\alpha x^\eta)^\xi}\,\mathrm{d}\,x=\sum_{n=0}^\infty\frac{(\xi)_n}{n!(1-\lambda)^{n+\xi}}\sum_{j=0}^n\binom{n}{j}(-\lambda)^{n-j}b_j(\alpha,\eta).$$

3. Proofs of main results

We are now in a position to prove our main results.

Proof of Theorem 1.1. Integrating by parts gives

$$\int_0^1 x^\lambda\ln C_x\,\mathrm{d}\,x=\frac{1}{\lambda+1}\int_0^1\ln C_x\,\mathrm{d}\,x^{\lambda+1}=-\frac{1}{\lambda+1}\int_0^1x^{\lambda+1}\frac{C_x'}{C_x}\,\mathrm{d}\,x.$$

On the other hand, applying Lemma 2.1 leads to

$$\frac{C_x'}{C_x}=2\ln2+\psi\biggl(x+\frac12\biggr)-\psi(x+2)=2\psi(2x)-2\psi(x)-\frac1x-\frac{1}{x+1}. \tag{3.1}$$

Multiplying by $x^{\lambda+1}$ and integrating from 0 to 1 on both sides of (3.1) result in

$$\int_0^1x^{\lambda+1}\frac{C_x'}{C_x}\,\mathrm{d}\,x=2\int_0^1x^{\lambda+1}\psi(2x)\,\mathrm{d}\,x-2\int_0^1x^{\lambda+1}\psi(x)\,\mathrm{d}\,x-\int_0^1x^\lambda\,\mathrm{d}\,x-\int_0^1\frac{x^{\lambda+1}}{x+1}\,\mathrm{d}\,x.$$

Using the representation

$$\psi(x)=-\gamma-\frac1x+\sum_{n=1}^\infty\frac{x}{n(n+x)}$$

in [21, p. 139, 5.7.6], where γ is the Euler-Mascheroni constant, arrives at

$$\begin{aligned}\int_0^1x^{\lambda+1}\psi(x)\,\mathrm{d}\,x&=-\gamma\int_0^1x^{\lambda+1}\,\mathrm{d}\,x-\int_0^1x^\lambda\,\mathrm{d}\,x+\sum_{n=1}^\infty\int_0^1\frac{x^{\lambda+2}}{n(n+x)}\,\mathrm{d}\,x\\&=-\frac{\gamma}{\lambda+2}-\frac{1}{\lambda+1}+\sum_{n=1}^\infty\frac{1}{n^2}\int_0^1\frac{x^{\lambda+2}}{1+x/n}\,\mathrm{d}\,x\\&=-\frac{\gamma}{\lambda+2}-\frac{1}{\lambda+1}+\sum_{n=1}^\infty\frac{1}{n^2}\int_0^1x^{\lambda+2}\sum_{k=0}^\infty(-1)^k\Bigl(\frac{x}{n}\Bigr)^k\,\mathrm{d}\,x\\&=-\frac{\gamma}{\lambda+2}-\frac{1}{\lambda+1}+\sum_{n=1}^\infty\frac{1}{n^2}\sum_{k=0}^\infty\frac{(-1)^k}{n^k}\frac{1}{\lambda+k+3}.\end{aligned} \tag{3.2}$$

Similar method results in

$$\begin{aligned}\int_0^1x^{\lambda+1}\psi(2x)\,\mathrm{d}\,x&=\frac{1}{2^{\lambda+2}}\int_0^2x^{\lambda+1}\psi(x)\,\mathrm{d}\,x\\&=-\frac{\gamma}{\lambda+2}-\frac{1}{2(\lambda+1)}+\frac{1}{2^{\lambda+1}}\sum_{n=1}^\infty\frac{1}{n^2}\sum_{k=0}^\infty\frac{(-1)^k}{n^k}\frac{2^{\lambda+k+3}}{\lambda+k+3}\end{aligned} \tag{3.3}$$

