 doi:10.20944/preprints201702.0058.v2

Exponential and polynomial decay for a laminated beam with Fourier's type heat conduction

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In this paper, we study the well-posedness and asymptotics of a one-dimensional thermoelastic laminated beam system either with or without structural damping, where the heat conduction is given by Fourier's law effective in the rotation angle displacements. We show that the system is well-posed by using Lumer-Philips theorem, and prove that the system is exponentially stable if and only if the wave speeds are equal, by using the perturbed energy method and Gearhart-Herbst-Prüss-Huang theorem. Furthermore, we show that the system with structural damping is polynomially stable provided that the wave speeds are not equal, by using the second-order energy method.

Keywords: laminated beam, Fourier's law, exponential stability, lack of exponential stability, polynomial stability.

AMS Subject Classification (2000): 35B40, 74F05, 93D20.

1 Introduction

With the increasing demand of advanced performance, the vibration suppression of the laminated beams has been one of the main research topics in smart materials and structures. These composite laminates usually have superior structural properties such as adaptability. The design of their piezoelectric materials can be used as both actuators and sensors [27]. Hansen and Spies in [12] derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, i.e.,

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi - \varphi_x) = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ 3I_\rho w_{tt} - 3Dw_{xx} + 3G(\psi - \varphi_x) + 4\gamma w + 4\beta w_t = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{1.1}$$

where $\rho, G, I_\rho, D, \gamma$ are positive constant coefficients, ρ is the density of the beams, G is the shear stiffness, I_ρ is the mass moment of inertia, D is the flexural rigidity, γ is the adhesive stiffness of the beams, and $\beta \geq 0$ is the adhesive damping parameter. The function φ denotes the transverse displacement of the beam which departs from its equilibrium position, ψ represents the rotation angle, w is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x, $3w-\psi$ denotes the effective rotation angle, $(1.1)_3$ describes the dynamics of the slip. If $\beta = 0$, $(1.1)_3$ describes the coupled laminated beams without structural damping at the interface. If $\beta \neq 0$, the adhesion at the interface supplies a restoring force proportion to the interfacial slip.

In recent years, an increasing interest has been developed to determine the asymptotic behavior of the solution of several laminated beam problems. For example, Wang et al. [27] considered system (1.1) with the cantilever boundary conditions and two different wave speeds ($\sqrt{G/\rho}$ and $\sqrt{D/I_\rho}$). The authors proved the well-posedness and pointed out that system (1.1) can obtain the asymptotic stability but it does not reach the exponential stability due to the action of the slip w. Furthermore, to achieve the exponential decay result, the authors added an additional boundary control such that the boundary conditions become

$$\varphi(0,t)=\xi(0,t)=w(0,t)=0,\ w_x(1,t)=0,$$

$$3w(1,t)-\xi(1,t)-\varphi_x(1,t)=u_1(t):=k_1\varphi_t(1,t),$$
$$\xi_x(1,t)=u_2(t):=-k_2\xi_t(1,t),$$

where $\xi=3w-\psi$. Cao et al. [9] considered the system (1.1) with following boundary conditions

$$\psi(0,t)-\varphi_x(0,t)=u_1(t):=-k_1\varphi_t(0,t)-\varphi(0,t),$$

$$3w_x(1,t)-\psi_x(1,t)=u_2(t):=-k_2\xi_t(1,t)-\xi(1,t),$$

where $\xi=3w-\psi$. The authors obtained an exponential stability result provided $k_1\neq\sqrt{\rho/G}$ and $k_2\neq\sqrt{I_\rho/D}$. More importantly, the authors proved that the dominant part of the system is itself exponentially stable. Raposo [25] considered system (1.1) with two frictional dampings of the form

$$\begin{cases}\rho\varphi_{tt}+G(\psi-\varphi_x)_x+k_1\varphi_t=0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_\rho(3w-\psi)_{tt}-D(3w-\psi)_{xx}-G(\psi-\varphi_x)+k_2(3w-\psi)_t=0, & (x,t)\in(0,1)\times(0,+\infty),\\ 3I_\rho w_{tt}-3Dw_{xx}+3G(\psi-\varphi_x)+4\gamma w+4\beta w_t=0, & (x,t)\in(0,1)\times(0,+\infty)\end{cases}\tag{1.2}$$

and obtained the exponential decay result under appropriate initial and boundary conditions. More recently, Apalara [3] investigated a laminated beam with structural damping under Cattaneos law of heat conduction, and proved the exponential and polynomial stability results depend on a stability number. However, the case of the absence of structural damping was left as an open problem. The present authors [18] studied the well-posedness and asymptotic stability of a thermoelastic laminated beam with past history. For the system with structural damping, without any restriction on the speeds of wave propagations, we proved the exponential and polynomial stabilities which depend on the behavior of the kernel function of the history term. For the system without structural damping, we proved the exponential and polynomial stabilities in case of equal speeds and lack of exponential stability in case of non-equal speeds.

It is easy to find that if the slip w is assumed to be identically zero, then the first two equations of system (1.1) can be reduced exactly to the Timoshenko beam system. For the case of the Timoshenko beam with Fourier's law, many authors have shown various decay estimates depending on the wave speeds. Rivera and Racke [20] studied the Timoshenko system with thermoelastic dissipation, i.e.,

$$\begin{cases}\rho_1\varphi_{tt}+k(\psi-\varphi_x)_x=0, & (x,t)\in(0,L)\times(0,+\infty),\\ \rho_2\psi_{tt}-b\psi_{xx}+k(\psi-\varphi_x)-\gamma\theta_x=0, & (x,t)\in(0,L)\times(0,+\infty),\\ \rho_3\theta_t-\kappa\theta_{xx}-\gamma\psi_{tx}=0, & (x,t)\in(0,L)\times(0,+\infty),\end{cases}\tag{1.3}$$

with positive constants $\rho_1,\rho_2,\rho_3,k,b,\gamma,\kappa,$ where θ models the temperature difference. The authors showed that the exponential stability holds if and only if the wave speeds are equal

$\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$. Júnior and Rivera [2] considered a new coupling to the thermoelastic Timoshenko beam of the form

$$\begin{cases} \rho_1\varphi_{tt} + k(\psi - \varphi_x)_x + \sigma\theta_x = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\psi - \varphi_x) + \sigma\theta = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_3\theta_t - \gamma\theta_{xx} - \sigma(\psi - \varphi_x)_t = 0, & (x,t) \in (0,L) \times (0,+\infty) \end{cases} \tag{1.4}$$

for $\sigma > 0$ from thermo-elasticity theory. The authors showed this system is exponentially stable if and only if the wave speeds are equal $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$. On the contrary, the authors obtained the polynomially stability depending on the different boundary conditions. For system (1.4) with Dirichlet boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi(t,0) = \psi(t,L) = \theta(t,0) = \theta(t,L) = 0,$$

the authors obtained that the semigroup decays as $\frac{1}{\sqrt[4]{t}}$. For system (1.4) with Dirichlet-Neumann boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta_x(t,0) = \theta_x(t,L) = 0,$$

the authors obtained that the semigroup decays as $\frac{1}{\sqrt{t}}$. We refer the reader to [1, 5, 6, 8, 10, 11, 14, 16, 17, 19, 24, 26] for some other related results.

Motivated by the above results, we intend to study the well-posedness and the asymptotic stability of the thermoelastic laminated beam system either with or without structural damping, where the heat flux is given by Fourier's law. The system is written as

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3w - \psi)_{tt} - D(3w - \psi)_{xx} - G(\psi - \varphi_x) + \sigma\theta_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ k\theta_t - \tau\theta_{xx} + \sigma(3w - \psi)_{tx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{1.5}$$

where $\rho, G, I_\rho, D, \sigma, \gamma, k, \tau$ are positive constant coefficients, $\beta \geq 0$. We consider following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \psi(x,0) = \psi_0(x), w(x,0) = w_0(x), \theta(x,0) = \theta_0(x), & x \in [0,1], \\ \varphi_t(x,0) = \varphi_1(x), \psi_t(x,0) = \psi_1(x), w_t(x,0) = w_1(x), & x \in [0,1], \\ \varphi(1,t) = \psi_x(1,t) = w_x(1,t) = \theta(1,t) = 0, & t \in [0,+\infty), \\ \varphi_x(0,t) = \psi(0,t) = w(0,t) = \theta_x(0,t) = 0, & t \in [0,+\infty). \end{cases} \tag{1.6}$$

By using Lumer-Philips theorem, we first prove the well-posedness result. By using the perturbed energy method and Gearhart-Herbst-Prüss-Huang theorem, we then prove that the system is exponentially stable if and only if $\frac{\rho}{G} = \frac{I_\rho}{D}$. Furthermore, by using the second-order energy method, we show that the system with structural damping is polynomially stable provided that $\frac{\rho}{G} \neq \frac{I_\rho}{D}$. The main difficulties in carry out this paper is the appearance of the Fourier's law of heat conduction and the possible absence of structural damping. For this purpose, we shall use appropriated multiplies to build equivalent Lyapunov functionals.

 doi:10.20944/preprints201702.0058.v2

We now briefly sketch the outline of the paper. In Section 2, we state and prove the well-posedness of problem (1.5)-(1.6). In Section 3, we establish an exponential stability result of the energy. In Section 4, the lack of exponential stability has been studied. In Section 5, we state and prove the polynomial stability. Section 6 is devoted to the conclusion and open problem. Throughout this paper, we use c_i or C_i to denote generic positive constants.

