Exponential and polynomial decay for a laminated beam with Fourier's type heat conduction

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In this paper, we study the well-posedness and the asymptotic behavior of a one-dimensional laminated beam system, where the heat conduction is given by Fourier's law effective in the rotation angle displacements. We show that the system is well-posed by using the Hille-Yosida theorem and prove that the system is exponentially stable if and only if the wave speeds are equal. Furthermore, we show that the system is polynomially stable provided that the wave speeds are not equal.

Keywords: laminated beam, Fourier's law, exponential stability, lack of exponential stability, polynomial stability.

1 Introduction

With the increasing demand of advanced performance, the vibration suppression of the laminated beams has been one of the main research topics in smart materials and structures. These composite laminates usually have superior structural properties such as adaptability. The design of their piezoelectric materials can be used as both actuators and sensors [1]. Hansen and Spies in [2] derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, the system is given by the following equations:

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ 3I_\rho w_{tt} + 3G(\psi - \varphi_x) + 4\gamma w + 4\beta w_t - 3Dw_{xx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{1.1}$$

where $\rho, G, I_\rho, D, \gamma, \beta$ are positive constant coefficients, ρ is the density of the beams, G is the shear stiffness, I_ρ is the mass moment of inertia, D is the flexural rigidity, γ is the adhesive stiffness of the beams, and β is the adhesive damping parameter. The function φ denotes the transverse displacement of the beam which departs from its equilibrium position, ψ represents the rotation angle, w is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x, $3w - \psi$ denotes the effective rotation angle, $(1.1)_3$ describes the dynamics of the slip.

In recent years, an increasing interest has been developed to determine the asymptotic behavior of the solution of several laminated beam problems. For example, Wang et al. [1] considered system (1.1) with the cantilever boundary conditions and two different wave speeds ($\sqrt{G/\rho}$ and $\sqrt{D/I_\rho}$). The authors proved the well-posedness and pointed out that system (1.1) can obtain the asymptotic stability but it does not reach the exponential stability due to the action of the slip w. Furthermore, to achieve the exponential decay result, the authors added an additional boundary control such that the boundary conditions become

$$\varphi(0,t) = \xi(0,t) = w(0,t) = 0, \; w_x(1,t) = 0,$$

$$3w(1,t) - \xi(1,t) - \varphi_x(1,t) = u_1(t) := k_1\varphi_t(1,t),$$

$$\xi_x(1,t) = u_2(t) := -k_2\xi_t(1,t),$$

where $\xi = 3w - \psi$. Cao et al. [3] considered the system (1.1) with following boundary conditions

$$\psi(0,t) - \varphi_x(0,t) = u_1(t) := -k_1\varphi_t(0,t) - \varphi(0,t),$$

$$3w_x(1,t) - \psi_x(1,t) = u_2(t) := -k_2\xi_t(1,t) - \xi(1,t),$$

where $\xi = 3w - \psi$. The authors obtained an exponential stability result provided $k_1 \neq \sqrt{\rho/G}$ and $k_2 \neq \sqrt{I_\rho/D}$. More importantly, the authors proved that the dominant part of the system is itself exponentially stable. Raposo [4] considered system (1.1) with two frictional dampings of the form

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x + k_1\varphi_t = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} + k_2(3w - \psi)_t = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ 3I_\rho w_{tt} + 3G(\psi - \varphi_x) + 4\gamma w + 4\beta w_t - 3Dw_{xx} = 0, & (x,t) \in (0,1) \times (0,+\infty) \end{cases} \tag{1.2}$$

and obtained the exponential decay result under appropriate initial and boundary conditions.