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and

$$\int_0^1 \frac{x^{\lambda+1}}{x+1} \, \mathrm{d}\, x = \int_0^1 x^{\lambda+1} \sum_{k=0}^{\infty} (-1)^k x^k \, \mathrm{d}\, x = \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda+k+2}. \tag{3.4}$$

Combining (3.2) and (3.3) with (3.4) reveals

$$\int_0^1 x^{\lambda+1} \frac{C_x'}{C_x} \, \mathrm{d}\, x = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{n^k} \frac{2^{k+2}-2}{\lambda+k+3} - \zeta_a(1,\lambda+2).$$

The proof of the formula (1.5) is complete.

Using Lemma 2.2 and by the same method as in the proof of (1.5), we have

$$\begin{aligned}
\int_0^1 x^{\lambda+1} \psi(2x) \, \mathrm{d}\, x &= \frac{1}{2^{\lambda+2}} \int_0^2 x^{\lambda+1} \psi(x) \, \mathrm{d}\, x \\
&= -\frac{\gamma}{\lambda+2} - \frac{1}{2(\lambda+1)} + \frac{1}{2^{\lambda+2}} \sum_{k=1}^{\infty} \left[\sum_{j=0}^{\lambda+1} (-1)^j k^{j-1} \frac{2^{\lambda-j+2}}{\lambda-j+2} \right. \\
&\quad \left. + \sum_{n=1}^{\infty} (-1)^{\lambda+2} k^{\lambda+1} \ln\left(1+\frac{2}{k}\right) \right] \\
&= -\frac{\gamma}{\lambda+2} - \frac{1}{2(\lambda+1)} + \frac{1}{2^{\lambda+2}} \sum_{k=1}^{\infty} \sum_{j=\lambda+3}^{\infty} (-1)^{\lambda+2} k^{\lambda+1} \frac{(-1)^{j+1}}{j} \frac{2^j}{k^j} \\
&= -\frac{\gamma}{\lambda+2} - \frac{1}{2(\lambda+1)} + \frac{(-1)^{\lambda+2}}{2^{\lambda+2}} \sum_{j=\lambda+3}^{\infty} \frac{(-1)^{j+1} 2^j}{j} \sum_{k=1}^{\infty} \frac{1}{k^{j-\lambda-1}} \\
&= -\frac{\gamma}{\lambda+2} - \frac{1}{2(\lambda+1)} + \frac{(-1)^{\lambda+2}}{2^{\lambda+2}} \sum_{j=\lambda+3}^{\infty} \frac{(-1)^{j+1} 2^j}{j} \zeta(j-\lambda-1),
\end{aligned} \tag{3.5}$$

$$\int_0^1 x^{\lambda+1} \psi(x) \, \mathrm{d}\, x = -\frac{\gamma}{\lambda+2} - \frac{1}{(\lambda+1)} + (-1)^{\lambda+2} \sum_{j=\lambda+3}^{\infty} \frac{(-1)^{j+1}}{j} \zeta(j-\lambda-1), \tag{3.6}$$

and

$$\int_0^1 \frac{x^{\lambda+1}}{x+1} \, \mathrm{d}\, x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{nk^n}, \tag{3.7}$$

where we utilized the well-known Maclaurin series

$$\ln\left(1+\frac{1}{k}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{nk^n}.$$

Combining (3.5) and (3.6) with (3.7) results in (1.6). The proof of Theorem 1.1 is complete. □

Remark 3.1. Taking $\lambda = 0$ and using the identity

$$\int_0^1 \ln \Gamma(x+a) \, \mathrm{d}\, x = \frac{1}{2} \ln(2\pi) + a \ln a - a$$

in [10, p. 124, (43a)], we easily obtain

$$\int_0^1 \ln C_x \, \mathrm{d}\, x = -\frac{3}{2} \ln 2 - \ln \sqrt{\pi} + \frac{3}{2}.$$

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Proof of Theorem 1.2. Using the Maclaurin series

$$\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n \tag{3.8}$$

and substituting $4x=t$ give

$$\frac{1}{\sqrt{1-t}}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{t^n}{4^n}. \tag{3.9}$$

Taking $\alpha=1$, $\xi=\frac{1}{2}$, $\eta=1$, and $g(x)=1$ in Lemma 2.3 and integrating on both sides of (3.9) from 0 to 1 arrive at the formula (1.7). The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. It is easy to verify that the series converges for $|x|<4$.

