Article

Evolutionary Optimization of Colebrook's Turbulent Flow Friction Approximations

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Abstract: Today, Colebrook's equation is mostly accepted as an informal standard for modeling of turbulent flow in hydraulically smooth and rough pipes including transient zone in between. The empirical Colebrook's equation relates the unknown flow friction factor (λ) with the known Reynolds number (R) and the known relative roughness of inner pipe surface (ε/D). It is implicit in unknown friction factor (λ). Implicit Colebrook's equation cannot be rearranged to derive friction factor (λ) directly and therefore it can be solved only iteratively [λ=f(λ, R, ε/D)] or using its explicit approximations [λ≈f(R, ε/D)]. Of course, approximations carry in certain error compared with the iterative solution where the highest level of accuracy can be reached after enough number of iterations. The explicit approximations give a relatively good prediction of the friction factor (λ) and can reproduce accurately Colebrook's equation and its Moody's plot. Usually, more complex models of approximations are more accurate and vice versa. In this paper, numerical values of parameters in various existing approximations are changed (optimized) using genetic algorithms to reduce maximal relative error. After this improvement computational burden stays unchanged while accuracy of approximations increases in some of the cases very significantly.

Keywords: colebrook equation; colebrook-white; moody diagram; turbulent flow; hydraulic resistance; darcy friction; pipes; genetic algorithms; optimization techniques; error analysis

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1. Introduction

In this paper more accurate explicit approximations of Colebrook's equation are presented. The Colebrook equation (1) relates hydraulic flow friction (λ) through Reynolds number (R) and relative roughness (ε/D) of inner pipe surface but in implicit way; λ=f(λ, R, ε/D) [1-18]. On the other hand, to express flow friction (λ) in implicit way a number of approximations can be used [19-44]. Accuracy of the approximations was increased using genetic algorithms [45-52] and results are presented here.

The Colebrook equation is empirical and hence its accuracy can be disputed. The friction factor curves derived from the Colebrook equation are said to be monotonic, i.e. the friction factor (λ) decreases continuously with increasing Reynolds number (R). Despite of that in some tests carried out on pipes that were artificially roughened with grains of sand the curves were inflectional in nature, i.e. the friction factor (λ) decreases to a minimum value with increasing Reynolds number (R) and then rises again to reach a constant value for complete turbulence [53,54], the Colebrook equation is still accepted in engineering practice as sufficiently accurate. It is still widely used in petroleum, mining, mechanical, civil and in all branches of engineering which deals with fluid flow.

Hydraulic resistance. Hydraulic resistance in general depends on flow rate [53-59]. To make things even more complex, hydraulic resistance is usually expressed through flow friction factor such as Darcy's (λ) where further pressure drop and flow rate is correlated with the well-known formula by Darcy and Weisbach. In the non-linear Darcy-Weisbach law for pipe flow, Darcy's friction factor (λ) is variable and always depends on flow. This assumption stands also if Fanning's friction is in use since its physical meaning is equal with Darcy's friction (λ). Darcy's friction factor,

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known also under the names of Moody or Darcy-Weisbach, is 4 times greater than Fanning's friction factor.

Colebrook Equation. To be more complex, widely used empirical and nonlinear Colebrook's equation (1) for calculation of Darcy's friction factor (λ) is iterative i.e. implicit in fluid flow friction factor since the unknown friction factor appears on the both sides of the equation [λ_0=f(λ_0, R, ε/D)] [2]. This unknown friction factor (λ) cannot be extracted to be on the left side of the equal sign analytically, i.e. with no use of some kind of mathematical simplifications. Better to say, it can be expressed explicitly only if approximate calculus takes place.

$$\frac{1}{\sqrt{\lambda_0}} = -2\cdot\log_{10}\left(\frac{2.51}{R\cdot\sqrt{\lambda_0}}+\frac{\varepsilon}{3.71\cdot D}\right), \quad (1)$$

λ_0 denotes high precision iterative solution of Colebrook's equation which is treated here as accurate, R denotes the Reynolds number while ε/D denotes relative roughness of inner pipe surfaces. All three mentioned values are dimensionless.

The Colebrook equation is somewhere known as the Colebrook-White equation or simply the CW equation [1,2]. This equation is valuable for determination of hydraulic resistances for turbulent regime in smooth and rough pipes including turbulent zone between them, but it is not valid for laminar regime. It describes a monotonic change in the friction factor (λ) during the turbulent flow in commercial pipes from smooth to fully rough. Moody's and Rouse's charts [3,4] represent the plots of the Colebrook equation over a very wide range of the Reynolds number (R from 2320 to 10^8) and relative roughness values (ε/D from 0 to 0.05). Beside of some of its shortcomings [54], today, Colebrook's equation is accepted as the informal standard of accuracy for calculation of hydraulic friction factor (λ).

Accuracy. As already noted, the Colebrook equation is empirical and therefore its accuracy can be disputed; equal sign '=' in 'λ_0=f(λ_0, R, ε/D)', i.e. in Eq. (1) instead of approximately equal sign "≈" can be treated as accurate only conditionally [48]. In this paper, iterative solution of Colebrook's equation (λ_0) after enough number of iterations will be treated as accurate and will be used for comparison as standard of accuracy where accuracy of friction factor (λ) calculated using the shown approximations will be compared with it.

Lambert W-function. The Colebrook equation can be rearranged in explicit form only approximately [λ≈f(R, ε/D)] where approach with the Lambert W-function can be treated as partial exemption from this rule [6-8,60-62], but also, further evaluation of the Lambert W-function function is approximate.

Looped Network of Pipes. The use of the accurate explicit approximations should be prioritized over the use of iterative solution in calculation of looped networks of pipes since in that way double iterative procedures, one for the Colebrook equation and one for the solution of the whole looped system of pipes, can be avoided [63-67].

Goal of the Study. The goal is to increase accuracy of already available explicit approximation of Colebrook's equation. This is accomplished using genetic algorithms.

2. Genetic algorithm optimization technique

Methodology. Genetic algorithms are one of the evolutionary computational intelligence techniques [45,46], inspired by Darwin's theory of biological evolution. Genetic algorithms provide solutions using randomly generated strings (chromosomes) for different types of problems, searching the most suitable among chromosomes that make the population in the potential solutions space. Genetic optimization is an alternative to the traditional optimal search approaches which make difficult finding the global optimum for nonlinear and multimodal optimization problems. Thus, genetic algorithms have been successful for example in solving combinatorial problems, control applications of parameter identification and control structure design, as well as in many other areas [47-52].

Used Optimization Approach. Here, genetic algorithms approach has been implemented to optimize parameters of available approximations of the Colebrook equation for hydraulic friction

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factor determination in order to improve their accuracy at the same time retaining previous complexity and computational burden of approximations. Small letters in the equations through paper corresponds to the numerical values before while capital letters to the numerical values after optimization through genetic algorithms as it is picturesquely presented in Figure 1.

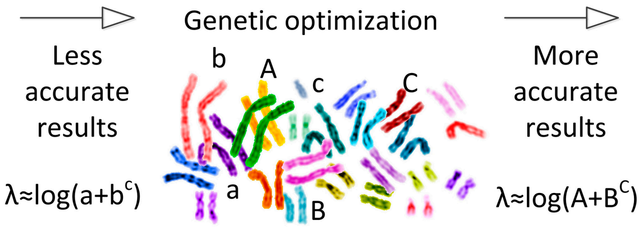

Figure 1. Picturesquely shown optimization using genetic algorithms

Genetic algorithms are very powerful tool for optimization. Samadianfard [47] uses genetic programming, a sort of genetic algorithms, to develop his own explicit approximations to the Colebrook equation. Also genetic algorithms can be used together with some other techniques of artificial intelligence such as neural networks [50-52].

Real coded genetic algorithms are used in this paper. The real coded genetic algorithms use the optimization designed cost function that minimizes maximal relative error, δ_{max} as follow (2):

$$\left.\begin{aligned} fitness &= \max_{\substack{i \\ i\in[1,n]}}(\delta) \\ \delta &= \left|\frac{\lambda-\lambda_0}{\lambda_0}\right|\cdot 100\% \end{aligned}\right\} \quad (2)$$

In (2), δ denotes relative (percentage) error, λ_0 denotes high precision iterative solution of Colebrook's equation which is treated as accurate here, λ denotes hydraulic friction factor solution calculated by each approximation considered, and n denotes number of pairs of λ_0 and λ used for optimization (in our case n=90,000).

Fitness function was evaluated in large number of 90 thousand points uniformly distributed in domains of the Reynolds number (R) and the relative roughness (ε/D). Subject of genetic optimization are coefficients in approximations, i.e. numeric coefficients in each approximation were changed by genetic algorithms in order to minimize the fitness function (2). In that way approximations are changed in order to match accuracy of iterative solution of Colebrook's equation as close as possible. Simultaneous optimization of all coefficients in each approximation was attempted, while the range of values of parameters in which optimal solutions were searched always in arbitrary neighborhood of initial values. Here we chose to present the results obtained with fitness function (2) in order to reduce maximal error of each approximation as much as possible (assumed that the reduction of the maximal relative error is of the highest importance for practical use of approximations). Genetic algorithms performance depends on its parameter values, so genetic algorithm parameters were carefully selected by conducting numerous experiments. In the implemented algorithm real-coded population of 100 individuals, an elitism of 10 individuals, and a scattered crossover function were used. All the members were subjected to adaptive feasible mutation except for the elite. The individuals were randomly selected by the Roulette method. Optimization with genetic algorithms was carried out in MATLAB by MathWorks. Practical domain of the Reynolds number (R) and relative roughness of inner pipe surface (ε/D) is coved by mesh of n=90,000 points for this optimization. In these 90 thousand points, iterative solution of the implicitly given Colebrook equation, λ_0 and non-iterative solution for every single observed approximation, λ, were calculated. The optimization of every single approximation lasts several hours. All evaluations of error were performed in MATLAB, with further confirmations in MS Excel to maintain full comparability with the study of Brkić [10] (For use of iterative calculus in MS Excel ver. 2007 see

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Brkić [11]; in Brkić and Tansković [68], MS Excel is also used for other extensive but non-iterative calculations). Mesh in MS Excel over the practical domain of the Reynolds number (R) and relative roughness of inner pipe surface (ε/D) consists of n=740 uniformly distributed points.