It is easy to find that if the slip w is assumed to be identically zero, then the first two equations of system (1.1) can be reduced exactly to the Timoshenko beam system. For the case of the Timoshenko beam with Fourier's law, many authors have shown various decay estimates depending on the wave speeds. Rivera and Racke [5] studied the Timoshenko system with thermoelastic dissipation, i.e.,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0, & (x,t) \in (0,L) \times (0,+\infty), \end{cases} \tag{1.3}$$

with positive constants $\rho_1, \rho_2, \rho_3, k, b, \gamma, \kappa$. The authors showed that the exponential stability holds if and only if the wave speeds are equal $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$. Júnior and Rivera [6] considered a new coupling to the thermoelastic Timoshenko beam of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \sigma\theta_x = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) - \sigma\theta = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_3\theta_t - \gamma\theta_{xx} + \sigma(\varphi_x + \psi)_t = 0, & (x,t) \in (0,L) \times (0,+\infty). \end{cases} \tag{1.4}$$

The authors showed this system is exponentially stable if and only if the wave speeds are equal $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$. On the contrary, the authors obtained the polynomially stable depending on the different boundary conditions. For system (1.4) with Dirichlet boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi(t,0) = \psi(t,L) = \theta(t,0) = \theta(t,L) = 0,$$

the authors obtained that the semigroup decay as $\dfrac{1}{\sqrt[4]{t}}$. For system (1.4) with Dirichlet-Neumann boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta_x(t,0) = \theta_x(t,L) = 0,$$

the authors obtained that the semigroup decay as $\dfrac{1}{\sqrt{t}}$. We refer the reader to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], for some other related results.

Motivated by the above results, we intend to study the well-posedness and the asymptotic stability of the laminated beam system where the heat flux is given by Fourier's law. The system is written as

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3w-\psi)_{tt} - G(\psi-\varphi_x) - D(3w-\psi)_{xx} + \sigma\theta_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho w_{tt} + G(\psi-\varphi_x) + \dfrac{4}{3}\gamma w + \dfrac{4}{3}\beta w_t - Dw_{xx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ k\theta_t - \tau\theta_{xx} + \sigma(3w-\psi)_{tx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{1.5}$$

where $\rho, G, I_\rho, D, \sigma, \gamma, \beta, k, \tau$ are positive constant coefficients. We consider following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in [0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in [0,1], \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in [0,1], \\ \theta(x,0) = \theta_0(x), & x \in [0,1], \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0, & t \in [0,+\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0, & t \in [0,+\infty). \end{cases} \tag{1.6}$$

By using Hille-Yosida theorem, we first prove the well-posedness result. By using the perturbed energy method, we then establish the exponential result if and only if $\frac{\rho}{G} = \frac{I_\rho}{D}$ and the polynomial stability if $\frac{\rho}{G} \neq \frac{I_\rho}{D}$. Furthermore, by using Gearhart-Herbst-Prüss-Huang theorem, we obtain the lack of exponential stability. The main difficulty in carry out this paper is the appearance for the Fourier's law of heat conduction. For this purpose, we use the appropriated multiplies and energy method to build an equivalent Lyapunov functional.

We now briefly sketch the outline of the paper. In Section 2, we state and prove the well-posedness of problem (1.5)-(1.6). In Section 3, we establish an exponential stability result of the energy. In Section 4, the lack of exponential stability has been studied. Finally, Section 5 is devoted to the statement and proof of the polynomial stability.

2 The well-posedness

In this Section, we prove the well-posedness of problem (1.5)-(1.6) by using Hille-Yosida theorem. Firstly, we introduce the vector function

$$U = (\varphi, \varphi_t, 3w-\psi, (3w-\psi)_t, w, w_t, \theta)^T.$$

Then system (1.5)-(1.6) can be written as

$$\begin{cases} \partial_t U = \mathscr{A}U, \\ U(x,0) = U^0(x) = (\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0)^T, \end{cases} \tag{2.1}$$

where $\mathscr{A}$ is a linear operator defined by

$$\mathscr{A}U = \begin{pmatrix} \varphi_t \\ -\frac{G}{\rho}(\psi - \varphi_x)_x \\ (3w-\psi)_t \\ \frac{G}{I_\rho}(\psi - \varphi_x) + \frac{D}{I_\rho}(3w-\psi)_{xx} - \frac{\sigma}{I_\rho}\theta_x \\ w_t \\ -\frac{G}{I_\rho}(\psi_- \varphi_x) - \frac{4\gamma}{3I_\rho}w - \frac{4\beta}{3I_\rho}w_t + \frac{D}{I_\rho}w_{xx} \\ \frac{\tau}{k}\theta_{xx} - \frac{\sigma}{k}(3w-\psi)_{tx} \end{pmatrix}.$$