2 Well-posedness (for $\beta \geq 0$)

In this section, we prove the well-posedness of problem (1.5)-(1.6) by using Lumer-Philips theorem. Firstly, we introduce the vector function

$$U = (\varphi, \varphi_t, 3w - \psi, (3w - \psi)_t, w, w_t, \theta)^T.$$

Then system (1.5)-(1.6) can be written as

$$\begin{cases} \partial_t U = \mathscr{A} U, \\ U(x,0) = U_0(x) = (\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0)^T, \end{cases} \tag{2.1}$$

where $\mathscr{A}$ is a linear operator defined by

$$\mathscr{A} U = \begin{pmatrix} \varphi_t \\ -\dfrac{G}{\rho}(\psi - \varphi_x)_x \\ (3w - \psi)_t \\ \dfrac{G}{I_\rho}(\psi - \varphi_x) + \dfrac{D}{I_\rho}(3w - \psi)_{xx} - \dfrac{\sigma}{I_\rho}\theta_x \\ w_t \\ -\dfrac{G}{I_\rho}(\psi_{-}\varphi_x) - \dfrac{4\gamma}{3I_\rho} w - \dfrac{4\beta}{3I_\rho} w_t + \dfrac{D}{I_\rho} w_{xx} \\ \dfrac{\tau}{k}\theta_{xx} - \dfrac{\sigma}{k}(3w - \psi)_{tx} \end{pmatrix}.$$

We consider the following spaces:

$$H_*^1(0,1) = \left\{ \eta \,\middle|\, \eta \in H^1(0,1) : \eta(0) = 0 \right\}, \quad \tilde{H}_*^1(0,1) = \left\{ \eta \,\middle|\, \eta \in H^1(0,1) : \eta(1) = 0 \right\},$$

$$H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1), \quad \tilde{H}_*^2(0,1) = H^2(0,1) \cap \tilde{H}_*^1(0,1),$$

$$\tilde{L}_*^2(0,1) = \left\{ \eta \,\middle|\, \eta \in L^2(0,1) : \eta(1) = 0 \right\},$$

and

$$\mathscr{H} = \tilde{H}_*^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times \tilde{L}_*^2(0,1), \tag{2.2}$$

equipped with the inner product

$$\left(U, \tilde{U}\right)_{\mathscr{H}} = \rho \int_0^1 \varphi_t \tilde{\varphi}_t \mathrm{d}x + I_\rho \int_0^1 (3w - \psi)_t (3\tilde{w} - \tilde{\psi})_t \mathrm{d}x + 3I_\rho \int_0^1 w_t \tilde{w}_t \mathrm{d}x + k \int_0^1 \theta \tilde{\theta} \mathrm{d}x$$

 doi:10.20944/preprints201702.0058.v2

$$
\begin{aligned}
&+ G\int_0^1 (\psi - \varphi_x)(\tilde{\psi} - \tilde{\varphi}_x)\mathrm{d}x + D\int_0^1 (3w-\psi)_x(3\tilde{w}-\tilde{\psi})_x\mathrm{d}x + 4\gamma\int_0^1 w\tilde{w}\mathrm{d}x\\
&+ 3D\int_0^1 w_x\tilde{w}_x\mathrm{d}x.
\end{aligned}
$$

Then, the domain of $\mathscr{A}$ is given by

$$
\begin{aligned}
D(\mathscr{A}) = \Big\{ & U\in\mathscr{H} \mid \varphi\in\tilde{H}^2_*(0,1), 3w-\psi, w\in H^2_*(0,1), \theta\in\tilde{H}^1_*(0,1), \varphi_t\in\tilde{H}^1_*(0,1),\\
& 3w_t-\psi_t, w_t\in H^1_*(0,1), \varphi_x(0,t)=\theta_x(0,t)=0, \psi_x(1,t)=w_x(1,t)=0\Big\}.
\end{aligned}
$$

The well-posedness of problem (2.1) is ensured by

Theorem 2.1 *Assume that $\beta\geq 0$ holds. Let $U_0\in\mathscr{H}$, then problem* (2.1) *exists a unique weak solution $U\in C(\mathbb{R}^+;\mathscr{H})$. Moreover, if $U_0\in D(\mathscr{A})$, then*

$$
U\in C(\mathbb{R}^+;D(\mathscr{A}))\cap C^1(\mathbb{R}^+;\mathscr{H}).
$$

Proof. To obtain the above result, we need to prove that $\mathscr{A}$: $D(\mathscr{A})\to\mathscr{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathscr{A}$ is dissipative and $Id-\mathscr{A}$ is surjective.

Step 1. $\mathscr{A}$ is dissipative.

For any $U\in D(\mathscr{A})$, by using the inner product and integration by parts, we have

$$
\begin{aligned}
\langle\mathscr{A}U,U\rangle_{\mathscr{H}} =& -\int_0^1 G\left(\psi-\varphi_x\right)_x\varphi_t\mathrm{d}x + \int_0^1 \left[D(3w-\psi)_{xx} + G(\psi-\varphi_x) - \sigma\theta_x\right](3w-\psi)_t\mathrm{d}x\\
&+ \int_0^1 \left[3Dw_{xx} - 3G(\psi-\varphi_x) - 4\gamma w - 4\beta w_t\right]w_t\mathrm{d}x + \int_0^1\left[\tau\theta_{xx} - \sigma(3w-\psi)_{tx}\right]\theta\mathrm{d}x\\
&+ G\int_0^1 (\psi-\varphi_x)(\psi_t-\varphi_{xt})\mathrm{d}x + D\int_0^1 (3w-\psi)_{xt}(3w-\psi)_x\mathrm{d}x + 4\gamma\int_0^1 w_t w\mathrm{d}x\\
&+ 3D\int_0^1 w_{xt}w_x\mathrm{d}x\\
=& -\tau\int_0^1\theta_x^2\mathrm{d}x - 4\beta\int_0^1 w_t^2\mathrm{d}x\\
\leq& 0.
\end{aligned}\tag{2.3}
$$

Hence, $\mathscr{A}$ is a dissipative operator.

Step 2. $Id-\mathscr{A}$ is surjective.

To prove that the operator $Id-\mathscr{A}$ is surjective, that is, for any $F=(f_1,\cdots,f_7)\in\mathscr{H}$, there exists $V=(v_1,\cdots,v_7)\in D(\mathscr{A})$ satisfying

$$
(Id-\mathscr{A})V = F, \tag{2.4}
$$

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which is equivalent to

$$\begin{cases} v_1 - v_2 = f_1, \\ \rho v_2 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho f_2, \\ v_3 - v_4 = f_3, \\ I_\rho v_4 + G\partial_x v_1 + Gv_3 - D\partial_{xx}v_3 - 3Gv_5 + \sigma\partial_x v_7 = I_\rho f_4, \\ v_5 - v_6 = f_5, \\ \left(I_\rho + \frac{4\beta}{3}\right) v_6 - G\partial_x v_1 - Gv_3 + \left(3G + \frac{4\gamma}{3}\right) v_5 - D\partial_{xx}v_5 = I_\rho f_6, \\ kv_7 - \tau\partial_{xx}v_7 + \sigma\partial_x v_4 = kf_7. \end{cases} \tag{2.5}$$

$(2.5)_1$, $(2.5)_3$ and $(2.5)_5$ give

$$\begin{cases} v_2 = v_1 - f_1, \\ v_4 = v_3 - f_3, \\ v_6 = v_5 - f_5. \end{cases} \tag{2.6}$$

Inserting (2.6) into $(2.5)_2$, $(2.5)_4$, $(2.5)_6$ and $(2.5)_7$, we get

$$\begin{cases} \rho v_1 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho(f_1 + f_2), \\ (I_\rho + G)v_3 + G\partial_x v_1 - D\partial_{xx}v_3 - 3Gv_5 + \sigma\partial_x v_7 = I_\rho(f_3 + f_4), \\ \left(I_\rho + 3G + \frac{4\beta}{3} + \frac{4\gamma}{3}\right) v_5 - G\partial_x v_1 - Gv_3 - D\partial_{xx}v_5 = I_\rho(f_5 + f_6) + \frac{4\beta}{3} f_5, \\ kv_7 + \sigma\partial_x v_3 - \tau\partial_{xx}v_7 = \sigma\partial_x f_3 + kf_7. \end{cases} \tag{2.7}$$