For $x\in(-4,0]$, a simple computation yields

$$\sum_{n=0}^{\infty}\frac{x^n}{C_n}=\sum_{n=0}^{\infty}\frac{n!(n+1)!}{(2n)!}x^n=1+\sum_{n=1}^{\infty}\frac{(n+1)!}{2^n(2n-1)!!}x^n=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n+2)!!}{(2n-1)!!}\left(\frac{x}{4}\right)^n.$$

Denoting

$$f(x)=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n+2)!!}{(2n-1)!!}\left(\frac{x}{4}\right)^n$$

and substitution $x=-4t$ for $t>0$ produce

$$2[f(-4t)-1]=\sum_{n=1}^{\infty}(-1)^n\frac{(2n+2)!!}{(2n-1)!!}t^n\triangleq g(t).$$

Integrating twice on both sides of

$$\sum_{n=1}^{\infty}\frac{(2n+2)!!}{(2n-1)!!}(-1)^nt^{2n}=g\bigl(t^2\bigr)$$

from 0 to t shows

$$h(t)\triangleq\int_0^t g\bigl(t^2\bigr)\,\mathrm{d}\,t=\sum_{n=1}^{\infty}\frac{(2n+2)!!}{(2n+1)!!}(-1)^nt^{2n+1}$$

and

$$\alpha(t)\triangleq\int_0^t h(t)\,\mathrm{d}\,t=\sum_{n=1}^{\infty}\frac{(2n)!!}{(2n+1)!!}(-1)^nt^{2n+2}.$$

Using the well-known Maclaurin series

$$\frac{\ln\bigl(x+\sqrt{1+x^2}\,\bigr)}{\sqrt{1+x^2}}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!!}{(2n+1)!!}x^{2n+1},\quad x\in[-1,1]$$

in [34, p. 292] gives

$$\alpha(t)=\frac{t\ln\bigl(t+\sqrt{1+t^2}\,\bigr)}{\sqrt{1+t^2}}-t,$$

$$h(t)=\frac{\mathrm{d}\,\alpha(t)}{\mathrm{d}\,t}=\frac{t\sqrt{1+t^2}\,-\ln\bigl(t+\sqrt{1+t^2}\,\bigr)}{\sqrt{(1+t^2)^3}}-1,$$

$$g\bigl(t^2\bigr)=\frac{\mathrm{d}\,h(t)}{\mathrm{d}\,t}=\frac{3t\ln\bigl(t+\sqrt{1+t^2}\,\bigr)}{\sqrt{(1+t^2)^5}}-\frac{t^2}{(1+t^2)^2},$$

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and

$$f(x)=\frac{1}{2}\left[g\left(-\frac{x}{4}\right)\right]+1.$$

The formula (1.8) is thus proved. The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. Differentiating on both sides of (3.8) results in

$$\frac{2}{(1-4x)^{3/2}}=\sum_{n=1}^{\infty}\binom{2n}{n}nx^{n-1}. \tag{3.10}$$

Multiplying by x^p and integrating from 0 to 1 on both sides of (3.10) and taking $\alpha=4$, $\xi=\frac{3}{2}$, $\eta=1$, and $g(x)=2x^p$ in Lemma 2.3 conclude (1.9). The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. Using the integral representation

$$C_n=\frac{2^{2n+5}}{\pi}\int_0^1\frac{x^2\left(1-x^2\right)^{2n}}{(1+x^2)^{2n+3}}\,\mathrm{d}\,x$$