We consider the following spaces:

$$H^1_*(0,1) = \left\{\eta \;\middle|\; \eta \in H^1(0,1) : \eta(0) = 0\right\}, \quad \tilde{H}^1_*(0,1) = \left\{\eta \;\middle|\; \eta \in H^1(0,1) : \eta(1) = 0\right\},$$

$$H^2_*(0,1) = H^2(0,1) \cap H^1_*(0,1), \quad \tilde{H}^2_*(0,1) = H^2(0,1) \cap \tilde{H}^1_*(0,1),$$

and

$$\mathscr{H} = H^1_*(0,1) \times L^2(0,1) \times \tilde{H}^1_*(0,1) \times L^2(0,1) \times \tilde{H}^1_*(0,1) \times L^2(0,1) \times L^2(0,1), \tag{2.2}$$

equipped with the inner product

$$\begin{aligned}\left(U, \tilde{U}\right)_{\mathscr{H}} =& \rho\int_0^1 \varphi_t\tilde{\varphi}_t \mathrm{d}x + I_\rho \int_0^1 (3w-\psi)_t(3\tilde{w} - \tilde{\psi})_t \mathrm{d}x + 3I_\rho \int_0^1 w_t\tilde{w}_t \mathrm{d}x + k\int_0^1 \theta\tilde{\theta}\mathrm{d}x \\ &+ G\int_0^1 (\psi - \varphi_x)(\tilde{\psi} - \tilde{\varphi}_x)\mathrm{d}x + D\int_0^1 (3w-\psi)_x(3\tilde{w} - \tilde{\psi})_x \mathrm{d}x + 4\gamma\int_0^1 w\tilde{w}\mathrm{d}x \\ &+ 3D\int_0^1 w_x\tilde{w}_x\mathrm{d}x.\end{aligned}$$

Then, the domain of $\mathscr{A}$ is given by

$$\begin{aligned}D(\mathscr{A}) = \Big\{ &U \in \mathscr{H} \mid \varphi \in H^2_*(0,1), 3w-\psi, w \in \tilde{H}^2_*(0,1), \theta \in H^1_*(0,1), \varphi_t \in H^1_*(0,1), \\ &3w_t - \psi_t, w_t \in \tilde{H}^1_*(0,1), \varphi_x(1,t) = 0, \psi_x(0,t) = w_x(0,t) = 0\Big\}.\end{aligned}$$

The well-posedness of problem (2.1) is ensured by

Theorem 2.1 *Let $U^0 \in \mathscr{H}$, then problem (2.1) exists a unique weak solution $U \in C(\mathbb{R}^+; \mathscr{H})$. Moreover, if $U^0 \in D(\mathscr{A})$, then*

$$U \in C(\mathbb{R}^+; D(\mathscr{A})) \cap C^1(\mathbb{R}^+; \mathscr{H}).$$

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Proof. To obtain the above result, we need to prove that $\mathscr{A}\colon D(\mathscr{A}) \to \mathscr{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathscr{A}$ is dissipative and $Id - \mathscr{A}$ is surjective.

Step 1. $\mathscr{A}$ is dissipative.

For any $U \in D(\mathscr{A})$, by using the inner product and integration by parts, we can imply that

$$\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\tau \int_0^1 \theta_x^2 \mathrm{d}x - 4\beta \int_0^1 w_t^2 \mathrm{d}x \le 0. \tag{2.3}$$

Hence, $\mathscr{A}$ is a dissipative operator.

Step 2. $Id - \mathscr{A}$ is surjective.