Alternative Opptimization Approaches. The main goal of the optimization in our case is to reduce the maximal error (δmax) of the every single observed approximation. This means, that sometimes the average (mean) relative error in the practical range of the Reynolds number (R) and the relative roughness of inner pipe surface (ε/D) increases compared to the model of the observed approximation with initial, non-optimized values of parameters. Of course, as it will be shown using genetic algorithm optimization with function defined to reduce maximal error, this error will be reduced more or less efficiently, which at the same time does not mean that average error will necessarily increased or decreased. Although the minimization of average error is not set as a goal by (2), it can be reduced also during the optimization. Instead of here already shown fitness function (2), it can be redefined to simultaneously reduce average and maximal error (3). In that way, both errors, i.e. maximal relative error and average (mean) relative error will be reduced simultaneously for sure. This requires more one-off computational efforts compared with the approach in which only one type of error is reduced; in our case this will be maximal relative error, δ_{max} while fitness function is defined by (2). In (3), the first term reduces average (mean) relative error δ_{avr}, the second term reduces maximal error δ, while weights k_1 and k_2 can be used to signify one of the terms and reduce influence of other. In that case compromise between reduction of maximal and average relative error is obtained.

$$\left.\begin{aligned} fitness &= k_1 \cdot (\delta_{avr}) + k_2 \cdot \max_{\substack{i \\ i\in[1,n]}}(\delta) \\ \delta_{avr} &= \frac{1}{n}\sum_{i=1}^{n}\left(\left|\frac{\lambda-\lambda_0}{\lambda_0}\right|\cdot 100\%\right)_i \\ \delta &= \left|\frac{\lambda-\lambda_0}{\lambda_0}\right|\cdot 100\% \end{aligned}\right\} \quad (3)$$

Also, fitness function can be set to reduce simultaneously mean square error δ_{MSE} and maximal relative error δ, as in (4). As already noted for (3), ratio between weight coefficients k_3 and k_4, determines influence of mean square error δ_{MSE} and maximal relative error δ in optimization. According to many different criterions, values of coefficients in existing explicit approximation to the Colebrook equation can be used. Using a lot computational resources, all three errors shown in our paper can be simultaneously reduced (5), but such procedure seems to be quite elusive.

$$\left.\begin{aligned} fitness &= k_3 \cdot (\delta_{MSE}) + k_4 \cdot \max_{\substack{i \\ i\in[1,n]}}(\delta) \\ \delta_{MSE} &= \frac{1}{n}\sum_{i=1}^{n}(\lambda-\lambda_0)_i^2 \\ \delta &= \left|\frac{\lambda-\lambda_0}{\lambda_0}\right|\cdot 100\% \end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned} fitness &= k_4 \cdot \max_{\substack{i \\ i\in[1,n]}}(\delta) + k_1 \cdot (\delta_{avr}) + k_3 \cdot (\delta_{MSE}) \\ \delta &= \left|\frac{\lambda-\lambda_0}{\lambda_0}\right|\cdot 100\% \\ \delta_{avr} &= \frac{1}{n}\sum_{i=1}^{n}\left(\left|\frac{\lambda-\lambda_0}{\lambda_0}\right|\cdot 100\%\right)_i \\ \delta_{MSE} &= \frac{1}{n}\sum_{i=1}^{n}(\lambda-\lambda_0)_i^2 \end{aligned}\right\} \quad (5)$$

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3. Explicit approximations of Colebrook's equation

Colebrook's equation (1) [2] suffers from being implicit in unknown friction factor (λ). It requires an iterative solution where convergence to the final accuracy of the observed approximation typically requires less than 7 iterations. As Brkić [10] proposed, we use here even few thousand iterations to be sure that sufficient value of accuracy for friction factor, λ_0, is reached.

As we already stated, implicit Colebrook's equation cannot be rearranged to derive friction factor directly in one step while iterative calculus can cause problem in simulation of flow in a pipe system in which it may be necessary to evaluate friction factor hundreds or thousands of times. This is the main reason for attempting to develop a relationship that is a reasonable and as possible accurate approximation for the Colebrook equation but which is explicit in friction factor. These approximations will be used for calculation of friction factor (λ), which will be compared with very accurate solution (λ_0) calculated using iterative procedure.

In this paper, 25 approximations will be optimized: Brkić [19, 20], Fang et al. [21], Ghanbari et al. [22], Papaevangelou et al. [23], Avci and Karagoz [24], Buzzelli [25], Sonnad and Goudar [26], Romeo et al. [27], Manadilli [28], Chen J.J.J. [29], Serghides [30], Haaland [31], Zigrang and Sylvester [32], Barr [33], Round [34], Shacham (available from [35]), Chen [36], Swamee and Jain [37], Eck [38], Wood [39] and, Moody [40]. Ćojbašić and Brkić [42] already optimized numerical values of parameters by Romeo et al. [27] and by Serghides [30].

Accuracy of existing approximations of Colebrook's equation was thoroughly checked by many researchers [10-16]. Yıldırım [14] conducted comprehensive analysis of existing correlations for single-phase friction but he used Techdig 2.0 software to read date from the Moody diagram which caused remarkable reading error. One must be always aware that the Moody diagram [3] was constructed using Colebrook's equation [2] and not opposite. After all, main conclusion of all papers [10-16] is that the relative error, δ, is non–uniformly distributed over the domain of the Reynolds number (R) and the relative roughness (ε/D).

The relative error δ is defined in (2-4) of this paper, the average (mean) relative error δ_{avr} in (3) and the mean square error δ_{MSE} in (4). All three types of error will be used in further text for estimation of accuracy of the examined explicit approximations of the Colebrook equation, but accent will be on minimization of the maximal relative error, δ_{max}.

Using shown genetic algorithm optimization technique, the values of existing parameters of the explicit approximations are improved compared to the iterative solution of Colebrook's equation. This means that the error of approximations decreases while computational burden stays unchanged. In this section, new parameters are shown and reduction of maximal relative error, δ_{max} is estimated. Relative error of the approximations shown in further text of this paper are calculated as $\delta=[(\lambda-\lambda_0)/\lambda_0]\cdot 100\%$, where λ is the Darcy friction factor calculated using the observed approximation while λ_0 is the iterative solution of Colebrook's equation which can be used as accurate after enough number of iterations (here set to the maximal available number of iterations in MS Excel which is 32767 as explained in [10]).

Every of 25 observed approximations is supplied with three diagrams; first is distribution of the relative error over the practical domain of applicability in engineering practice; second is same as the first but with the relative error distribution after optimization; and third is comparative diagram. For the first two mentioned diagrams, entire practical domain of the Reynolds number (R) and the relative roughness of inner pipe surface (ε/D) is covered with 740 point-mesh (diagrams produced in MS Excel). For the first two Figures with approximations, same pace of error is used for non-optimized and for optimized approximation, to provide more easily comparison with exceptions of Appr. 10, Appr. 11 and Appr. 14, where the optimization was extremely successfully performed. Mesh of 740 points is formed in MS Excel using 20 values of the relative roughness (ε/D) [shown in the related Figures] and using 37 values of the Reynolds number (R); from 10^4 to 10^5 with pace 10^4, from 10^5 to 10^6 with pace 10^5, from 10^6 to 10^7 with pace 10^6, and from 10^7 to 10^8 with pace 10^7.

According to Winning and Coole [16], using the value of mean square error δ_{MSE} defined in (4), all approximation can be classified in four groups (very small error is lower than 10^{-11}, small is

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between 10^{-11} and 10^{-8}, medium is between 10^{-8} and $5\cdot10^{-6}$, and large is above $5\cdot10^{-6}$). This criterion will also be used in further evaluation.

Regarding accuracy, it should be noted that inner roughness of pipe, ε, cannot be determined easily [17], so physical interpretation of the relative roughness of inner pipe surface (ε/D) is not subject of this study.

For genetic algorithm optimization, MATLAB 2010a by MatWorks was used. For this purpose, mesh of 90 thousand points over the entire practical domain of the Reynolds number (R) and the relative roughness of inner pipe surface (ε/D), is generated. For this 90 thousand pairs of Reynolds number (R) and the relative roughness of inner pipe surface (ε/D), friction factor (λ_0) is very accurately calculated to be used as a pattern during the procedure of optimization.

Efficiency of computing in computer environment stays unchanged between non-optimized and related optimized approximations, since the model of the approximation stays unchanged; i.e. number of logarithmic and power expression stays unchanged [9,18]. Only change of integer power to non-integer power in some approximation can increase computational burden, but even than not significantly.

In the following Figures 2-51, symbols and zones with green and red colors represent: Δδ-decreased level of maximal relative error δ_{max}; 1. Zone of increased relative error δ (red), 2. Zone of decreased relative error δ (green).

Brkić approximation [Appr. 1]. Relevant parameters and errors related to Approximation by Brkić [19] after and before optimization (6); [Appr. 1], are given in Figures 2-3.

$$\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx -2\cdot\log_{10}\left(\frac{2.18\cdot a_1}{R}+\frac{1}{3.71}\cdot\frac{\varepsilon}{D}\right)\\ a_1 &\approx \ln\frac{R}{1.816\cdot\ln\left(\frac{1.1\cdot R}{\ln(1+1.1\cdot R)}\right)}\end{aligned}\right\}\rightarrow\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx -2.013\cdot\log_{10}\left(\frac{2.261\cdot A_1}{R}+\frac{1}{3.71}\cdot\frac{\varepsilon}{D}\right)\\ A_1 &\approx \ln\frac{R}{2.479\cdot\ln\left(\frac{1.1\cdot R}{\ln(1+1.1\cdot R)}\right)}\end{aligned}\right\} \tag{6}$$

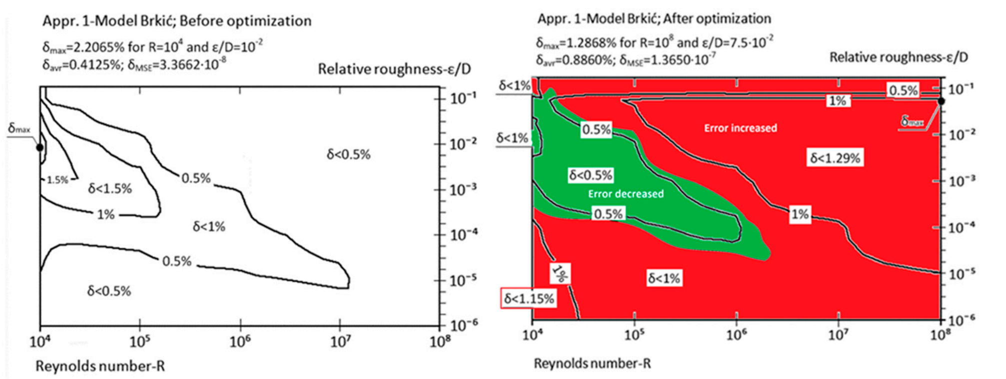

Figure 2. Relative error Brkić [Appr. 1; (6)] before and after optimization

δ_{max}: 2.2065% → 1.2868%

δ_{avr}: 0.4125% → 0.8860%

δ_{MSE}: $3.3662\cdot10^{-8}$ → $1.3650\cdot10^{-7}$

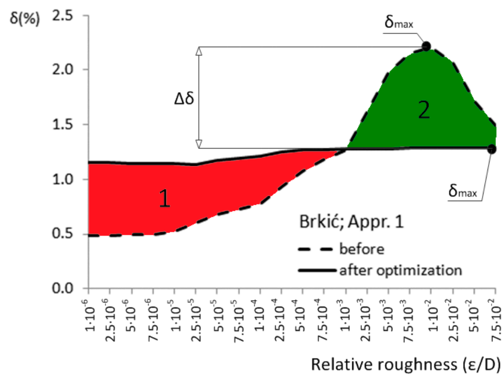

Figure 3. Performed genetic algorithm optimization of Brkić [Appr. 1; (6)]

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Brkić approximation [Appr. 2]. Relevant parameters and errors related to Approximation by Brkić [19] after and before optimization (7); [Appr. 2], are given in Figures 4-5.

$$\left.\begin{array}{l}\dfrac{1}{\sqrt{\lambda}}\approx -2\cdot\log_{10}\left(10^{-0.4343\cdot a_2}+\dfrac{1}{3.71}\cdot\dfrac{\varepsilon}{D}\right)\\ a_2\approx\ln\dfrac{R}{1.816\cdot\ln\left(\dfrac{1.1\cdot R}{\ln(1+1.1\cdot R)}\right)}\end{array}\right\}\rightarrow\left.\begin{array}{l}\dfrac{1}{\sqrt{\lambda}}\approx -2.013\cdot\log_{10}\left(10^{-0.43\cdot A_2}+\dfrac{1}{3.71}\cdot\dfrac{\varepsilon}{D}\right)\\ A_2\approx\ln\dfrac{R}{1.895\cdot\ln\left(\dfrac{1.1\cdot R}{\ln(1+1.1\cdot R)}\right)}\end{array}\right\}\qquad(7)$$