Multiplying $(2.7)_1$-$(2.7)_4$ by $\tilde{v}_1$, $\tilde{v}_3$, $3\tilde{v}_5$ and $\tilde{v}_7$ respectively, and integrating over $(0, 1)$, we arrive at

$$\begin{cases} \int_0^1 \rho v_1\tilde{v}_1 \mathrm{d}x - \int_0^1 G\partial_{xx}v_1\tilde{v}_1 \mathrm{d}x - \int_0^1 G\partial_x v_3\tilde{v}_1 \mathrm{d}x + \int_0^1 3G\partial_x v_5\tilde{v}_1 \mathrm{d}x = \int_0^1 \rho(f_1 + f_2)\tilde{v}_1 \mathrm{d}x, \\ \int_0^1 (I_\rho + G)v_3\tilde{v}_3 \mathrm{d}x + \int_0^1 G\partial_x v_1\tilde{v}_3 \mathrm{d}x - \int_0^1 D\partial_{xx}v_3\tilde{v}_3 \mathrm{d}x - \int_0^1 3Gv_5\tilde{v}_3 \mathrm{d}x + \int_0^1 \sigma\partial_x v_7\tilde{v}_3 \mathrm{d}x \\ = \int_0^1 I_\rho(f_3 + f_4)\tilde{v}_3 \mathrm{d}x, \\ \int_0^1 (3I_\rho + 9G + 4\beta + 4\gamma)\, v_5\tilde{v}_5 \mathrm{d}x - \int_0^1 3G\partial_x v_1\tilde{v}_5 \mathrm{d}x - \int_0^1 3Gv_3\tilde{v}_5 \mathrm{d}x - \int_0^1 3D\partial_{xx}v_5\tilde{v}_5 \mathrm{d}x \\ = \int_0^1 3I_\rho(f_5 + f_6)\tilde{v}_5 \mathrm{d}x + \int_0^1 4\beta f_5\tilde{v}_5 \mathrm{d}x, \\ \int_0^1 kv_7\tilde{v}_7 \mathrm{d}x + \int_0^1 \sigma\partial_x v_3\tilde{v}_7 \mathrm{d}x - \int_0^1 \tau\partial_{xx}v_7\tilde{v}_7 \mathrm{d}x = \int_0^1 \sigma\partial_x f_3\tilde{v}_7 \mathrm{d}x + \int_0^1 kf_7\tilde{v}_7 \mathrm{d}x. \end{cases} \tag{2.8}$$

The sum of the equations in (2.8) gives the following variational formulation:

$$a\left((v_1, v_3, v_5, v_7)^T, (\tilde{v}_1, \tilde{v}_3, \tilde{v}_5, \tilde{v}_7)^T\right) = \tilde{a}\left((\tilde{v}_1, \tilde{v}_3, \tilde{v}_5, \tilde{v}_7)^T\right),$$

$$\forall\ (\tilde v_1,\tilde v_3,\tilde v_5,\tilde v_7)^T\in \tilde H^1_*(0,1)\times H^1_*(0,1)\times H^1_*(0,1)\times \tilde L^2_*(0,1), \tag{2.9}$$

where

$$\begin{aligned}
&a\left((v_1,v_3,v_5,v_7)^T,(\tilde v_1,\tilde v_3,\tilde v_5,\tilde v_7)^T\right)\\
=&\int_0^1 G(-\partial_x v_1-v_3+3v_5)(-\partial_x\tilde v_1-\tilde v_3+3\tilde v_5)\mathrm{d}x+\int_0^1\rho v_1\tilde v_1\mathrm{d}x+\int_0^1 I_\rho v_3\tilde v_3\mathrm{d}x\\
&+\int_0^1(3I_\rho+4\gamma+4\beta)v_5\tilde v_5\mathrm{d}x+\int_0^1 kv_7\tilde v_7\mathrm{d}x+\int_0^1 D\partial_x v_3\partial_x\tilde v_3\mathrm{d}x+\int_0^1 3D\partial_x v_5\partial_x\tilde v_5\mathrm{d}x\\
&+\int_0^1\tau\partial_x v_7\partial_x\tilde v_7\mathrm{d}x+\int_0^1\sigma(\partial_x v_7)\tilde v_3\mathrm{d}x+\int_0^1\sigma(\partial_x v_3)\tilde v_7\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\tilde a\left((\tilde v_1,\tilde v_3,\tilde v_5,\tilde v_7)^T\right)\\
=&\int_0^1\left[\rho(f_1+f_2)\tilde v_1+I_\rho(f_3+f_4)\tilde v_3+3I_\rho(f_5+f_6)\tilde v_5+4\beta f_5\tilde v_5+\sigma\partial_x f_3\tilde v_7+kf_7\tilde v_7\right]\mathrm{d}x.
\end{aligned}$$

Now, we introduce the Hilbert space $V=\tilde H^1_*(0,1)\times H^1_*(0,1)\times H^1_*(0,1)\times\tilde L^2_*(0,1)$ equipped with the norm

$$\|(v_1,v_3,v_5,v_7)\|_V^2=\|-\partial_x v_1-v_3+3v_5\|_2^2+\|v_1\|_2^2+\|\partial_x v_3\|_2^2+\|\partial_x v_5\|_2^2+\|\partial_x v_7\|_2^2.$$

It is clear that $a(\cdot,\cdot)$ and $\tilde a(\cdot)$ are bounded. Furthermore, we can obtain that there exists a positive constant m such that

$$\begin{aligned}
&a\left((v_1,v_3,v_5,v_7)^T,(v_1,v_3,v_5,v_7)^T\right)\\
=&\int_0^1 G(-\partial_x v_1-v_3+3v_5)^2\mathrm{d}x+\int_0^1\rho v_1^2\mathrm{d}x+\int_0^1 I_\rho v_3^2\mathrm{d}x+\int_0^1(3I_\rho+4\gamma+4\beta)v_5^2\mathrm{d}x\\
&+\int_0^1 kv_7^2\mathrm{d}x+\int_0^1 D(\partial_x v_3)^2\mathrm{d}x+\int_0^1 3D(\partial_x v_5)^2\mathrm{d}x+\tau\int_0^1(\partial_x v_7)^2\mathrm{d}x\\
\geq& m\|(v_1,v_3,v_5,v_7)\|_V^2,
\end{aligned}$$

which implies that $a(\cdot,\cdot)$ is coercive.

Hence, we assert that $a(\cdot,\cdot)$ is a bilinear continuous coercive form on $V\times V$, and $\tilde a(\cdot)$ is a linear continuous form on V. Applying Lax-Milgram theorem [22], we obtain that (2.8) has a unique solution $(v_1,v_3,v_5,v_7)^T\in V$. Then, by substituting v_1,v_3,v_5 into (2.6), we obtain

$$v_2\in\tilde H^1_*(0,1),v_4\in H^1_*(0,1),v_6\in H^1_*(0,1).$$

Next, we turn to show that

$$v_1\in\tilde H^2_*(0,1),v_3\in H^2_*(0,1),v_5\in H^2_*(0,1),v_7\in\tilde H^1_*(0,1),$$

$$\partial_x v_1(0)=\partial_x v_3(1)=\partial_x v_5(1)=\partial_x v_7(0)=0.$$

Furthermore, if $(\tilde v_3,\tilde v_5,\tilde v_7)\equiv(0,0,0)\in H^1_*(0,1)\times H^1_*(0,1)\times\tilde L^2_*(0,1)$, then (2.9) reduces to

$$\int_0^1 G\partial_{xx}v_1\tilde v_1\mathrm{d}x=\int_0^1\rho v_1\tilde v_1\mathrm{d}x-\int_0^1 G\partial_x v_3\tilde v_1\mathrm{d}x+\int_0^1 3G\partial_x v_5\tilde v_1\mathrm{d}x-\int_0^1\rho(f_1+f_2)\tilde v_1\mathrm{d}x, \tag{2.10}$$

for all $\tilde{v}_1 \in \tilde{H}^1_*(0,1)$, which implies

$$G\partial_{xx}v_1 = \rho v_1 - G\partial_x v_3 + 3G\partial_x v_5 - \rho(f_1+f_2) \in L^2(0,1). \tag{2.11}$$

Thus, by the L^2 theory for the linear elliptic equations, we obtain that

$$v_1 \in \tilde{H}^2_*(0,1).$$

Moreover, (2.10) is also true for any $\phi \in C^1([0,1]) \subset \tilde{H}^1_*(0,1)$ ($\phi(1)=0$). Hence, we get

$$\int_0^1 G\partial_x v_1 \partial_x \phi \mathrm{d}x + \int_0^1 \rho v_1 \phi \mathrm{d}x - \int_0^1 G(\partial_x v_3)\phi \mathrm{d}x + \int_0^1 3G(\partial_x v_5)\phi \mathrm{d}x = \int_0^1 \rho(f_1+f_2)\phi \mathrm{d}x.$$

By using the integration by parts, we have

$$\partial_x v_1(0)\phi(0) = 0, \ \forall \phi \in C^1([0,1]), \ \phi(1)=0.$$

Therefore,

$$\partial_x v_1(0) = 0.$$

In the same way, we get

$$v_3 \in H^2_*(0,1), v_5 \in H^2_*(0,1), v_7 \in \tilde{H}^1_*(0,1), \partial_x v_3(1) = \partial_x v_5(1) = \partial_x v_7(0) = 0.$$

Finally, the application of the classical regularity theory for linear elliptic equations guarantees the existence of unique solution $V \in D(\mathscr{A})$ which satisfies (2.4). Hence, the operator $Id - \mathscr{A}$ is surjective. Moreover, it is easy to see that $D(\mathscr{A})$ is dense in $\mathscr{H}$.

At last, by Lumer-Philips theorem (see [7, 15]) we have the well-posedness result stated in Theorem 2.1. $\square$

3 Exponential stability (for $\beta \geq 0$ and $\frac{\rho}{G} = \frac{I_\rho}{D}$)

In this Section, we prove the exponential decay for problem (1.5)-(1.6) either with structural damping ($\beta > 0$) or without structural damping ($\beta = 0$), in case of equal wave speeds ($\frac{\rho}{G} = \frac{I_\rho}{D}$). It will be achieved by using the perturbed energy method.

We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) := E(u(t)) =& \frac{1}{2}\Big(\rho \int_0^1 \varphi_t^2 \mathrm{d}x + I_\rho \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + 3I_\rho \int_0^1 w_t^2 \mathrm{d}x + G\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ &+ D\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + 3D\int_0^1 w_x^2 \mathrm{d}x + 4\gamma \int_0^1 w^2 \mathrm{d}x + k\int_0^1 \theta^2 \mathrm{d}x\Big) \end{aligned} \tag{3.1}$$

We have the following exponentially stable result.