in [20, p. 10] and letting $\alpha=1$, $\xi=-2n$, $\eta=2$, $g(x)=\frac{x^2}{(1+x^2)^{2n+3}}$ in Lemma 2.3 produce

$$\begin{aligned}b_j&=\int_0^1\frac{t^{j+2}}{(1+t^2)^{2n+3}}\,\mathrm{d}\,x=\frac{1}{2}\int_0^1u^{(j+1)/2}(1+u)^{-2n-3}\,\mathrm{d}\,u\\&=\sum_{k=0}^{\infty}\binom{-2n-3}{k}\frac{1}{2}\int_0^1u^{(j+1+2k)/2}\,\mathrm{d}\,u=\sum_{k=0}^{\infty}\binom{-2n-3}{k}\frac{1}{j+2k+3}.\end{aligned}$$

By virtue of the substitution $t^2=u$, the formula (1.10) follows immediately. The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. Using the Fourier series

$$x=\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\frac{\sin^{2n+1}x}{2n+1}, \tag{3.11}$$

multiplying (1.11) by $x\cot x$, and integrating over the interval $\left[0,\frac{\pi}{2}\right]$ reveals

$$\int_0^{\pi/2}x^2\cot x\,\mathrm{d}\,x=\sum_{n=0}^{\infty}\frac{(n+1)C_n}{4^n(2n+1)}\int_0^{\pi/2}x\sin^{2n+1}x\cot x\,\mathrm{d}\,x$$

Applying the identity

$$\int_0^{\pi/2}t\cos^{p-1}t\sin at\,\mathrm{d}\,t=\frac{\pi}{2^{p+1}}\Gamma(p)\frac{\psi((p+a+1)/2)-\psi((p-a+1)/2)}{\Gamma((p+a+1)/2)\Gamma((p-a+1)/2)}.$$

in [8, p. 26, (3,8a)] gives

$$\begin{aligned}\int_0^{\pi/2}x\sin^{2n+1}x\cot x\,\mathrm{d}\,x&\overset{x=\pi/2-t}{=}\int_0^{\pi/2}\left(\frac{\pi}{2}-t\right)\cos^{2n}t\sin t\,\mathrm{d}\,t\\&=\frac{\pi}{2(2n+1)}-\frac{\pi}{2^{2n+2}}\Gamma(2n+1)\frac{\psi((2n+3)/2)-\psi((2n+1)/2)}{\Gamma((2n+3)/2)\Gamma((2n+1)/2)}\\&=\frac{\pi}{2(2n+1)}-\frac{1}{2n+1}\frac{(2n)!!}{(2n+1)!!}.\end{aligned}$$

On the other hand, applying the identity

$$\int_0^{\pi/2} x\ln\sin x\,\mathrm{d}\,x = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\ln 2$$

in [10, p. 144, (7.16)] gives

$$\int_0^{\pi/2} x^2\cot x\,\mathrm{d}\,x = -2\int_0^{\pi/2} x\ln\sin x\,\mathrm{d}\,x = -\frac{7}{8}\zeta(3) + \frac{\pi^2}{4}\ln 2.$$

To prove the formula (1.11), we just need to prove

$$\sum_{n=0}^{\infty}\frac{(n+1)C_n}{4^n(2n+1)^2} = \frac{\pi}{2}\ln 2.$$

In fact, multiplying (3.11) by $\cot x$ results in

$$\sum_{n=0}^{\infty}\frac{(n+1)C_n}{4^n(2n+1)^2} = \int_0^{\pi/2} x\cot x\,\mathrm{d}\,x = -\int_0^{\pi/2}\ln\sin x\,\mathrm{d}\,x = \frac{\pi}{2}\ln 2.$$

The proof of Theorem 1.6 is complete. □

Remark 3.2. There are some results and applications of properties for the Riemannian zeta function $\zeta(s)$ in the papers [17, 18, 22] and the closely related references therein.

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