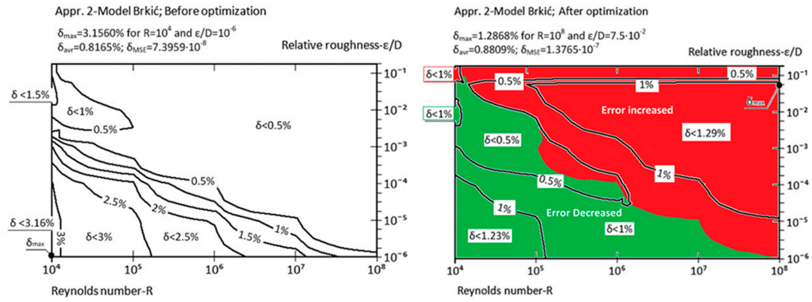

Figure 4. Relative error of Brkić [Appr. 2; Eq. (7)] before and after optimization

δ_{max}: 3.1560% → 1.2868%

δ_{avr}: 0.8165% → 0.8809%

δ_{MSE}: $7.3959\cdot10^{-8}$ → $1.3765\cdot10^{-7}$

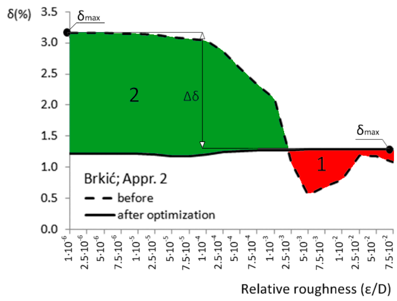

Figure 5. Performed genetic algorithm optimization of Brkić [Appr. 2; Eq. (7)]

Brkić approximation [Appr. 3]. Relevant parameters and errors related to Approximation by Brkić [20] after and before optimization (8); [Appr. 3], are given in Figures 6-7.

$$\left.\begin{array}{l}\dfrac{1}{\sqrt{\lambda}}\approx -2\cdot\log_{10}\left(a_3+\dfrac{1}{3.71}\cdot\dfrac{\varepsilon}{D}\right)\\ a_3\approx\dfrac{150.39}{R^{0.98865}}-\dfrac{152.66}{R}\end{array}\right\}\rightarrow\left.\begin{array}{l}\dfrac{1}{\sqrt{\lambda}}\approx -2.011\cdot\log_{10}\left(A_3+\dfrac{1}{3.71}\cdot\dfrac{\varepsilon}{D}\right)\\ A_3\approx\dfrac{147.21}{R^{0.98865}}-\dfrac{149.243}{R}\end{array}\right\}\qquad(8)$$

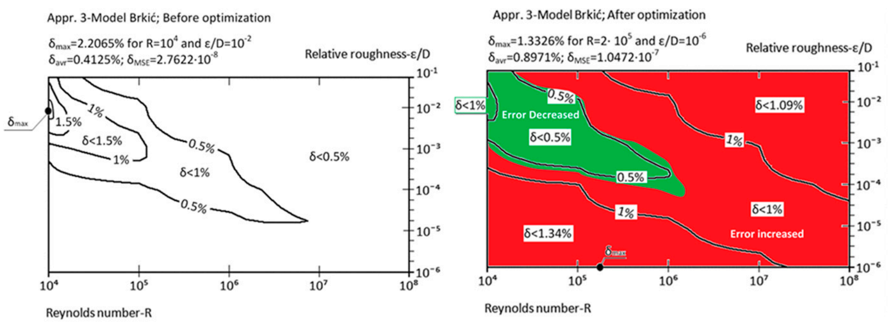

Figure 6. Relative error of Brkić [Appr. 3; Eq. (8)] before and after optimization

δ_{max}: 2.0715% → 1.3326%

δ_{avr}: 0.3101% → 0.8971%

δ_{MSE}: $2.7622\cdot10^{-8}$ → $1.0472\cdot10^{-7}$

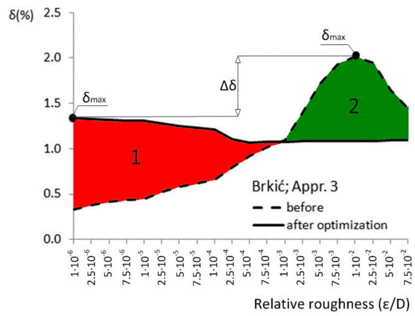

Figure 7. Performed genetic algorithm optimization of Brkić [Appr. 3; Eq. (8)]

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Brkić approximation [Appr. 4]. Relevant parameters and errors related to Approximation by Brkić [20] after and before optimization (9); [Appr. 4], are given in Figures 8-9.

$$\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx-2\cdot\log_{10}\left(\frac{1.25603}{R\cdot\sqrt{a_4}}+\frac{1}{3.71}\cdot\frac{\varepsilon}{D}\right)\\ a_4\approx\frac{-0.0015702}{\ln(R)}+\frac{0.3942031}{\ln^2(R)}+\frac{2.5341533}{\ln^3(R)}\end{array}\right\}\rightarrow\left\{\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx-2.013\cdot\log_{10}\left(\frac{1.216}{R\cdot\sqrt{A_4}}+\frac{1}{3.71}\cdot\frac{\varepsilon}{D}\right)\\ A_4\approx\frac{-0.013}{\ln(R)}+\frac{0.383}{\ln^2(R)}+\frac{2.997}{\ln^3(R)}\end{array}\right\} \quad (9)$$

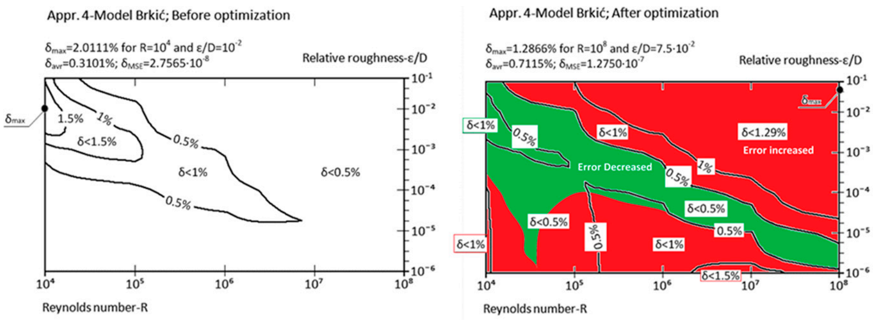

Figure 9. Relative error of Brkić [Appr. 4; Eq. (9)] before and after optimization

δ_{max}: 2.0111% → 1.2866%

δ_{avr}: 0.3101% → 0.7115%

δ_{MSE}: $2.7565\cdot10^{-8}$ → $1.2750\cdot10^{-7}$

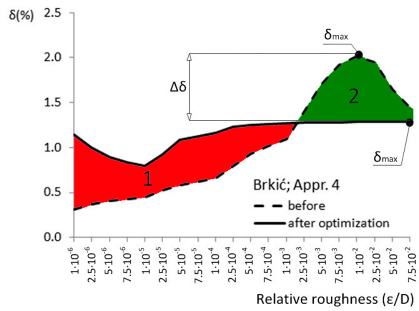

Figure 9. Performed genetic algorithm optimization of by Brkić [Appr. 4; Eq. (9)]

Fang et al. approximation [Appr. 5]. Relevant parameters and errors related to Approximation by Fang et al. [21] after and before optimization (10); [Appr. 5], are given in Figures 10-11.

$$\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx\left(1.613\cdot\left(\ln\left(0.234\cdot\left(\frac{\varepsilon}{D}\right)^{1.1007}-a_5\right)\right)^{-2}\right)^{-2}\\ a_5\approx\frac{60.525}{R^{1.1105}}+\frac{56.291}{R^{1.0712}}\end{array}\right\}\rightarrow\left\{\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx\left(1.61\cdot\left(\ln\left(0.234\cdot\left(\frac{\varepsilon}{D}\right)^{1.1007}-A_5\right)\right)^{-2}\right)^{-2}\\ A_5\approx\frac{61.948}{R^{1.1105}}+\frac{57.449}{R^{1.0712}}\end{array}\right\} \quad (10)$$

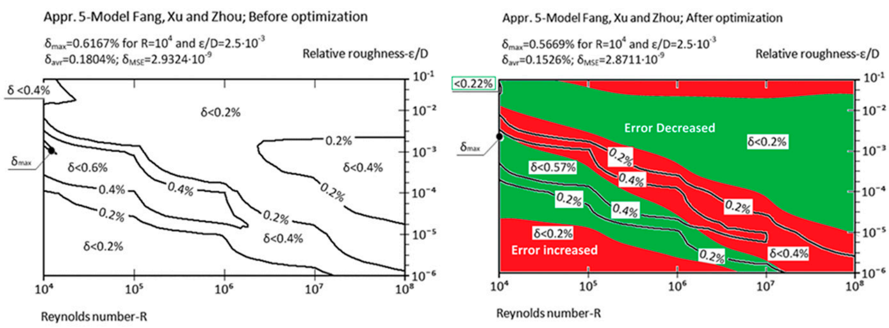

Figure 10. Relative error of Fang et al. [Appr. 5; Eq. (10)] before and after optimization

δ_{max}: 0.6167% → 0.5669%

δ_{avr}: 0.3101% → 0.1526%

δ_{MSE}: $2.9324\cdot10^{-9}$ → $2.8711\cdot10^{-9}$

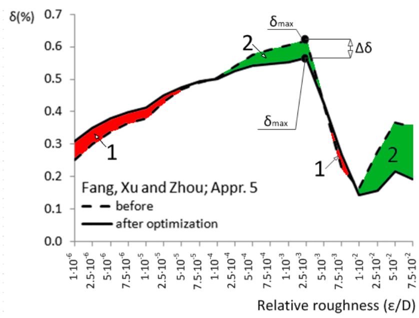

Figure 11. Performed genetic algorithm optimization of Fang et al. [Appr. 5; Eq. (10)]

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Ghanbari, Farshad and Rieke approximation [Appr. 6]. Relevant parameters and errors related to Approximation by Ghanbari et al. [22] after and before optimization (11); [Appr. 6], are given in Figures 12-13.

$$\left.\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(\left(-1.52 \cdot \log_{10}(a_6)\right)^{-2.169}\right)^{-2} \\ a_6 \approx \left(\frac{1}{7.21} \cdot \frac{\varepsilon}{D}\right)^{1.042} + \left(\frac{2.731}{R}\right)^{0.9152} \end{array}\right\} \rightarrow \left\{\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(\left(-1.606 \cdot \log_{10}(A_6)\right)^{-2.195}\right)^{-2} \\ A_6 \approx \left(\frac{1}{7.03} \cdot \frac{\varepsilon}{D}\right)^{0.967} + \left(\frac{2.629}{R}\right)^{0.858} \end{array}\right\} \quad (11)$$

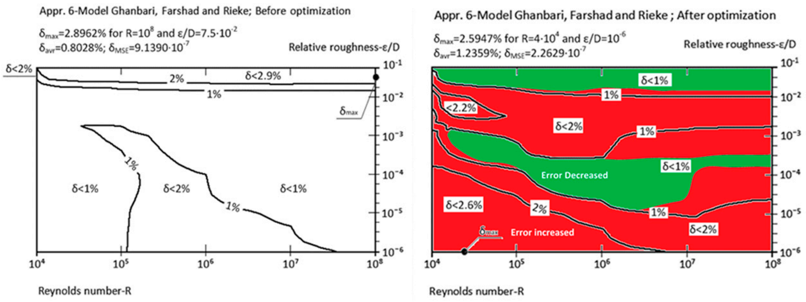

Figure 12. Relative error of Ghanbari et al. [Appr. 6; Eq. (11)] before and after optimization