Theorem 3.1 *Assume that $\beta \geq 0$ and $\frac{\rho}{G} = \frac{I_\rho}{D}$ hold. Let $U_0 \in \mathscr{H}$, then there exists positive constants c_0, c_1 such that the energy $E(t)$ associated with problem* (1.5)-(1.6) *satisfies*

$$E(t) \leq c_0 E(0) e^{-c_1 t}, \ t \geq 0. \tag{3.2}$$

To prove this result, we will state and prove some useful lemmas in advance.

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Lemma 3.2 *Let $(\varphi, \psi, w, \theta)$ be the solution of (1.5)-(1.6). Then the energy functional satisfies*

$$\frac{d}{dt}E(t) = -4\beta\int_0^1 w_t^2\mathrm{d}x - \tau\int_0^1 \theta_x^2\mathrm{d}x \le 0, \ \forall\, t \ge 0. \tag{3.3}$$

Proof. First, multiplying $(1.5)_1$ by φ_t, integrating over $(0, 1)$, using integration by parts and the boundary conditions in (1.6), we have

$$\frac{d}{dt}\left\{\frac{1}{2}\rho\int_0^1 \varphi_t^2\mathrm{d}x\right\} - G\int_0^1 (\psi - \varphi_x)\varphi_{xt}\mathrm{d}x = 0. \tag{3.4}$$

Note that

$$\begin{aligned} G\int_0^1 (\psi - \varphi_x)\varphi_{xt}\mathrm{d}x =& - G\int_0^1 (\psi - \varphi_x)(\psi - \varphi_x - \psi)_t\mathrm{d}x \\ =&\frac{d}{dt}\left\{-\frac{1}{2}G\int_0^1 (\psi - \varphi_x)^2\mathrm{d}x\right\} + G\int_0^1 (\psi - \varphi_x)\psi_t\mathrm{d}x. \end{aligned}$$

Hence, equation (3.4) becomes

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\rho\int_0^1 \varphi_t^2\mathrm{d}x + G\int_0^1 (\psi - \varphi_x)^2\mathrm{d}x\right)\right\} = G\int_0^1 (\psi - \varphi_x)\psi_t\mathrm{d}x. \tag{3.5}$$

Similarly, multiplying $(1.5)_2$, $(1.5)_3$, $(1.5)_4$ by $3(w - \psi)_t$, $3w_t$, θ and integrating over $(0, 1)$, using integration by parts the boundary conditions in (1.6), we can get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\left(I_\rho\int_0^1 (3w_t - \psi_t)^2\mathrm{d}x + D\int_0^1 (3w_x - \psi_x)^2\mathrm{d}x\right)\right\} \\ =&G\int_0^1 (\psi - \varphi_x)(3w - \psi)_t\mathrm{d}x - \sigma\int_0^1 \theta_x(3w - \psi)_t\mathrm{d}x, \qquad (3.6) \\ &\frac{d}{dt}\left\{\frac{1}{2}\left(3I_\rho\int_0^1 w_t^2\mathrm{d}x + 4\gamma\int_0^1 w^2\mathrm{d}x + 3D\int_0^1 w_x^2\mathrm{d}x\right)\right\} \\ =& - 3G\int_0^1 (\psi - \varphi_x)w_t\mathrm{d}x - 4\beta\int_0^1 w_t^2\mathrm{d}x, \qquad (3.7) \\ &\frac{d}{dt}\left\{\frac{1}{2}k\int_0^1 \theta^2\mathrm{d}x\right\} = \sigma\int_0^1 (3w - \psi)_t\theta_x\mathrm{d}x - \tau\int_0^1 \theta_x^2\mathrm{d}x. \qquad (3.8) \end{aligned}$$

Finally, adding (3.5)-(3.8), we obtain (3.3), which completes the proof. □

Next, in order to construct suitable Lyapunov functionals equivalent to the energy, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.3 *Let $(\varphi, \psi, w, \theta)$ be the solution of (1.5)-(1.6). Then the functional*

$$I_1(t) = -\rho\int_0^1 \varphi\varphi_t\mathrm{d}x$$

satisfies the estimate

$$I_1'(t) \le -\rho\int_0^1 \varphi_t^2\mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 (\psi - \varphi_x)^2\mathrm{d}x + \varepsilon_1\int_0^1 (3w_x - \psi_x)^2\mathrm{d}x + \varepsilon_1\int_0^1 w_x^2\mathrm{d}x, \tag{3.9}$$

for any $\varepsilon_1 > 0$.

Proof. By differentiating $I_1(t)$ with respect to t, using $(1.5)_1$ and integrating by parts, we obtain

$$I_1'(t) = -\rho\int_0^1 \varphi_t^2 \mathrm{d}x - G\int_0^1 \varphi_x(\psi-\varphi_x)\mathrm{d}x.$$

Note that

$$-G\int_0^1(\psi-\varphi_x)\varphi_x \mathrm{d}x = G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x - G\int_0^1 \psi(\psi-\varphi_x)\mathrm{d}x.$$

Then, we deduce that

$$I_1'(t) = -\rho\int_0^1 \varphi_t^2\mathrm{d}x + G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x - G\int_0^1\psi(\psi-\varphi_x)\mathrm{d}x.$$

Making use of Young's inequality with $\varepsilon_1 > 0$, we obtain

$$I_1'(t) \le -\rho\int_0^1 \varphi_t^2\mathrm{d} + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1(\psi-\varphi_x)^2\mathrm{d}x + \varepsilon_1\int_0^1\psi_x^2\mathrm{d}x.$$

Note that

$$\int_0^1\psi_x^2\mathrm{d}x = \int_0^1(\psi_x - 3w_x + 3w_x)^2\mathrm{d}x \le 2\int_0^1(3w_x-\psi_x)^2\mathrm{d}x + 18\int_0^1 w_x^2\mathrm{d}x.$$

Then estimate (3.9) is established. $\square$

Lemma 3.4 *Let* (φ,ψ,w,θ) *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_2(t) = I_\rho\int_0^1(3w-\psi)(3w-\psi)_t\mathrm{d}x$$

satisfies the estimate

$$I_2'(t) \le -\frac{D}{2}\int_0^1(3w_x-\psi_x)^2\mathrm{d}x + I_\rho\int_0^1(3w_t-\psi_t)^2\mathrm{d}x + c\int_0^1(\psi-\varphi_x)^2\mathrm{d}x + c\int_0^1\theta_x^2\mathrm{d}x. \quad (3.10)$$

Proof. Taking the derivative of $I_2(t)$ with respect to t, using $(1.5)_2$ and integrating by parts, we get

$$\begin{aligned} I_2'(t) =& -D\int_0^1(3w_x-\psi_x)^2\mathrm{d}x + I_\rho\int_0^1(3w_t-\psi_t)^2\mathrm{d}x \\ &+ G\int_0^1(\psi-\varphi_x)(3w-\psi)\mathrm{d}x + \sigma\int_0^1(3w-\psi)_x\theta\mathrm{d}x. \end{aligned}$$

Then, using Young's inequality, we arrive at (3.10). $\square$

Lemma 3.5 *Let* (φ,ψ,w,θ) *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_3(t) = I_\rho\int_0^1 w w_t\mathrm{d}x$$

satisfies the estimate

$$I_3'(t) \le -\frac{2\gamma}{3}\int_0^1 w^2\mathrm{d}x - D\int_0^1 w_x^2\mathrm{d}x + c\int_0^1 w_t^2\mathrm{d}x + c\int_0^1(\psi-\varphi_x)^2\mathrm{d}x. \quad (3.11)$$

Proof. By differentiating $I_3(t)$ with respect to t, using $(1.5)_3$ and integrating by parts, we obtain

$$I_3'(t) = -\frac{4\gamma}{3}\int_0^1 w^2\mathrm{d}x - D\int_0^1 w_x^2\mathrm{d}x + I_\rho\int_0^1 w_t^2\mathrm{d}x - G\int_0^1 w(\psi-\varphi_x)\mathrm{d}x - \frac{4\beta}{3}\int_0^1 ww_t\mathrm{d}x.$$

We then use Young's inequality to obtain (3.11). □

Lemma 3.6 *Let (φ,ψ,w,θ) be the solution of* (1.5)-(1.6). *Then the functional*

$$I_4(t) = \frac{kI_\rho}{\sigma}\int_0^1 (3w-\psi)_t\int_0^x \theta\mathrm{d}y\mathrm{d}x$$

satisfies the estimate

$$\begin{aligned} I_4'(t) \le & -\frac{I_\rho}{2}\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 \theta_x^2\mathrm{d}x + \varepsilon_4\int_0^1(\psi-\varphi_x)^2\mathrm{d}x \\ & + \varepsilon_4\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x, \end{aligned} \tag{3.12}$$

for any $\varepsilon_4 > 0$.