δmax: 2.8962% → 2.5947%

δavr: 0.8028% → 1.2359%

δMSE: $9.1390 \cdot 10^{-7} \rightarrow 2.2629 \cdot 10^{-7}$

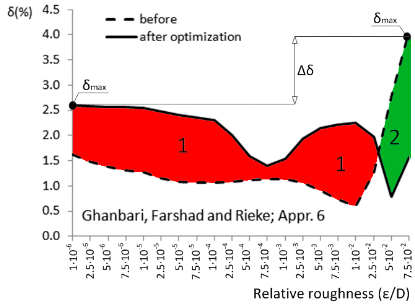

Figure 13. Performed genetic algorithm optimization of Ghanbari et al. [Appr. 6; Eq. (11)]

Papaevangelou, Evangelides and Tzimopoulos approximation [Appr. 7]. Relevant parameters and errors related to Approximation by Papaevangelou et al. [23] after and before optimization (12); [Appr. 7], are given in Figures 14-15.

$$\left.\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(\frac{0.2479 - 0.0000947 \cdot (a_7)^4}{\left(\log_{10}\left(\frac{1}{3.615} \cdot \frac{\varepsilon}{D} + \frac{7.366}{R^{0.9142}}\right)\right)^2}\right)^{-2} \\ a_7 \approx 7 - \log_{10}(R) \end{array}\right\} \rightarrow \left\{\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(\frac{0.249 - 0.0000974 \cdot |A_7|^{3.769}}{\left(\log_{10}\left(\frac{1}{3.646} \cdot \frac{\varepsilon}{D} + \frac{7.484}{R^{0.919}}\right)\right)^2}\right)^{-2} \\ A_7 \approx 7.122 - \log_{10}(R) \end{array}\right\} \quad (12)$$

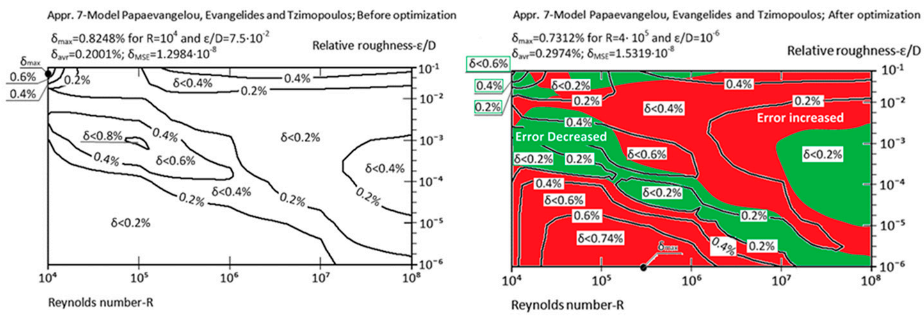

Figure 14. Relative error of Papaevangelou et al. [Appr. 7; Eq. (12)] before and after optimization

δmax: 0.8248% → 0.7312%

δavr: 0.2001% → 0.2974%

δMSE: $1.2984 \cdot 10^{-8} \rightarrow 1.5319 \cdot 10^{-8}$

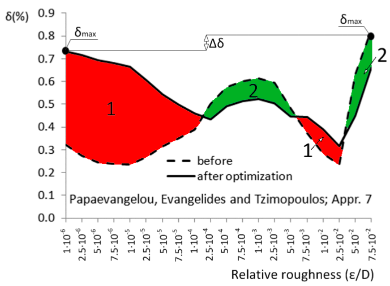

Figure 15. Performed genetic algorithm optimization of Papaevangelou et al. [Appr. 7; Eq. (12)]

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Avci and Karagoz approximation [Appr. 8]. Relevant parameters and errors related to Approximation by Avci and Karagoz [24] after and before optimization (13); [Appr. 8], are given in Figures 16-17.

$$\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx\left(\frac{6.4}{\left(\ln(R)-\ln(1+0.01\cdot a_8)\right)^{2.4}}\right)^{-2}\\ a_8\approx R\cdot\frac{\varepsilon}{D}\cdot\left(1+10\cdot\sqrt{\frac{\varepsilon}{D}}\right)\end{array}\right\}\rightarrow\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx\left(\frac{6.264}{\left(\ln(R)-\ln(1+0.009\cdot A_8)\right)^{2.383}}\right)^{-2}\\ A_8\approx R\cdot\frac{\varepsilon}{D}\cdot\left(1+10\cdot\sqrt{\frac{\varepsilon}{D}}\right)\end{array}\right\} \qquad (13)$$

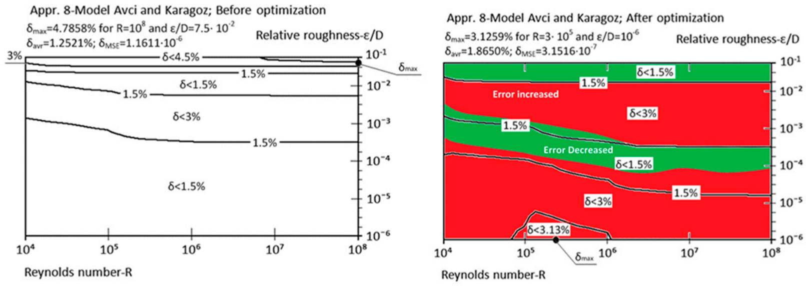

Figure 16. Relative error of Avci and Karagoz [Appr. 8; Eq. (13)] before and after optimization

δmax: 4.7858% → 3.1259%

δavr: 1.2521% → 1.8650%

δMSE: $1.1611\cdot10^{-6}\rightarrow3.1516\cdot10^{-7}$

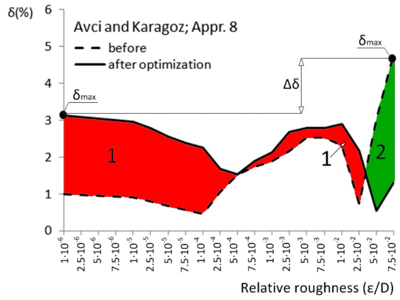

Figure 17. Performed genetic algorithm optimization of Avci and Karagoz [Appr. 8; Eq. (13)]

Buzzelli approximation [Appr. 9]. Relevant parameters and errors related to Approximation by Buzzelli [25] after and before optimization (14); [Appr. 9], are given in Figures 18-19.

$$\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx a_9-\left(\frac{a_9+2\cdot\log_{10}\left(\frac{a_{10}}{R}\right)}{1+\frac{2.18}{a_{10}}}\right)\\ a_9\approx\frac{(0.774\cdot\ln(R))-1.41}{\left(1+1.32\cdot\sqrt{\frac{\varepsilon}{D}}\right)}\\ a_{10}\approx\frac{1}{3.7}\cdot\frac{\varepsilon}{D}\cdot R+2.51\cdot a_9\end{array}\right\}\rightarrow\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx A_9-\left(\frac{A_9+1.9999\cdot\log_{10}\left(\frac{A_{10}}{R}\right)}{0.9996+\frac{2.1018}{A_{10}}}\right)\\ A_9\approx\frac{(0.7314\cdot\ln(R))-1.3163}{\left(1.0025+1.2435\cdot\sqrt{\frac{\varepsilon}{D}}\right)}\\ A_{10}\approx\frac{1}{3.7165}\cdot\frac{\varepsilon}{D}\cdot R+2.5137\cdot A_9\end{array}\right\} \qquad (14)$$

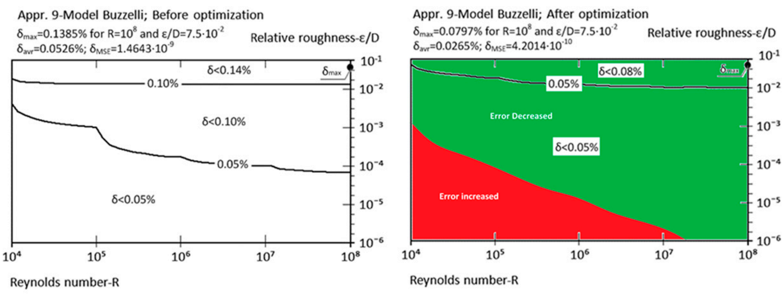

Figure 18. Relative error of Buzzelli [Appr. 9; Eq. (14)] before and after optimization

δmax: 0.1385% → 0.0797%

δavr: 0.0526% → 0.0265%

δMSE: $1.4643\cdot10^{-9}\rightarrow4.2014\cdot10^{-10}$

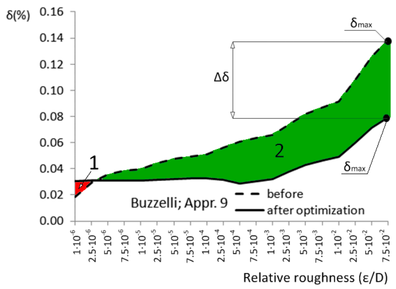

Figure 19. Performed genetic algorithm optimization of Buzzelli [Appr. 9; Eq. (14)]

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Sonnad and Goudar approximation [Appr. 10]. Relevant parameters and errors related to Approximation by Sonnad and Goudar [26] after and before optimization (15); [Appr. 10], are given in Figures 20-21. Vatankhah and Kouchakzadeh [43,44] changing parameter α_{11} to A_{11}-0.31 slightly change model of Sonnad and Goudar [26]. They used line fitting tool for optimization. We failed with further optimization using genetic algorithms.

$$\left.\begin{aligned} &\frac{1}{\sqrt{\lambda}} \approx 0.8686\cdot\ln\left(\frac{0.4587\cdot R}{a_{11}^{\,a_{12}}}\right)\\ &a_{11}\approx 0.124\cdot R\cdot\frac{\varepsilon}{D}+\ln(0.4587\cdot R)\\ &a_{12}\approx\frac{a_{11}}{a_{11}+1}\end{aligned}\right\}\rightarrow\left.\begin{aligned} &\frac{1}{\sqrt{\lambda}} \approx 0.8686\cdot\ln\left(\frac{0.4587\cdot R}{(A_{11}-0.31)^{A_{12}}}\right)\\ &A_{11}\approx 0.124\cdot R\cdot\frac{\varepsilon}{D}+\ln(0.4587\cdot R)\\ &A_{12}\approx\frac{A_{11}}{A_{11}+0.9633}\end{aligned}\right\} \tag{15}$$

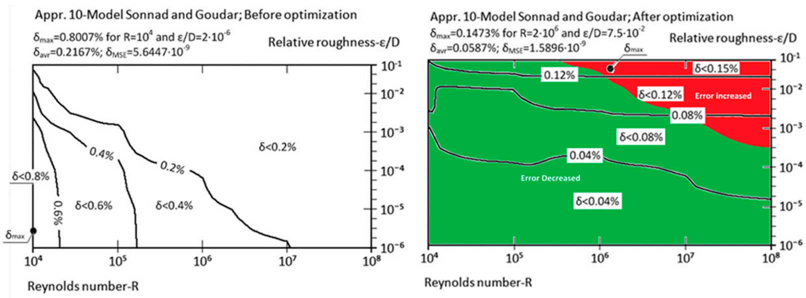

Figure 20. Relative error of Sonnad and Goudar [Appr. 10; Eq. (15)] before and after optimization; optimized by Vatankhah and Kouchakzadeh [43,44]

δ_{max}: 0.8007% → 0.1473%

δ_{avr}: 0.2167% → 0.0587%

δ_{MSE}: $5.6447\cdot10^{-9}$ → $1.5896\cdot10^{-9}$

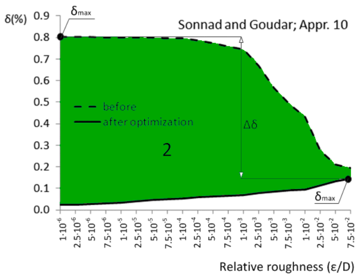

Figure 21. Performed genetic algorithm optimization of Sonnad and Goudar and optimized by Vatankhah and Kouchakzadeh [Appr. 10; Eq. (15)]