Proof. Taking the derivative of $I_4(t)$ with respect to t, using $(1.5)_2$, $(1.5)_4$ and integrating by parts, we get

$$\begin{aligned} I_4'(t) = & -I_\rho\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + \frac{kG}{\sigma}\int_0^1 (\psi-\varphi_x)\int_0^x \theta\mathrm{d}y\mathrm{d}x - \frac{kD}{\sigma}\int_0^1 (3w-\psi)_x\theta\mathrm{d}x \\ & + k\int_0^1 \theta^2\mathrm{d}x + \frac{\tau I_\rho}{\sigma}\int_0^1 (3w-\psi)_t\theta_x\mathrm{d}x. \end{aligned}$$

Using Young's inequality with $\varepsilon_4 > 0$, we establish (3.12). □

Lemma 3.7 *Let (φ,ψ,w,θ) be the solution of* (1.5)-(1.6). *Then the functional*

$$I_5(t) = -I_\rho\int_0^1 (3w-\psi)_t(\psi-\varphi_x)\mathrm{d}x + \frac{\rho D}{G}\int_0^1 \varphi_t(3w-\psi)_x\mathrm{d}x$$

satisfies the estimate

$$I_5'(t) \le -\frac{G}{2}\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x + c\int_0^1 \theta_x^2\mathrm{d}x + \varepsilon_5\int_0^1 w_t^2\mathrm{d}x + c\left(1+\frac{1}{\varepsilon_5}\right)\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x, \tag{3.13}$$

for any $\varepsilon_5 > 0$.

Proof. By $(1.5)_1$, $(1.5)_2$ and integrating by parts, we get

$$\begin{aligned} I_5'(t) = & -G\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x + \sigma\int_0^1 \theta_x(\psi-\varphi_x)\mathrm{d}x - I_\rho\int_0^1 (3w-\psi)_t\psi_t\mathrm{d}x \\ & + \left(\frac{\rho D}{G} - I_\rho\right)\int_0^1 (3w-\psi)_{xt}\varphi_t\mathrm{d}x. \end{aligned} \tag{3.14}$$

Using Young's inequality with $\varepsilon_5 > 0$ and $\frac{\rho}{G} = \frac{I_\rho}{D}$, we get (3.13). □

Now, we turn to prove the main result in this section.

Proof of Theorem 3.1. We devide it into two cases: $\beta > 0$ and $\beta = 0$.

Case 1. $\beta > 0$.

Let $N, N_4, N_5 > 0$ and $\frac{\rho}{G} = \frac{I_\rho}{D}$, we define a Lyapunov function as follows.

$$L_1(t) = NE(t) + I_1(t) + I_2(t) + I_3(t) + N_4 I_4(t) + N_5 I_5(t). \tag{3.15}$$

Gathering the estimates in the previous lemmas, we obtain

$$\begin{aligned} L_1'(t) \leq & -\rho \int_0^1 \varphi_t^2 \mathrm{d}x - \left[\frac{I_\rho N_4}{2} - I_\rho - c\left(1 + \frac{1}{\varepsilon_5}\right) N_5\right] \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ & - (4\beta N - c - \varepsilon_5 N_5) \int_0^1 w_t^2 \mathrm{d}x - \left[\frac{GN_5}{2} - c\left(1 + \frac{1}{\varepsilon_1}\right) - c - c - \varepsilon_4 N_4\right] \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ & - \left(\frac{D}{2} - \varepsilon_1 - \varepsilon_4 N_4\right) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \frac{2\gamma}{3} \int_0^1 w^2 \mathrm{d}x - (D - \varepsilon_1) \int_0^1 w_x^2 \mathrm{d}x \\ & - \left[\tau N - c - c\left(1 + \frac{1}{\varepsilon_4}\right) N_4 - cN_5\right] \int_0^1 \theta_x^2 \mathrm{d}x. \end{aligned} \tag{3.16}$$

At this point, we need to choose our constants very carefully. First, we choose

$$\varepsilon_1 = \frac{D}{8}, \varepsilon_4 = \frac{D}{8N_4}, \varepsilon_5 = \frac{2\beta N}{N_5}$$

so that

$$\begin{aligned} L_1'(t) \leq & -\rho \int_0^1 \varphi_t^2 \mathrm{d}x - \left[\frac{I_\rho N_4}{2} - I_\rho - c\left(1 + \frac{N_5}{N}\right) N_5\right] \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x - (2\beta N - c) \int_0^1 w_t^2 \mathrm{d}x \\ & - \left(\frac{GN_5}{2} - c\right) \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x - \frac{D}{4} \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \frac{2\gamma}{3} \int_0^1 w^2 \mathrm{d}x - \frac{7D}{8} \int_0^1 w_x^2 \mathrm{d}x \\ & - \left[\tau N - c - c\left(1 + N_4\right) N_4 - cN_5\right] \int_0^1 \theta_x^2 \mathrm{d}x. \end{aligned} \tag{3.17}$$

Then, we select N_5 large enough so that

$$\frac{GN_5}{2} - c > 0.$$

Next, we choose N_4 large enough so that

$$\frac{I_\rho N_4}{2} - I_\rho - c\left(1 + \frac{N_5}{N}\right) N_5 > 0.$$

Finally, we select N large enough so that

$$2\beta N - c > 0, \ \tau N - c - c\left(1 + N_4\right) N_4 - cN_5 > 0.$$

From the above, we deduce that there exist a positive constant C_1 such that (3.17) becomes

$$L_1'(t) \leq -C_1 E(t). \tag{3.18}$$

Using (3.1), Young's and Cauchy-Schwarz inequalities, we can easily deduce that there exists a positive constant c such that

$$|L_1(t) - NE(t)| \leq cE(t).$$

Choosing N large enough, we obtain that there exist positive constants $\mu_1 := N-c$ and $\mu_2 := N+c$ such that

$$\mu_1 E(t) \le L_1(t) \le \mu_2 E(t). \tag{3.19}$$

Then by (3.18) and (3.19), we get

$$L_1'(t) \le -C_1 E(t) \le -C_2 L_1(t), \tag{3.20}$$

where $C_2 = \frac{C_1}{\mu_1}$. Then, a simple integration of (3.20) over $(0,t)$ yields

$$L_1(t) \le L_1(0) e^{-C_2 t}, \ \forall\, t \ge 0. \tag{3.21}$$

At last, estimate (3.21) gives the desired result (3.2) when combined with (3.19).

Case 2. $\beta = 0$.

In this case, we prove the exponential decay result (3.2) for problem (1.5)-(1.6) without structural damping ($\beta = 0$). To overcome the difficulty (see (3.16)) brought by the absence of structural damping, we construct the multiplier $I_6(t)$ below to estimate $\int_0^1 w_t^2 \mathrm{d}x$.

Lemma 3.8 *Let* $(\varphi, \psi, w, \theta)$ *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_6(t) = -I_\rho \int_0^1 w_t(\psi - \varphi_x)\mathrm{d}x + I_\rho \int_0^1 \varphi_t w_x \mathrm{d}x$$

satisfies the estimate

$$I_6'(t) \le -\frac{3I_\rho}{2} \int_0^1 w_t^2 \mathrm{d}x + c\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + \varepsilon_6 \int_0^1 w^2 \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_6}\right)\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x, \tag{3.22}$$

for any $\varepsilon_6 > 0$.

Proof. By $(1.5)_1$, $(1.5)_3$ and integrating by parts, we get

$$\begin{aligned} I_6'(t) =& -3I_\rho \int_0^1 w_t^2 \mathrm{d}x + G\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + I_\rho \int_0^1 (3w - \psi)_t w_t \mathrm{d}x + \frac{4\gamma}{3}\int_0^1 w(\psi - \varphi_x)\mathrm{d}x \\ &+ \left(D - \frac{I_\rho G}{\rho}\right)\int_0^1 w_x(\psi - \varphi_x)_x \mathrm{d}x. \end{aligned} \tag{3.23}$$

Using Young's inequality and $\frac{\rho}{G} = \frac{I_\rho}{D}$, we get (3.22). $\square$

Let $N, N_4, N_5, N_6 > 0$ and $\frac{\rho}{G} = \frac{I_\rho}{D}$, we define the following Lyapunov functional

$$L_2(t) = NE(t) + I_1(t) + I_2(t) + I_3(t) + N_4 I_4(t) + N_5 I_5(t) + N_6 I_6(t). \tag{3.24}$$

From (3.9), (3.10), (3.11), (3.12), (3.13) and (3.22), we can obtain

$$\begin{aligned} L_2'(t) \le& -\rho\int_0^1 \varphi_t^2 \mathrm{d}x - \left[\frac{I_\rho N_4}{2} - I_\rho - c\left(1 + \frac{1}{\varepsilon_5}\right)N_5 - cN_6\right]\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ &- \left(\frac{3I_\rho}{2}N_6 - c - \varepsilon_5 N_5\right)\int_0^1 w_t^2 \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{GN_5}{2} - c\left(1+\frac{1}{\varepsilon_1}\right) - cN_6\left(1+\frac{1}{\varepsilon_6}\right) - c - \varepsilon_4 N_4 \right] \int_0^1 (\psi-\varphi_x)^2 \mathrm{d}x \\
&- \left(\frac{D}{2} - \varepsilon_1 - N_4\varepsilon_4\right) \int_0^1 (3w_x-\psi_x)^2 \mathrm{d}x - \left(\frac{2\gamma}{3} - N_6\varepsilon_6\right) \int_0^1 w^2 \mathrm{d}x \\
&- (D-\varepsilon_1) \int_0^1 w_x^2 \mathrm{d}x - \left[\tau N - c - c\left(1+\frac{1}{\varepsilon_4}\right) N_4 - cN_5\right] \int_0^1 \theta_x^2 \mathrm{d}x. \qquad (3.25)
\end{aligned}
$$