Romeo, Royo and Monzón approximation [Appr. 11]. Relevant parameters and errors related to Approximation by Romeo et al. [27] after and before optimization (16); [Appr. 11], are given in Figures 22-23. This optimization is already shown in the form of preliminary note in Ćojbašić and Brkić [42].

$$\left.\begin{aligned} &\frac{1}{\sqrt{\lambda}} \approx -2\cdot\log_{10}\left(\frac{1}{3.7065}\cdot\frac{\varepsilon}{D}-\frac{5.0272}{R}\cdot a_{13}\right)\\ &a_{13}\approx\log_{10}\left(\frac{1}{3.827}\cdot\frac{\varepsilon}{D}-\frac{4.567}{R}\cdot a_{14}\right)\\ &a_{14}\approx\log_{10}\left(\left(\frac{1}{7.7918}\cdot\frac{\varepsilon}{D}\right)^{0.9924}+\left(\frac{5.3326}{208.815+R}\right)^{0.9345}\right)\end{aligned}\right\}\rightarrow\left.\begin{aligned} &\frac{1}{\sqrt{\lambda}} \approx -2\cdot\log_{10}\left(\frac{1}{3.7106}\cdot\frac{\varepsilon}{D}-\frac{5}{R}\cdot A_{13}\right)\\ &A_{13}\approx\log_{10}\left(\frac{1}{3.8597}\cdot\frac{\varepsilon}{D}-\frac{4.795}{R}\cdot A_{14}\right)\\ &A_{14}\approx\log_{10}\left(\left(\frac{1}{7.646}\cdot\frac{\varepsilon}{D}\right)^{0.9685}+\left(\frac{4.9755}{206.2795+R}\right)^{0.8759}\right)\end{aligned}\right\} \tag{16}$$

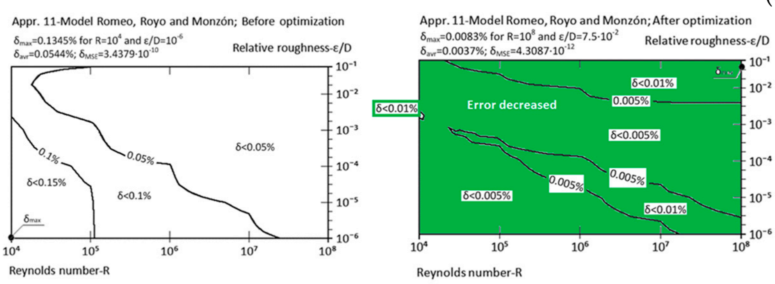

Figure 22. Relative error of Romeo, Royo and Monzón [Appr. 11; Eq. (16)] before and after optimization; optimized by Ćojbašić and Brkić [42]

δ_{max}: 0.1345% → 0.0083%

δ_{avr}: 0.0544% → 0.0037%

δ_{MSE}: $3.4379\cdot10^{-10}$ → $4.3087\cdot10^{-12}$

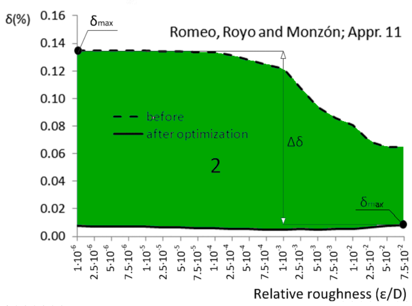

Figure 23. Performed genetic algorithm optimization of Romeo et al. [Appr. 11; Eq. (16)]

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Manadilli approximation [Appr. 12]. Relevant parameters and errors related to Approximation by Manadilli [28] after and before optimization (17); [Appr. 12], are given in Figures 24-25.

$$\left.\begin{matrix}\frac{1}{\sqrt{\lambda}}\approx -2\cdot\log_{10}\left(a_{15}+\frac{1}{3.7}\cdot\frac{\varepsilon}{D}\right)\\ a_{15}\approx\frac{95}{R^{0.983}}-\frac{96.82}{R}\end{matrix}\right\}\rightarrow\left\{\begin{matrix}\frac{1}{\sqrt{\lambda}}\approx -1.98\cdot\log_{10}\left(A_{15}+\frac{1}{3.949}\cdot\frac{\varepsilon}{D}\right)\\ A_{15}\approx\frac{95.974}{R^{0.986}}-\frac{96.02}{R}\end{matrix}\right\} \quad (17)$$

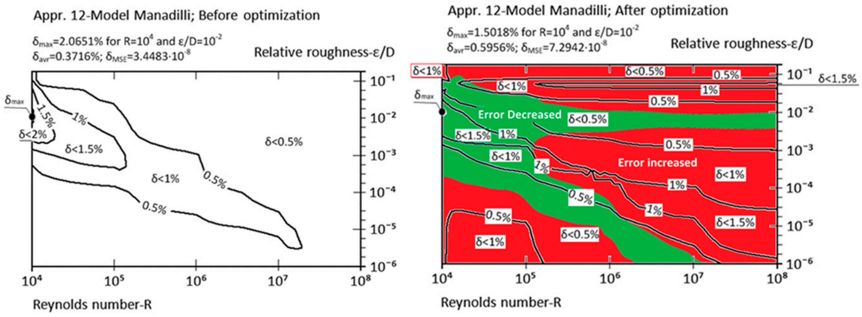

Figure 24. Relative error of Manadilli [Appr. 12; Eq. (17)] before and after optimization

δmax: 2.0651% → 1.5018%

δavr: 0.3716% → 0.5956%

δMSE: 3.4483·10-8 → 7.2942·10-8

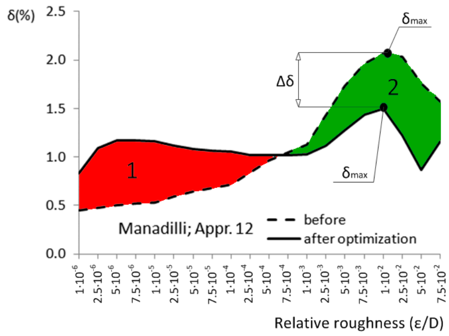

Figure 25. Performed genetic algorithm optimization of Manadilli [Appr. 12; Eq. (17)]

Chen J.J.J. approximation [Appr. 13]. Relevant parameters and errors related to Approximation by Chen [29] after and before optimization (18); [Appr. 13], are given in Figures 26-27.

$$\left.\begin{matrix}\frac{1}{\sqrt{\lambda}}\approx\left(0.184\cdot\left(a_{16}+0.7\cdot\frac{\varepsilon}{D}\right)^{0.3}\right)^{-2}\\ a_{16}\approx\frac{1}{R^{0.67}}\end{matrix}\right\}\rightarrow\left\{\begin{matrix}\frac{1}{\sqrt{\lambda}}\approx\left(0.208\cdot\left(A_{16}+0.697\cdot\frac{\varepsilon}{D}\right)^{0.315}\right)^{-2}\\ A_{16}\approx\frac{0.321}{R^{0.541}}\end{matrix}\right\} \quad (18)$$

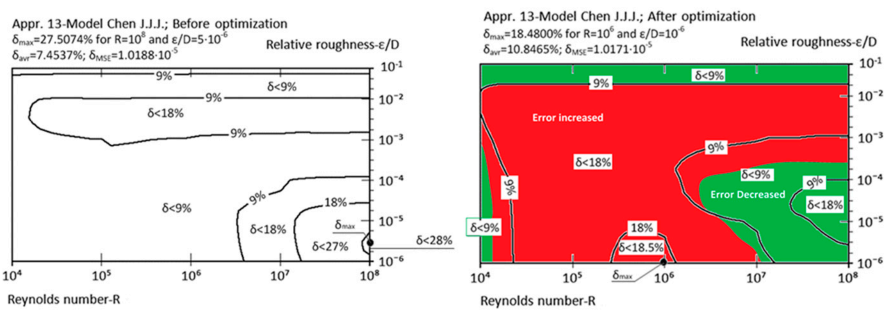

Figure 26. Relative error of Chen J.J.J. [Appr. 13; Eq. (18)] before and after optimization

δmax: 27.5074% → 18.4800%

δavr: 7.4537% → 10.8465%

δMSE: 1.0188·10-5 → 1.0171·10-5

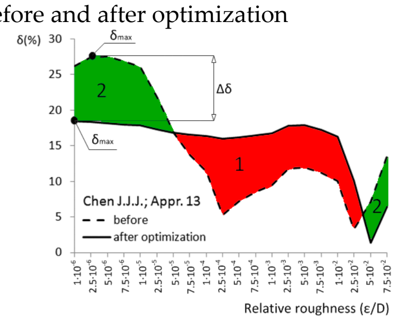

Figure 27. Performed genetic algorithm optimization of Chen J.J.J. [Appr. 13; Eq. (18)]

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Serghides approximation [Appr. 14]. Relevant parameters and errors related to Approximation by Serghides [30] after and before optimization (19); [Appr. 14], are given in Figures 28-29. This optimization is already shown in the form of preliminary note in Ćojbašić and Brkić [42].

$$\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx a_{17}-\frac{(a_{18}-a_{17})^2}{a_{19}-2\cdot a_{18}+a_{17}}\\ a_{17} &\approx -2\cdot\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}+\frac{12}{R}\right)\\ a_{18} &\approx -2\cdot\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}+\frac{2.51\cdot a_{17}}{R}\right)\\ a_{19} &\approx -2\cdot\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}+\frac{2.51\cdot a_{18}}{R}\right)\end{aligned}\right\}\rightarrow\left\{\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx A_{17}-\frac{(A_{18}-A_{17})^2}{A_{19}-2\cdot A_{18}+A_{17}}\\ A_{17} &\approx -2\cdot\log_{10}\left(\frac{1}{3.71}\cdot\frac{\varepsilon}{D}+\frac{12.585}{R}\right)\\ A_{18} &\approx -2\cdot\log_{10}\left(\frac{1}{3.71}\cdot\frac{\varepsilon}{D}+\frac{2.51\cdot A_{17}}{R}\right)\\ A_{19} &\approx -2\cdot\log_{10}\left(\frac{1}{3.71}\cdot\frac{\varepsilon}{D}+\frac{2.51\cdot A_{18}}{R}\right)\end{aligned}\right\} \quad (19)$$

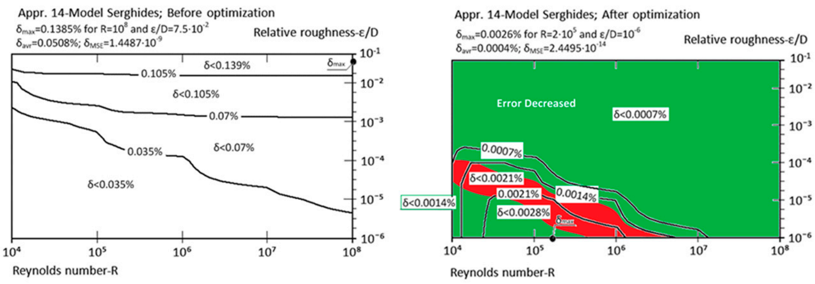

Figure 28. Relative error of Serghides [Appr. 14; Eq. (19)] before and after optimization; optimized by Ćojbašić and Brkić [42]

δ_{max}: 0.1385% → 0.0026%

δ_{avr}: 0.508% → 0.0004%

δ_{MSE}: $1.4487\cdot10^{-9}$ → $2.4495\cdot10^{-14}$

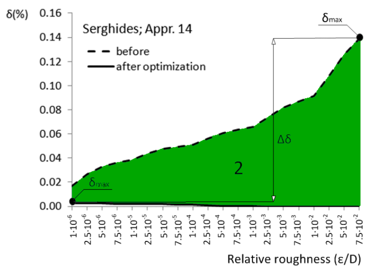

Figure 29. Performed genetic algorithm optimization of Serghides [Appr. 14; Eq. (19)]