At this point, we need to choose our constants very carefully. First, we choose

$$
\varepsilon_1 = \frac{D}{8},\ \varepsilon_4 = \frac{D}{8N_4},\ \varepsilon_5 = \frac{3I_\rho N_6}{4N_5},\ \varepsilon_6 = \frac{\gamma}{3N_6},
$$

so that

$$
\begin{aligned}
L_2'(t) \leq &- \rho \int_0^1 \varphi_t^2 \mathrm{d}x - \left[\frac{I_\rho N_4}{2} - I_\rho - c\left(1+\frac{N_5}{N_6}\right) N_5 - cN_6\right] \int_0^1 (3w_t-\psi_t)^2 \mathrm{d}x \\
&- \left(\frac{3I_\rho}{4}N_6 - c\right) \int_0^1 w_t^2 \mathrm{d}x - \left[\frac{GN_5}{2} - cN_6\left(1+N_6\right) - c\right] \int_0^1 (\psi-\varphi_x)^2 \mathrm{d}x \\
&- \frac{D}{4} \int_0^1 (3w_x-\psi_x)^2 \mathrm{d}x - \frac{\gamma}{3} \int_0^1 w^2 \mathrm{d}x - \frac{7D}{8} \int_0^1 w_x^2 \mathrm{d}x \\
&- \left[\tau N - c - c\left(1+N_4\right) N_4 - cN_5\right] \int_0^1 \theta_x^2 \mathrm{d}x. \qquad (3.26)
\end{aligned}
$$

Then, we select N_6 large enough so that

$$
\frac{3I_\rho N_6}{4} - c > 0.
$$

Next, we select N_5 large enough so that

$$
\frac{GN_5}{2} - cN_6\left(1+N_6\right) - c > 0.
$$

Furthermore, we select N_4 large enough so that

$$
\frac{I_\rho N_4}{2} - I_\rho - c\left(1+\frac{N_5}{N_6}\right) N_5 - cN_6 > 0.
$$

Finally, we select N large enough so that

$$
N - c > 0,\ \tau N - c - c\left(1+N_4\right) N_4 - cN_5 > 0.
$$

Using (3.1), we obtain that there exists a positive constants c_1 such that (3.26) becomes

$$
L_2'(t) \leq -c_1 E(t),\ \forall\, t \geq 0. \qquad (3.27)
$$

Using (3.1), Young's and Cauchy-Schwarz inequalities, we can easily deduce that there exists a positive constant c such that

$$
|L_2(t) - NE(t)| \leq cE(t).
$$

Choosing N large enough, we obtain that there exist positive constants $\mu_1 := N-c$ and $\mu_2 := N+c$ such that

$$
\mu_1 E(t) \leq L_2(t) \leq \mu_2 E(t). \qquad (3.28)
$$

 doi:10.20944/preprints201702.0058.v2

Then by (3.27) and (3.28), we obtain

$$L_2'(t) \leq -c_2 E(t), \ \forall\, t \geq 0, \tag{3.29}$$

where $c_2 = \frac{c_1}{\mu_2}$. Then, a simple integration of (3.29) over $(0, t)$ yields

$$L_2(t) \leq L_2(0)e^{-c_2 t}, \ \forall\, t \geq 0. \tag{3.30}$$

At last, estimate (3.30) gives the desired result (3.2) when combined with (3.28). This completes the proof. □

4 Lack of exponential stability (for $\beta \geq 0$ and $\frac{\rho}{G} \neq \frac{I_\rho}{D}$)

This section is concerning the lack of exponential stability for problem (1.5)-(1.6) either with structural damping ($\beta > 0$) or without structural damping ($\beta = 0$), in case of non-equal wave speeds ($\frac{\rho}{G} \neq \frac{I_\rho}{D}$). It will be achieved by using Gearhart-Herbst-Prüss-Huang theorem to dissipative systems, see Prüss [23] and Huang [13].

Theorem 4.1 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supset \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of the differential operator $\mathcal{A}$.

Next, we state and prove the main result of this section.

Theorem 4.2 *Assume that $\beta \geq 0$ and $\frac{\rho}{G} \neq \frac{I_\rho}{D}$ hold. Then the semigroup associated to problem* (1.5)-(1.6) *is not exponentially stable.*

Proof. We will prove that there exists a sequence of imaginary number λ_μ and function $F_\mu \in \mathcal{H}$ with $\|F_\mu\|_{\mathcal{H}} \leq 1$ such that $\|(\lambda_\mu I - \mathcal{A})^{-1}F_\mu\|_{\mathcal{H}} = \|U_\mu\|_{\mathcal{H}} \to \infty$, where

$$\lambda_\mu U_\mu - \mathcal{A}U_\mu = F_\mu, \tag{4.1}$$

with $U_\mu = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)^T$ not bounded. Rewriting spectral equation (4.1) in term of its components, we have for $\lambda_\mu = \lambda$

$$\begin{cases} \lambda v_1 - v_2 = g_1, \\ \rho\lambda v_2 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2, \\ \lambda v_3 - v_4 = g_3, \\ I_\rho \lambda v_4 + G\partial_x v_1 + Gv_3 - D\partial_{xx}v_3 - 3Gv_5 + \sigma\partial_x v_7 = I_\rho g_4, \\ \lambda v_5 - v_6 = g_5, \\ I_\rho \lambda v_6 + \dfrac{4\beta}{3}v_6 - G\partial_x v_1 - Gv_3 + \left(3G + \dfrac{4\gamma}{3}\right) v_5 - D\partial_{xx}v_5 = I_\rho g_6, \\ k\lambda v_7 - \tau\partial_{xx}v_7 + \sigma\partial_x v_4 = kg_7, \end{cases} \tag{4.2}$$

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where $\lambda \in \mathbb{R}$ and $F = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T \in \mathcal{H}$. Taking $g_1 = g_3 = g_5 = 0$, then the above system becomes

$$\begin{cases} \rho\lambda^2 v_1 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2, \\ I_\rho\lambda^2 v_3 + G\partial_x v_1 + Gv_3 - D\partial_{xx}v_3 - 3Gv_5 + \sigma\partial_x v_7 = I_\rho g_4, \\ I_\rho\lambda^2 v_5 + \dfrac{4\beta}{3}\lambda v_5 - G\partial_x v_1 - Gv_3 + \left(3G + \dfrac{4\gamma}{3}\right) v_5 - D\partial_{xx}v_5 = I_\rho g_6, \\ k\lambda v_7 - \tau\partial_{xx}v_7 + \lambda\sigma\partial_x v_3 = kg_7. \end{cases} \tag{4.3}$$

Because of the boundary conditions in (1.6), we can suppose that

$$v_1 = A\cos\left(\frac{\mu\pi}{2}x\right), \; v_3 = B\sin\left(\frac{\mu\pi}{2}x\right), \; v_5 = C\sin\left(\frac{\mu\pi}{2}x\right), \; v_7 = E\cos\left(\frac{\mu\pi}{2}x\right).$$

Now, choosing

$$g_2 = \frac{1}{\rho}\cos\left(\frac{\mu\pi}{2}x\right), \; g_4 = g_6 = g_7 = 0,$$

we arrive at

$$\begin{cases} \left(\rho\lambda^2 + G\left(\dfrac{\mu\pi}{2}\right)^2\right) A - G\left(\dfrac{\mu\pi}{2}\right) B + 3G\left(\dfrac{\mu\pi}{2}\right) C = 1, \\ -G\left(\dfrac{\mu\pi}{2}\right) A + \left(I_\rho\lambda^2 + G + D\left(\dfrac{\mu\pi}{2}\right)^2\right) B - 3GC - \sigma\left(\dfrac{\mu\pi}{2}\right) E = 0, \\ G\left(\dfrac{\mu\pi}{2}\right) A - GB + \left(I_\rho\lambda^2 + \dfrac{4\beta}{3}\lambda + 3G + \dfrac{4\gamma}{3} + D\left(\dfrac{\mu\pi}{2}\right)^2\right) C = 0, \\ \lambda\sigma\left(\dfrac{\mu\pi}{2}\right) B + \left(k\lambda + \tau\left(\dfrac{\mu\pi}{2}\right)^2\right) E = 0. \end{cases} \tag{4.4}$$

Now, we take $\lambda = \lambda_\mu := i\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right)$ such that

$$\rho\lambda^2 + G\left(\frac{\mu\pi}{2}\right)^2 = 0,$$

then the above system can be written as

$$\begin{cases} -G\left(\dfrac{\mu\pi}{2}\right) B + 3G\left(\dfrac{\mu\pi}{2}\right) C = 1, \\ -G\left(\dfrac{\mu\pi}{2}\right) A + \left(G + I_\rho\left(\dfrac{D}{I_\rho} - \dfrac{G}{\rho}\right)\left(\dfrac{\mu\pi}{2}\right)^2\right) B - 3GC - \sigma\left(\dfrac{\mu\pi}{2}\right) E = 0, \\ G\left(\dfrac{\mu\pi}{2}\right) A - GB + \left(\dfrac{4\beta}{3}\lambda + 3G + \dfrac{4\gamma}{3} + I_\rho\left(\dfrac{D}{I_\rho} - \dfrac{G}{\rho}\right)\left(\dfrac{\mu\pi}{2}\right)^2\right) C = 0, \\ i\sigma\sqrt{\dfrac{G}{\rho}}\left(\dfrac{\mu\pi}{2}\right)^2 B + \left(ik\sqrt{\dfrac{G}{\rho}}\left(\dfrac{\mu\pi}{2}\right) + \tau\left(\dfrac{\mu\pi}{2}\right)^2\right) E = 0. \end{cases} \tag{4.5}$$

Adding $(4.5)_2$ to $(4.5)_3$, we get

$$I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 B - \sigma\left(\frac{\mu\pi}{2}\right) E + \left(\frac{4\beta}{3}\lambda + \frac{4\gamma}{3} + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2\right) C = 0. \tag{4.6}$$