Serghides approximation (simpler) [Appr. 15]. Relevant parameters and errors related to Approximation by Serghides -simpler [30] after and before optimization (20); [Appr. 15], are given in Figures 30-31.

$$\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx 4.781-\frac{(a_{20}-4.781)^2}{a_{21}-2\cdot a_{20}+4.781}\\ a_{20} &\approx -2\cdot\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}+\frac{12}{R}\right)\\ a_{21} &\approx -2\cdot\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}+\frac{2.51\cdot a_{20}}{R}\right)\end{aligned}\right\}\rightarrow\left\{\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx 4.83-\frac{(A_{20}-4.83)^2}{A_{21}-2\cdot A_{20}+4.83}\\ A_{20} &\approx -2\cdot\log_{10}\left(\frac{1}{3.71}\cdot\frac{\varepsilon}{D}+\frac{12.585}{R}\right)\\ A_{21} &\approx -2\cdot\log_{10}\left(\frac{1}{3.71}\cdot\frac{\varepsilon}{D}+\frac{2.51\cdot A_{20}}{R}\right)\end{aligned}\right\} \quad (20)$$

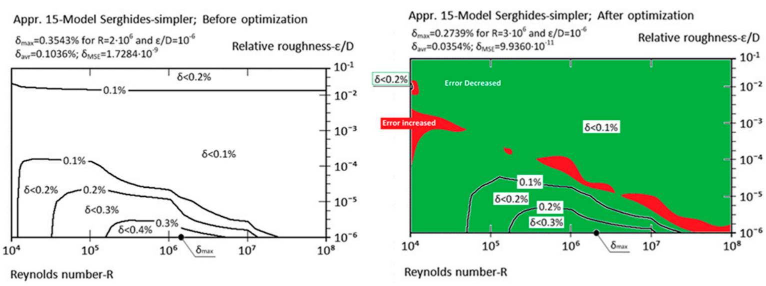

Figure 30. Relative error of simpler version of Serghides [Appr. 15; Eq. (20)] before and after optimization

δ_{max}: 0.3543% → 0.2739%

δ_{avr}: 0.1036% → 0.0354%

δ_{MSE}: $1.7284\cdot10^{-9}$ → $9.9360\cdot10^{-11}$

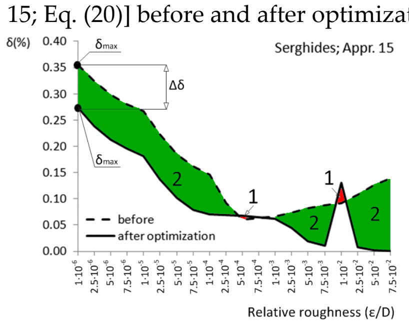

Figure 31. Performed genetic algorithm optimization of the simpler version of Serghides [Appr. 15; Eq. (20)]

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Haaland approximation [Appr. 16]. Relevant parameters and errors related to Approximation by Haaland [31] after and before optimization (21); [Appr. 16], are given in Figures 32-33.

$$\left.\begin{aligned}&\frac{1}{\sqrt{\lambda}}\approx -1.8\cdot\log_{10}\left(\frac{6.9}{R}+a_{22}\right)\\&a_{22}\approx\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}\right)^{1.11}\end{aligned}\right\}\rightarrow\left.\begin{aligned}&\frac{1}{\sqrt{\lambda}}\approx -1.798\cdot\log_{10}\left(\frac{6.891}{R}+A_{22}\right)\\&A_{22}\approx\left(\frac{1}{3.755}\cdot\frac{\varepsilon}{D}\right)^{1.106}\end{aligned}\right\}\qquad(21)$$

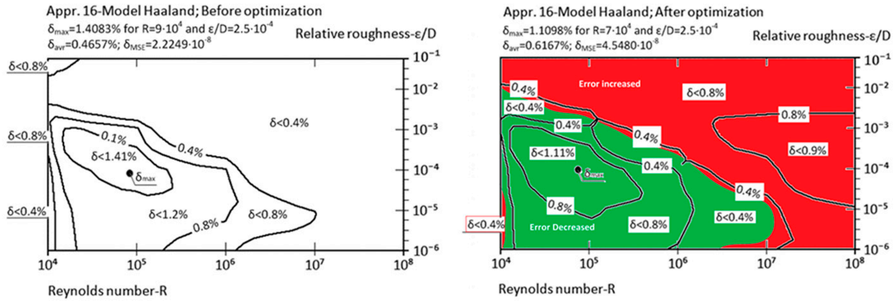

Figure 32. Relative error of Haaland [Appr. 16; Eq. (21)] before and after optimization

δ_{max}: 1.4083% → 1.1098%

δ_{avr}: 0.4657% → 0.6167%

δ_{MSE}: $2.2249\cdot 10^{-8}\rightarrow 4.5480\cdot 10^{-8}$

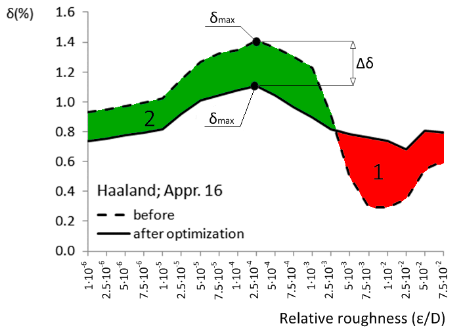

Figure 33. Performed genetic algorithm optimization of Haaland [Appr. 16; Eq. (21)]

Zigrang and Sylvester approximation [Appr. 17]. Relevant parameters and errors related to Approximation by Zigrang and Sylvester [32] after and before optimization (22); [Appr. 17], are given in Figures 34-35.

$$\left.\begin{aligned}&\frac{1}{\sqrt{\lambda}}\approx -2\cdot\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}-\frac{5.02}{R}\cdot a_{23}\right)\\&a_{23}\approx\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}-\frac{5.02}{R}\cdot a_{24}\right)\\&a_{24}\approx\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}+\frac{13}{R}\right)\end{aligned}\right\}\rightarrow\left.\begin{aligned}&\frac{1}{\sqrt{\lambda}}\approx -2.0012\cdot\log_{10}\left(\frac{1}{3.7027}\cdot\frac{\varepsilon}{D}-\frac{5.0605}{R}\cdot A_{23}\right)\\&A_{23}\approx\log_{10}\left(\frac{1}{3.7027}\cdot\frac{\varepsilon}{D}-\frac{5.0605}{R}\cdot A_{24}\right)\\&A_{24}\approx\log_{10}\left(\frac{1}{3.7027}\cdot\frac{\varepsilon}{D}+\frac{12.513}{R}\right)\end{aligned}\right\}\qquad(22)$$

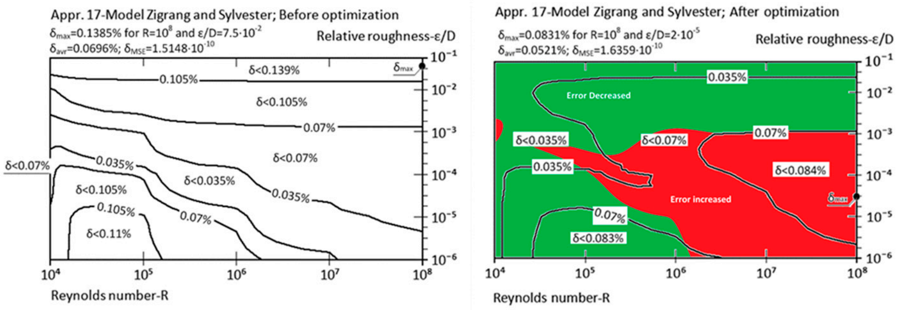

Figure 34. Relative error of Zigrang and Sylvester [Appr. 17; Eq. (22)] before and after optimization

δ_{max}: 0.1385% → 0.0831%

δ_{avr}: 0.0696% → 0.0521%

δ_{MSE}: $1.5148\cdot 10^{-10}\rightarrow 1.6359\cdot 10^{-10}$

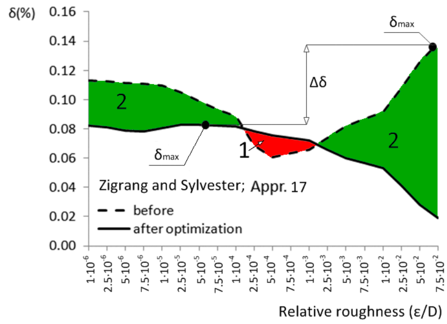

Figure 35. Performed genetic algorithm optimization of Zigrang and Sylvester [Appr. 17; Eq. (22)]

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Zigrang and Sylvester approximation (simpler) [Appr. 18]. Relevant parameters and errors related to Approximation by Zigrang and Sylvester -simpler [32] after and before optimization (23); [Appr. 18], are given in Figures 36-37.

$$\left.\begin{aligned} \frac{1}{\sqrt{\lambda}} &\approx -2\cdot\log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}-\frac{5.02}{R}\cdot a_{25}\right) \\ a_{25} &\approx \log_{10}\left(\frac{1}{3.7}\cdot\frac{\varepsilon}{D}+\frac{13}{R}\right) \end{aligned}\right\} \to \left\{\begin{aligned} \frac{1}{\sqrt{\lambda}} &\approx -2.0012\cdot\log_{10}\left(\frac{1}{3.7027}\cdot\frac{\varepsilon}{D}-\frac{5.0605}{R}\cdot A_{25}\right) \\ A_{25} &\approx \log_{10}\left(\frac{1}{3.7027}\cdot\frac{\varepsilon}{D}+\frac{15.202}{R}\right) \end{aligned}\right\} \quad (23)$$

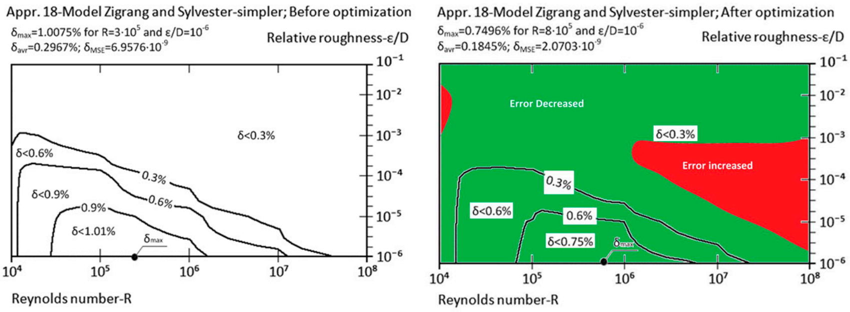

Figure 36. Relative error of simpler version of Zigrang and Sylvester [Appr. 18; Eq. (23)] before and after optimization

δmax: 1.0075% → 0.7496%

δavr: 0.2967% → 0.1845%

δMSE: $6.9576\cdot10^{-9} \to 2.0703\cdot10^{-9}$

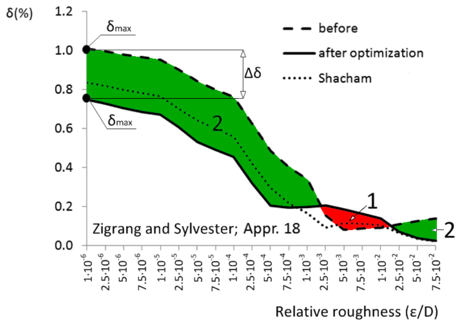

Figure 37. Performed genetic algorithm optimization of the simpler version of Zigrang and Sylvester [Appr. 18; Eq. (23)]

Barr approximation [Appr. 19]. Relevant parameters and errors related to Approximation by Barr [33] after and before optimization (24); [Appr. 19], are given in Figures 38-39.