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From $(4.5)_4$, we get

$$E = -\frac{i\sigma\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right)}{ik\sqrt{\frac{G}{\rho}} + \tau\left(\frac{\mu\pi}{2}\right)} B.$$

Substituting E into (4.6), we get

$$C = -\frac{\Lambda_\mu}{\Gamma_\mu} B,$$

where

$$\Lambda_\mu = I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 + \frac{i\sigma^2\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right)^2}{ik\sqrt{\frac{G}{\rho}} + \tau\left(\frac{\mu\pi}{2}\right)},$$

$$\Gamma_\mu = \frac{4\beta}{3}\lambda + \frac{4\gamma}{3} + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2.$$

Substituting C into $(4.5)_3$, we get

$$A = \frac{G\Gamma_\mu + \Lambda_\mu\Gamma_\mu + 3G\Lambda_\mu}{G\left(\frac{\mu\pi}{2}\right)\Gamma_\mu} B.$$

Similarly, substituting C into $(4.5)_1$, we get

$$B = -\frac{\Gamma_\mu}{G\left(\frac{\mu\pi}{2}\right)(\Gamma_\mu + 3\Lambda_\mu)}.$$

Let $\mu \to \infty$, we get

$$\left(\frac{\mu\pi}{2}\right) B \to -\frac{1}{4G}.$$

Substituting this expression into A, C and E, we obtain for $\mu \to \infty$,

$$A \to -\frac{D}{4\rho G}\left(\frac{\rho}{G} - \frac{I_\rho}{D}\right), \quad C \to O\left(\frac{1}{\mu}\right), \quad E \to O\left(\frac{1}{\mu}\right).$$

Thus

$$\begin{aligned}\|U_\mu\|^2_{\mathcal{H}} \geq G\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x &= G\left[3C - B + \left(\frac{\mu\pi}{2}\right)A\right]^2 \int_0^1 \sin^2\left(\frac{\mu\pi}{2}x\right)\mathrm{d}x \\ &= \frac{1}{2}G\left[3C - B + \left(\frac{\mu\pi}{2}\right)A\right]^2 \to \infty, \ \ as \ \mu \to \infty.\end{aligned}$$

This implies that

$$\|U_\mu\|_{\mathcal{H}} \to \infty, \ \ as \ \mu \to \infty.$$

Therefore, there is no exponential stability. This completes the proof. □

5 Polynomial stability (for $\beta > 0$ and $\frac{\rho}{G} \neq \frac{I_\rho}{D}$)

In this section, we consider the polynomial stability for problem (1.5)-(1.6) with structural damping ($\beta > 0$), in case of non-equal wave speeds ($\frac{\rho}{G} \neq \frac{I_\rho}{D}$). The case $\beta = 0$ and $\frac{\rho}{G} \neq \frac{I_\rho}{D}$ will be left as an open problem.

Theorem 5.1 *Assume that* $\beta > 0$ *and* $\frac{\rho}{G} \neq \frac{I_\rho}{D}$ *hold. Let* $U_0 \in \mathscr{H}$*, then there exists a positive constant* c_2 *such that the energy* $E(t)$ *associated with problem* (1.5)-(1.6) *satisfies*

$$E(t) \leq \frac{c_2}{t}, \ t > 0. \tag{5.1}$$

Proof. In this regard, we establish a polynomial decay result. As we will see, due to the presence of the $\int_0^1 (3w - \psi)_{xt}\varphi_t \mathrm{d}x$, we cannot directly perform the same proof as for the case where $\frac{\rho}{G} = \frac{I_\rho}{D}$. To overcome this difficulty, the second-order energy method is needed. The second-order energy is defined by

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\bigg(\rho \int_0^1 \varphi_{tt}^2 \mathrm{d}x + I_\rho \int_0^1 (3w_{tt} - \psi_{tt})^2 \mathrm{d}x + 3I_\rho \int_0^1 w_{tt}^2 \mathrm{d}x + G\int_0^1 (\psi_t - \varphi_{xt})^2 \mathrm{d}x \\ &+ D\int_0^1 (3w_{xt} - \psi_{xt})^2 \mathrm{d}x + 3D\int_0^1 w_{xt}^2 \mathrm{d}x + 4\gamma \int_0^1 w_t^2 \mathrm{d}x + k\int_0^1 \theta_t^2 \mathrm{d}x\bigg).\end{aligned}$$

A simple calculation (Similar to (3.3)) implies that

$$\mathcal{E}'(t) = -4\beta \int_0^1 w_{tt}^2 \mathrm{d}x - \tau \int_0^1 \theta_{xt}^2 \mathrm{d}x. \tag{5.2}$$

As in (3.15), we also define a Lyapunov functional $L_3(t)$ as follows:

$$\begin{aligned}L_3(t) =& N(E(t) + \mathcal{E}(t)) + I_1(t) + I_2(t) + I_3(t) + N_4 I_4(t) \\ &+ N_5\left[I_5(t) + \frac{\tau}{\sigma}\left(\frac{\rho D}{G} - I_\rho\right)\int_0^1 \theta_x \varphi_x \mathrm{d}x\right],\end{aligned} \tag{5.3}$$

where $I_i(t), i = 1,2,3,4,5$ remain as defined in Lemma 3.3-Lemma 3.7 with derivatives of $I_1(t)$-$I_4(t)$ remain the same while the derivative of $I_5(t)$ is given as

$$\begin{aligned}I_5'(t) \leq& -\frac{G}{2}\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + c\int_0^1 \theta_x^2 \mathrm{d}x + \varepsilon_5 \int_0^1 w_t^2 \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_5}\right)\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ &+ \left(\frac{\rho D}{G} - I_\rho\right)\int_0^1 (3w - \psi)_{xt}\varphi_t \mathrm{d}x,\end{aligned} \tag{5.4}$$

for any $\varepsilon_5 > 0$.

Now, we handle the last term in the right-hand side of (5.4), using $(1.5)_4$ as follows.

$$\begin{aligned}\int_0^1 (3w - \psi)_{xt}\varphi_t \mathrm{d}x =& \frac{\tau}{\sigma}\int_0^1 \theta_{xx}\varphi_t \mathrm{d}x - \frac{k}{\sigma}\int_0^1 \theta_t \varphi_t \mathrm{d}x \\ =& -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\tau}{\sigma}\int_0^1 \theta_x \varphi_x \mathrm{d}x\right] + \frac{\tau}{\sigma}\int_0^1 \theta_{xt}\varphi_x \mathrm{d}x - \frac{k}{\sigma}\int_0^1 \theta_t \varphi_t \mathrm{d}x \\ \leq& -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\tau}{\sigma}\int_0^1 \theta_x \varphi_x \mathrm{d}x\right] + \varepsilon \int_0^1 [\varphi_x^2 + \varphi_t^2] \mathrm{d}x + c_\varepsilon \int_0^1 [\theta_{xt}^2 + \theta_t^2] \mathrm{d}x.\end{aligned}$$

Note that

$$\int_0^1 \varphi_x^2 \mathrm{d}x \leq 2\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + 2\int_0^1 \psi_x^2 \mathrm{d}x \leq 2\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + 4\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + 36\int_0^1 w_x^2 \mathrm{d}x,$$

$$\int_0^1 \theta_t^2 \mathrm{d}x \leq \int_0^1 \theta_{xt}^2 \mathrm{d}x,$$

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then we can get

$$\int_0^1 (3w-\psi)_{xt}\varphi_t \mathrm{d}x \le -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\tau}{\sigma}\int_0^1 \theta_x\varphi_x \mathrm{d}x\right] + \varepsilon\int_0^1 [(\psi-\varphi_x)^2 + (3w_x-\psi_x)^2 + w_x^2 + \varphi_t^2]\mathrm{d}x$$
$$+ c_\varepsilon \int_0^1 \theta_{xt}^2 \mathrm{d}x. \tag{5.5}$$

Combining (5.4) with (5.5), we obtain

$$I_5'(t) \le -\frac{G}{4}\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x - \frac{\tau}{\sigma}\left(\frac{\rho D}{G} - I_\rho\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_x\varphi_x\mathrm{d}x + c\left(1+\frac{1}{\varepsilon_5}\right)\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x$$
$$+ \varepsilon_5\int_0^1 [(3w_x-\psi_x)^2 + w_x^2 + \varphi_t^2 + w_t^2]\mathrm{d}x + c\int_0^1 \theta_x^2\mathrm{d}x + c_{\varepsilon_5}\int_0^1 \theta_{xt}^2\mathrm{d}x. \tag{5.6}$$

Next, differentiating $L_3(t)$, we obtain

$$L_3'(t) \le -(\rho-\varepsilon_5 N_5)\int_0^1 \varphi_t^2\mathrm{d}x - \left[\frac{I_\rho N_4}{2} - I_\rho - c\left(1+\frac{1}{\varepsilon_5}\right)N_5\right]\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x$$
$$-(4\beta N - c - \varepsilon_5 N_5)\int_0^1 w_t^2\mathrm{d}x - \left[\frac{GN_5}{4} - c\left(1+\frac{1}{\varepsilon_1}\right) - c - c - \varepsilon_4 N_4\right]\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x$$
$$-\left(\frac{D}{2} - \varepsilon_1 - \varepsilon_4 N_4 - \varepsilon_5 N_5\right)\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x - (D-\varepsilon_1-\varepsilon_5 N_5)\int_0^1 w_x^2\mathrm{d}x$$
$$-\frac{2\gamma}{3}\int_0^1 w^2\mathrm{d}x - \left[\tau N - c - c\left(1+\frac{1}{\varepsilon_4}\right)N_4 - cN_5\right]\int_0^1 \theta_x^2\mathrm{d}x$$
$$-(\tau N - c_{\varepsilon_5}N_5)\int_0^1 \theta_{xt}^2\mathrm{d}x - 4\beta N\int_0^1 w_{tt}^2\mathrm{d}x. \tag{5.7}$$