$$\left.\begin{aligned} \frac{1}{\sqrt{\lambda}} &\approx -2\cdot\log_{10}\left(\frac{4.518\cdot\log_{10}\left(\frac{R}{7}\right)}{a_{26}}+\frac{1}{3.7}\cdot\frac{\varepsilon}{D}\right) \\ a_{26} &\approx R\cdot\left(1+\frac{R^{0.52}}{29}\cdot\left(\frac{\varepsilon}{D}\right)^{0.7}\right) \end{aligned}\right\} \to \left\{\begin{aligned} \frac{1}{\sqrt{\lambda}} &\approx -1.998\cdot\log_{10}\left(\frac{4.509\cdot\log_{10}\left(\frac{R}{7.049}\right)}{A_{26}}+\frac{1}{3.737}\cdot\frac{\varepsilon}{D}\right) \\ A_{26} &\approx R\cdot\left(0.999+\frac{R^{0.525}}{28.102}\cdot\left(\frac{\varepsilon}{D}\right)^{0.721}\right) \end{aligned}\right\} \quad (24)$$

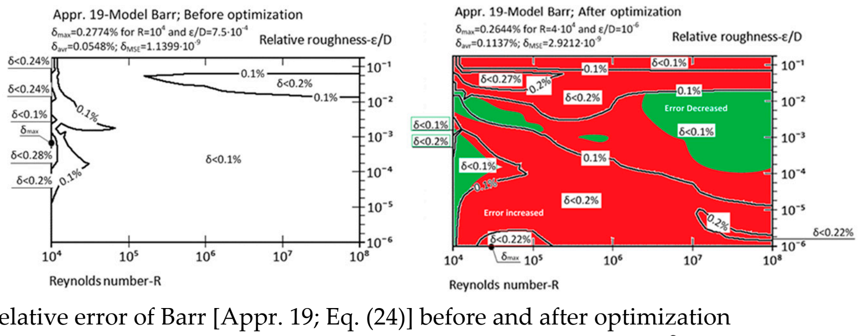

Figure 38. Relative error of Barr [Appr. 19; Eq. (24)] before and after optimization

δmax: 0.2774% → 0.2644%

δavr: 0.0548% → 0.1137%

δMSE: $1.1399\cdot10^{-9} \to 2.9212\cdot10^{-9}$

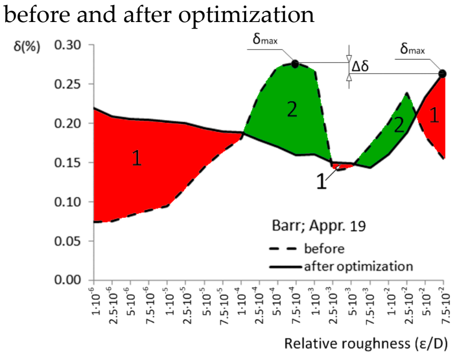

Figure 39. Performed genetic algorithm optimization of Barr [Appr. 19; Eq. (24)]

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Round approximation [Appr. 20]. Relevant parameters and errors related to Approximation by Round [34] after and before optimization (25); [Appr. 20], are given in Figures 40-41.

$$\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx 1.8\cdot\log_{10}\left(\frac{R}{a_{27}+6.5}\right)\\ a_{27}\approx 0.135\cdot R\cdot\frac{\varepsilon}{D}\end{array}\right\}\rightarrow\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx 1.898\cdot\log_{10}\left(\frac{R}{A_{27}+9.779}\right)\\ A_{27}\approx 0.202\cdot R\cdot\frac{\varepsilon}{D}\end{array}\right\} \quad (25)$$

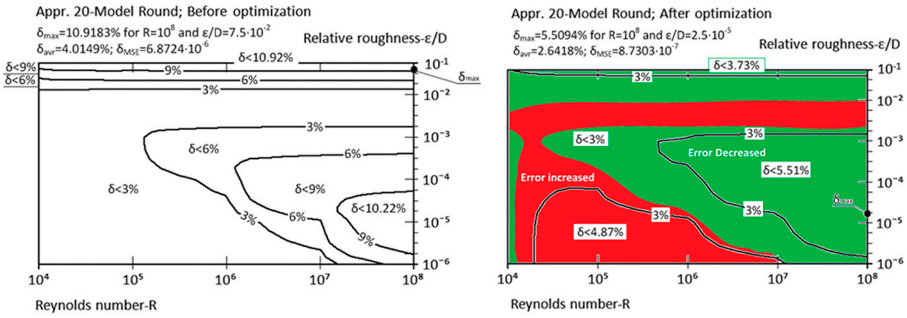
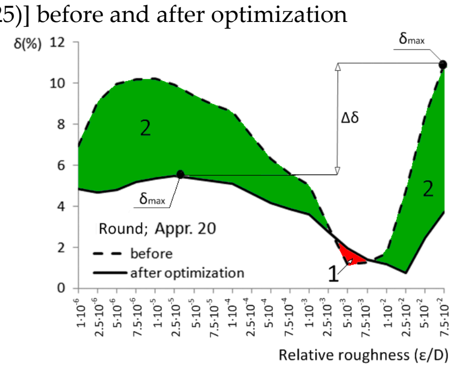


Figure 40. Relative error of Round [Appr. 20; Eq. (25)] before and after optimization



δmax: 10.9183% → 5.5094%

δavr: 4.0149% → 2.6418%

δMSE: 6.8724·10^-6 → 8.7303·10^-7

Figure 41. Performed genetic algorithm optimization of Barr [Appr. 20; Eq. (25)]

Chen approximation [Appr. 21]. Relevant parameters and errors related to Approximation by Chen [36] after and before optimization (26); [Appr. 21], are given in Figures 42-43.

$$\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx -2\cdot\log_{10}\left(\frac{1}{3.7065}\cdot\frac{\varepsilon}{D}-\frac{5.0452}{R}\cdot a_{28}\right)\\ a_{28}\approx\log_{10}\left(\frac{1}{2.8257}\cdot\left(\frac{\varepsilon}{D}\right)^{1.1098}+\frac{5.8506}{R^{0.8981}}\right)\end{array}\right\}\rightarrow\left.\begin{array}{l}\frac{1}{\sqrt{\lambda}}\approx -2.003\cdot\log_{10}\left(\frac{1}{3.689}\cdot\frac{\varepsilon}{D}-\frac{4.933}{R}\cdot A_{28}\right)\\ A_{28}\approx\log_{10}\left(\frac{1}{2.762}\cdot\left(\frac{\varepsilon}{D}\right)^{1.109}+\frac{5.89}{R^{0.923}}\right)\end{array}\right\} \quad (26)$$

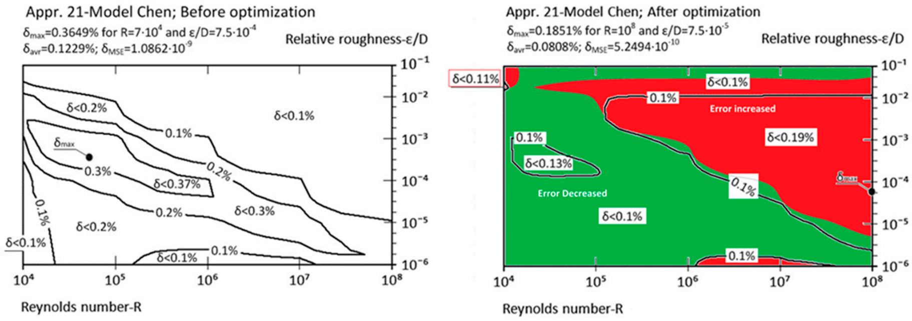
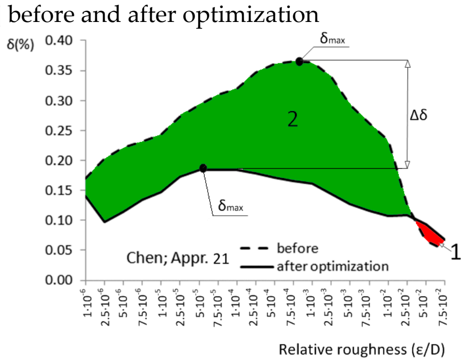


Figure 42. Relative error of Chen [Appr. 21; Eq. (26)] before and after optimization



δmax: 0.3649% → 0.1851%

δavr: 0.1229% → 0.0808%

δMSE: 1.0862·10^-9 → 5.2494·10^-10

Figure 43. Performed genetic algorithm optimization of Chen [Appr. 21; Eq. (26)]

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Swamee and Jain approximation [Appr. 22]. Relevant parameters and errors related to Approximation by Swamee and Jain [37] after and before optimization (27); [Appr. 22], are given in Figures 44-45.

$$\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx -2\cdot\log_{10}\left(\frac{5.74}{R^{0.9}}+a_{29}\right)\\ a_{29} &\approx \frac{1}{3.7}\cdot\frac{\varepsilon}{D}\end{aligned}\right\}\rightarrow\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx -1.972\cdot\log_{10}\left(\frac{5.828}{R^{0.916}}+A_{29}\right)\\ A_{29} &\approx \frac{1}{4.04}\cdot\frac{\varepsilon}{D}\end{aligned}\right\} \quad (27)$$

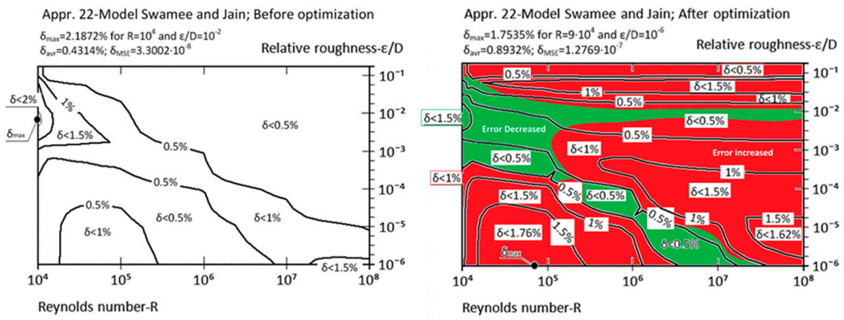

Figure 44. Relative error of Swamee and Jain [Appr. 22; Eq. (27)] before and after optimization

δmax: 2.1872% → 1.7535%

δavr: 0.4314% → 0.8932%

δMSE: $3.3002\cdot10^{-8} \rightarrow 1.2769\cdot10^{-7}$

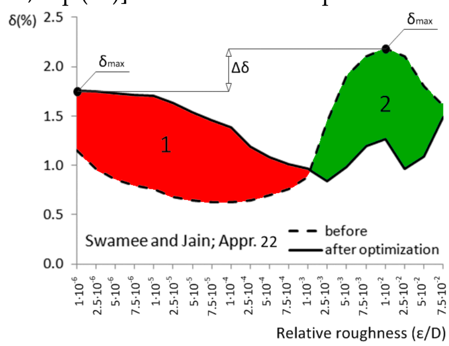

Figure 45. Performed genetic algorithm optimization of Swamee and Jain [Appr. 22; Eq. (27)]

Eck approximation [Appr. 23]. Relevant parameters and errors related to Approximation by Eck [38] after and before optimization (28); [Appr. 23], are given in Figures 46-47.

$$\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx -2\cdot\log_{10}\left(\frac{15}{R}+a_{30}\right)\\ a_{30} &\approx \frac{1}{3.715}\cdot\frac{\varepsilon}{D}\end{aligned}\right\}\rightarrow\left.\begin{aligned}\frac{1}{\sqrt{\lambda}} &\approx -1.963\cdot\log_{10}\left(\frac{14.064}{R}+A_{30}\right)\\ A_{30} &\approx \frac{1}{4.034}\cdot\frac{\varepsilon}{D}\end{aligned}\right\} \quad (28)$$