At this point, we need to choose our constants very carefully. First, we choose

$$\varepsilon_1 = \frac{D}{8}, \varepsilon_4 = \min\left\{\frac{GN_5}{8N_4}, \frac{D}{8N_4}\right\}, \varepsilon_5 = \min\left\{\frac{\rho}{2N_5}, \frac{2\beta N}{N_5}, \frac{D}{8N_5}\right\}$$

so that

$$L_3'(t) \le -\frac{\rho}{2}\int_0^1 \varphi_t^2\mathrm{d}x - \left[\frac{I_\rho N_4}{2} - I_\rho - c\left(1+\frac{1}{\varepsilon_5}\right)N_5\right]\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x$$
$$-(2\beta N - c)\int_0^1 w_t^2\mathrm{d}x - \left(\frac{GN_5}{8} - c\right)\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x - \frac{D}{8}\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x$$
$$-\frac{3D}{4}\int_0^1 w_x^2\mathrm{d}x - \frac{2\gamma}{3}\int_0^1 w^2\mathrm{d}x - \left[\tau N - c - c\left(1+\frac{1}{\varepsilon_4}\right)N_4 - cN_5\right]\int_0^1 \theta_x^2\mathrm{d}x$$
$$-(\tau N - c_{\varepsilon_5}N_5)\int_0^1 \theta_{xt}^2\mathrm{d}x - 4\beta N\int_0^1 w_{tt}^2\mathrm{d}x. \tag{5.8}$$

Then, we select N_5 large enough so that

$$\frac{GN_5}{8} - c > 0.$$

Next, we choose N_4 large enough so that

$$\frac{I_\rho N_4}{2} - I_\rho - c\left(1+\frac{1}{\varepsilon_5}\right)N_5 > 0.$$

Finally, we select N large enough so that

$$2\beta N - c > 0,\ \tau N - c - c\left(1+\frac{1}{\varepsilon_4}\right)N_4 - cN_5 > 0,\ \tau N - c_{\varepsilon_5}N_5 > 0.$$

Thus, we deduce that there exist positive constants C_3 such that (5.8) becomes

$$L_3'(t) \leq -\,C_3E(t). \tag{5.9}$$

A simple integration of (5.9) over $(0,t)$, recalling that $E(t)$ is non-increasing, yields

$$tE(t) \leq \int_0^t E(s)\mathrm{d}s \leq \frac{1}{C_3}(L_3(0) - L_3(t)) \leq \frac{L_3(0)}{C_3}. \tag{5.10}$$

Finally, for a positive constant $c_2 := \frac{L_3(0)}{C_3}$, we have

$$E(t) \leq \frac{c_2}{t},\ \forall\, t > 0,$$

which completes the proof. $\square$

6 Conclusion and open problem

In this paper, we first prove the well-posedness for a thermoelastic laminated beam both with structural damping ($\beta > 0$) and without structural damping ($\beta = 0$), where the heat conduction is given by Fourier's law effective in the rotation angle displacements. And then we prove that the system is exponentially stable if and only if the wave speeds are equal ($\frac{\rho}{G} = \frac{I_\rho}{D}$). Furthermore, we show that the system with structural damping ($\beta > 0$) is polynomially stable provided that the wave speeds are not equal ($\frac{\rho}{G} \neq \frac{I_\rho}{D}$). For the case of the wave speeds are not equal $\left(\frac{\rho}{G} \neq \frac{I_\rho}{D}\right)$, the problem of whether it is possible to get the polynomial stability for system (1.5)-(1.6) without structural damping ($\beta = 0$) is still an interesting **open problem**. In fact, when $\frac{\rho}{G} \neq \frac{I_\rho}{D}$, the presence of $\int_0^1(3w-\psi)_t\varphi_{xt}\mathrm{d}x$ in the proof of Lemma 3.7 (see (3.14)) and $\int_0^1 w_x(\psi-\varphi_x)_x\mathrm{d}x$ in the proof of Lemma 3.8 (see (3.23)) couldn't be estimated by using the usual *the second-order energy method* (see [3, 4, 21]).

Acknowledgments

This work was supported by the National Natural Science Foundation of China [grant number 11771216], the Natural Science Foundation of Jiangsu Province [grant number BK20151523], the Six Talent Peaks Project in Jiangsu Province [grant number 2015-XCL-020] and the Qing Lan Project of Jiangsu Province.

References

[1] A. M. Alharbi and N. H. Scott, Wave stability in anisotropic generalized temperature-rate-dependent thermoelasticity, IMA J. Appl. Math. **81** (2016), no. 5, 750–778.

[2] D. S. Almeida Júnior, M. L. Santos and J. E. Muñoz Rivera, Stability to 1-D thermoelastic Timoshenko beam acting on shear force, Z. Angew. Math. Phys. **65** (2014), no. 6, 1233–1249.

[3] T. A. Apalara, Uniform stability of a laminated beam with structural damping and second sound, Z. Angew. Math. Phys. **68** (2017), no. 2, 68:41.

[4] T. A. Apalara and S. A. Messaoudi, An exponential stability result of a Timoshenko system with thermoelasticity with second sound and in the presence of delay, Appl. Math. Optim. **71** (2015), no. 3, 449–472.

[5] J. Bajkowski et al., A thermoviscoelastic beam model for brakes, European J. Appl. Math. **15** (2004), no. 2, 181–202.

[6] F. Boulanouar and S. Drabla, General boundary stabilization result of memory-type thermoelasticity with second sound, Electron. J. Differential Equations **2014** (2014), no. 202, 18 pp.

[7] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York, 2011.

[8] A. D. S. Campelo, D. S. Almeida Júnior and M. L. Santos, Stability to the dissipative Reissner-Mindlin-Timoshenko acting on displacement equation, European J. Appl. Math. **27** (2016), no. 2, 157–193.

[9] X.-G. Cao, D.-Y. Liu and G.-Q. Xu, Easy test for stability of laminated beams with structural damping and boundary feedback controls, J. Dyn. Control Syst. **13** (2007), no. 3, 313–336.

[10] M. M. Cavalcanti et al., Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping, Z. Angew. Math. Phys. **65** (2014), no. 6, 1189–1206.

[11] M. M. Chen, W. J. Liu and W. C. Zhou, Existence and general stabilization of the Timoshenko system of thermo-viscoelasticity of type III with frictional damping and delay terms, Adv. Nonlinear Anal. **7** (2018), no. 4, 547–569.

[12] S. W. Hansen and R. Spies, Structural damping in a laminated beams due to interfacial slip, J. Sound Vibration. **204** (1997), no. 2, 183–202.

[13] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations **1** (1985), no. 1, 43–56.

[14] A. A. Keddi, T. A. Apalaras and S. A. Messaoudi, Exponential and polynomial decay in a thermoelastic-Bresse system with second sound, Appl. Math. Optim., in press. DOI: 10.1007/s00245-016-9376-y

[15] V. Komornik, Exact controllability and stabilization, RAM: Research in Applied Mathematics, Masson, Paris, 1994.

[16] W. J. Liu, K. W. Chen and J. Yu, Existence and general decay for the full von Kármán beam with a thermo-viscoelastic damping, frictional dampings and a delay term, IMA J. Math. Control Inform. **34** (2017), no. 2, 521–542.

[17] W. J. Liu, K. W. Chen and J. Yu, Asymptotic stability for a non-autonomous full von Kármán beam with thermo-viscoelastic damping, Appl. Anal. **97** (2018), no. 3, 400–414.

[18] W. J. Liu and W. F. Zhao, Stabilization of a thermoelastic laminated beam with past history, Appl. Math. Optim., in press. doi: 10.1007/s00245-017-9460-y

[19] S. A. Messaoudi and A. Fareh, Energy decay in a Timoshenko-type system of thermoelasticity of type III with different wave-propagation speeds, Arab. J. Math. (Springer) **2** (2013), no. 2, 199–207.

[20] J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems—global existence and exponential stability, J. Math. Anal. Appl. **276** (2002), no. 1, 248–278.

[21] J. E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems, Discrete Contin. Dyn. Syst. **9** (2003), no. 6, 1625–1639.

[22] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, 44, Springer, New York, 1983.

[23] J. Prüss, On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc. **284** (1984), no. 2, 847–857.

[24] Y. Qin, X.-G. Yang and Z. Ma, Global existence of solutions for the thermoelastic Bresse system, Commun. Pure Appl. Anal. **13** (2014), no. 4, 1395–1406.

[25] C. A. Raposo, Exponential stability for a structure with interfacial slip and frictional damping, Appl. Math. Lett. **53** (2016), 85–91.

[26] F. Tahamtani and A. Peyravi, Asymptotic behavior and blow-up of solution for a nonlinear viscoelastic wave equation with boundary dissipation, Taiwanese J. Math. **17** (2013), no. 6, 1921–1943.

[27] J.-M. Wang, G.-Q. Xu and S.-P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, SIAM J. Control Optim. **44** (2005), no. 5, 1575–1597.