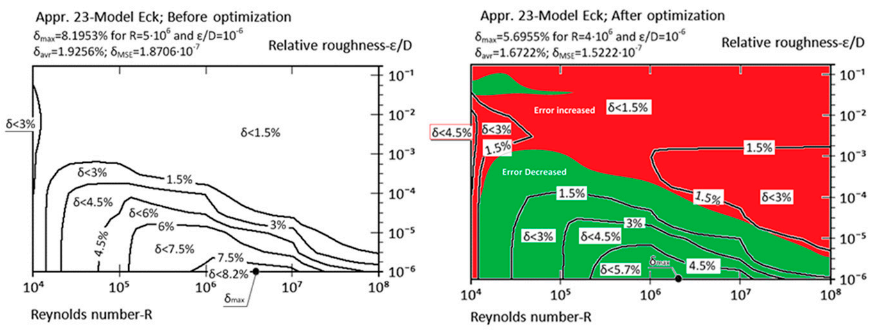

Figure 46. Relative error of Eck [Appr. 23; Eq. (28)] before and after optimization

δmax: 8.1953% → 5.6955%

δavr: 1.9256% → 1.6722%

δMSE: $1.18706\cdot10^{-7} \rightarrow 1.5222\cdot10^{-7}$

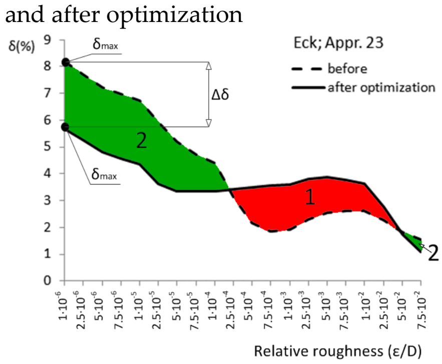

Figure 47. Performed genetic algorithm optimization of Eck [Appr. 23; Eq. (28)]

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Wood approximation [Appr. 24]. Relevant parameters and errors related to Approximation by Wood [39] after and before optimization (29); [Appr. 24], are given in Figures 48-49.

$$\left.\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(a_{31}+88\cdot\left(\frac{\varepsilon}{D}\right)^{0.44}\cdot R^{-a_{32}}\right)^{-2} \\ a_{31}=0.094\cdot\left(\frac{\varepsilon}{D}\right)^{0.225}+0.53\cdot\frac{\varepsilon}{D} \\ a_{32}\approx 1.62\cdot\left(\frac{\varepsilon}{D}\right)^{0.134} \end{array}\right\} \rightarrow \left.\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(A_{31}+85.005\cdot\left(\frac{\varepsilon}{D}\right)^{0.33}\cdot R^{-A_{32}}\right)^{-2} \\ A_{31}=0.094\cdot\left(\frac{\varepsilon}{D}\right)^{0.209}+0.376\cdot\frac{\varepsilon}{D} \\ A_{32}\approx 1.501\cdot\left(\frac{\varepsilon}{D}\right)^{0.101} \end{array}\right\} \quad (29)$$

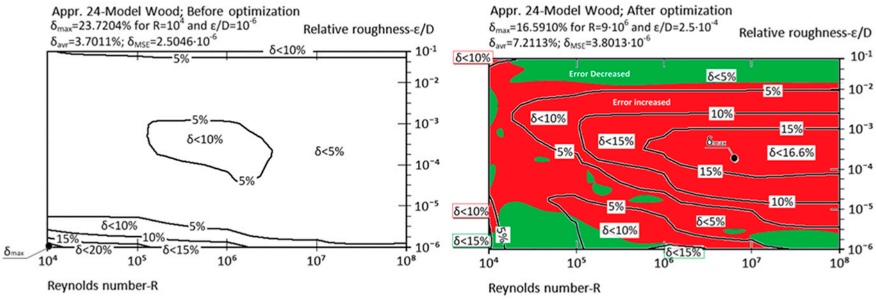
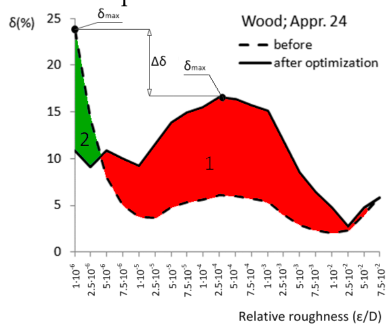


Figure 48. Relative error of Wood [Appr. 24; Eq. (29)] before and after optimization



δmax: 23.7204% → 16.5910%

δavr: 3.7011% → 7.2113%

δMSE: $2.5046\cdot10^{-6}$ → $3.8013\cdot10^{-6}$

Figure 49. Performed genetic algorithm optimization of Wood [Appr. 24; Eq. (29)]

Moody approximation [Appr. 25]. Relevant parameters and errors related to Approximation by Moody [40] after and before optimization (30); [Appr. 25], are given in Figures 50-51.

$$\left.\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(0.0055\cdot\left(1+a_{33}\right)\right)^{-2} \\ a_{33}\approx\left(2\cdot10^{4}\cdot\frac{\varepsilon}{D}+\frac{10^{6}}{R}\right)^{0.333} \end{array}\right\} \rightarrow \left.\begin{array}{l} \frac{1}{\sqrt{\lambda}} \approx \left(0.006\cdot\left(0.775+A_{33}\right)\right)^{-2} \\ A_{33}\approx\left(2.443\cdot10^{4}\cdot\frac{\varepsilon}{D}+\frac{10^{6}}{R}\right)^{0.343} \end{array}\right\} \quad (30)$$

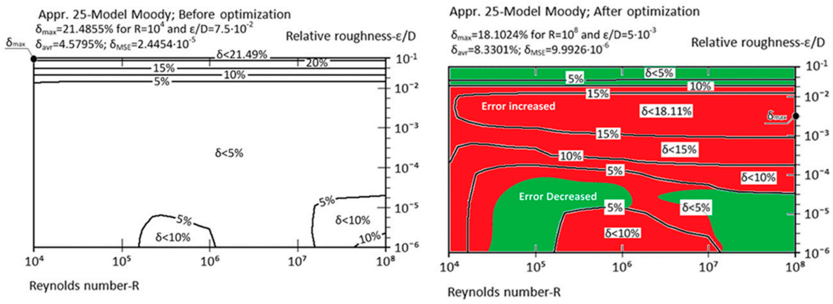
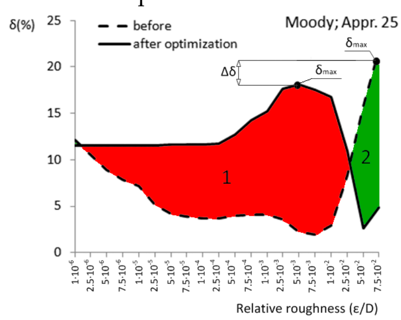


Figure 50. Relative error of Moody [Appr. 25; Eq. (30)] before and after optimization



δmax: 21.4855% → 18.1024%

δavr: 4.5795% → 8.3301%

δMSE: $2.4454\cdot10^{-5}$ → $9.9926\cdot10^{-6}$

Figure 51. Performed genetic algorithm optimization of Moody [Appr. 25; Eq. (30)]

4. Conclusions

Using genetic algorithms in order to increase the accuracy of available approximations of the Colebrook equation for flow friction, the numerical values of empirical parameters in 25 existing models of approximations were changed while computation burden remains the same. Using the value of decreased maximal relative error, Δδ, and change of relative error over the entire domain of the Reynolds number (R) and relative roughness of inner pipe surface (ε/D), success of genetic optimization is summarized in Table 1.

Table 1. Maximal relative error of the explicit approximations of the Colebrook-White equation before and after genetic optimization

Approximation No.	With original parameters	After genetic optimization	Estimation of improvement	Source
Appr. 11 - Eq. (16)	0.1345%	0.0083%	extremely successful	Romeo et al. [27,42]
Appr. 14 - Eq. (19)	0.1385%	0.0026%	extremely successful	Serghides [30,42]
Appr. 10 - Eq. (15)	0.8007%	0.1473%	successful	Sonnad and Goudar [26,43]
Appr. 2 - Eq. (7)	3.1560%	1.2871%	successful	Brkić [19]
Appr. 9 - Eq. (14)	0.1385%	0.0797%	successful	Buzzelli [25]
Appr. 15 - Eq. (20)	0.3543%	0.2739%	successful	Serghides [30]
Appr. 17 - Eq. (22)	0.1385%	0.0831%	successful	Zigrang and Sylvester [32]
Appr. 18 - Eq. (23)	1.0075%	0.7496%	successful	Zigrang and Sylvester [32]
Appr. 20 - Eq. (25)	10.9183%	5.5094%	successful	Round [34]
Appr. 21 - Eq. (26)	0.3649%	0.1851%	successful	Chen [36]
Appr. 1 - Eq. (6)	2.2065%	1.2868%	moderately successful	Brkić [19]
Appr. 3 - Eq. (8)	2.0715%	1.3326%	moderately successful	Brkić [20]
Appr. 4 - Eq. (9)	2.0111%	1.2866%	moderately successful	Brkić [20]
Appr. 12 - Eq. (17)	2.0651%	1.5018%	moderately successful	Manadilli [28]
Appr. 13 - Eq. (18)	27.5074%	18.4800%	moderately successful	Chen J.J.J. [29]
Appr. 16 - Eq. (21)	1.4083%	1.1098%	moderately successful	Haaland [31]
Appr. 22 - Eq. (27)	2.1872%	1.7535%	moderately successful	Swamee and Jain [37]
Appr. 23 - Eq. (28)	8.1953%	5.6955%	moderately successful	Eck [21]
Appr. 5 - Eq. (10)	0.6167%	0.5669%.	not very successful	Fang et al. [38]
Appr. 6 - Eq. (11)	2.8962%	2.5947%	not very successful	Ghanbari et al. [22]
Appr. 7 - Eq. (12)	0.8248%	0.7312%	not very successful	Papaevangelou et al. [23]
Appr. 8 - Eq. (13)	4.7858%	3.1259%	not very successful	Avci and Karagoz [24]
Appr. 19 - Eq. (24)	0.2774%	0.2644%	not very successful	Barr [33]
Appr. 24 - Eq. (29)	23.7204%	16.5910%	not very successful	Wood [39]
Appr. 25 - Eq. (30)	21.4855%	18.1024%	not very successful	Moody [40]

During this study, it was found that criterion from Winning and Coole [16] about the accuracy of approximations using value of mean square error should be modified as: very small error is lower than 10^{-10}, small is between 10^{-10} and 10^{-8}, medium is between 10^{-8} and $5\cdot10^{-7}$, and large is above $5\cdot10^{-7}$. Criterion of accuracy using value of maximal relative error δ_{max} should be set as: very small error is lower than 0.2%, small is between 0.2% and 1%, medium is between 1% and 3%, and large is above 3% (extremely large above 5%). Also it was found that error distribution, set as a criterion in Winning and Coole [16], does not depend only on the model of approximation, but it changes equally with change of values of parameters.

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Aside for the Colebrook equation, the presented methodology can be used to fit with the raw and updated measured data, all similar empirical equations which cover the same region of turbulent flow [54,55].

The results are relevant for all engineering fields which deal with fluid flow through pipes and related calculation of hydraulic flow friction.

Supplementary Materials: Excel and MATLAB codes of the approximations presented in this paper are available.

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Author Contributions: Dejan Brkić has scientific interest in hydraulics and he prepared data related for flow friction, checked accuracy of the final results, drawn diagrams and wrote the manuscript. Žarko Ćojbašić has scientific interest in artificial intelligence and he performed optimization using genetic algorithms.

Conflicts of Interest: The authors declare no conflict of interest.

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