To prove that the operator $Id - \mathscr{A}$ is surjective, that is, for any $F = (f_1, \cdots, f_7) \in \mathscr{H}$, there exists $V = (v_1, \cdots, v_7) \in D(\mathscr{A})$ satisfying

$$(Id - \mathscr{A})V = F, \tag{2.4}$$

which is equivalent to

$$\begin{cases} v_1 - v_2 = f_1, \\ \rho v_2 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho f_2, \\ v_3 - v_4 = f_3, \\ I_\rho v_4 + G\partial_x v_1 + Gv_3 - D\partial_{xx}v_3 - 3Gv_5 + \sigma\partial_x v_7 = I_\rho f_4, \\ v_5 - v_6 = f_5, \\ \left(I_\rho + \dfrac{4\beta}{3}\right) v_6 - G\partial_x v_1 - Gv_3 + \left(3G + \dfrac{4\gamma}{3}\right) v_5 - D\partial_{xx}v_5 = I_\rho f_6, \\ kv_7 - \tau\partial_{xx}v_7 + \sigma\partial_x v_4 = kf_7. \end{cases} \tag{2.5}$$

$(2.5)_1$, $(2.5)_3$ and $(2.5)_5$ give

$$\begin{cases} v_2 = v_1 - f_1, \\ v_4 = v_3 - f_3, \\ v_6 = v_5 - f_5. \end{cases} \tag{2.6}$$

Inserting (2.6) into $(2.5)_2$, $(2.5)_4$, $(2.5)_6$ and $(2.5)_7$, we get

$$\begin{cases} \rho v_1 - G\partial_{xx}v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho(f_1 + f_2), \\ (I_\rho + G)v_3 + G\partial_x v_1 - D\partial_{xx}v_3 - 3Gv_5 + \sigma\partial_x v_7 = I_\rho(f_3 + f_4), \\ \left(I_\rho + 3G + \dfrac{4\beta}{3} + \dfrac{4\gamma}{3}\right) v_5 - G\partial_x v_1 - Gv_3 - D\partial_{xx}v_5 = I_\rho(f_5 + f_6) + \dfrac{4\beta}{3} f_5, \\ kv_7 + \sigma\partial_x v_3 - \tau\partial_{xx}v_7 = \sigma\partial_x f_3 + kf_7. \end{cases} \tag{2.7}$$

Multiplying $(2.7)_1$-$(2.7)_4$ by $\tilde{v}_1$, $\tilde{v}_3$, $3\tilde{v}_5$ and $\tilde{v}_7$ respectively, and integrating over $(0,1)$, we arrive at

$$\begin{cases} \int_0^1 \rho v_1\tilde{v}_1 \mathrm{d}x - \int_0^1 G\partial_{xx}v_1\tilde{v}_1 \mathrm{d}x - \int_0^1 G\partial_x v_3\tilde{v}_1 \mathrm{d}x + \int_0^1 3G\partial_x v_5\tilde{v}_1 \mathrm{d}x = \int_0^1 \rho(f_1+f_2)\tilde{v}_1 \mathrm{d}x, \\ \int_0^1 (I_\rho + G)v_3\tilde{v}_3 \mathrm{d}x + \int_0^1 G\partial_x v_1\tilde{v}_3 \mathrm{d}x - \int_0^1 D\partial_{xx}v_3\tilde{v}_3 \mathrm{d}x - \int_0^1 3Gv_5\tilde{v}_3 \mathrm{d}x + \int_0^1 \sigma\partial_x v_7\tilde{v}_3 \mathrm{d}x \\ = \int_0^1 I_\rho(f_3+f_4)\tilde{v}_3 \mathrm{d}x, \\ \int_0^1 (3I_\rho + 9G + 4\beta + 4\gamma)\, v_5\tilde{v}_5 \mathrm{d}x - \int_0^1 3G\partial_x v_1\tilde{v}_5 \mathrm{d}x - \int_0^1 3Gv_3\tilde{v}_5 \mathrm{d}x - \int_0^1 3D\partial_{xx}v_5\tilde{v}_5 \mathrm{d}x \\ = \int_0^1 3I_\rho(f_5+f_6)\tilde{v}_5 \mathrm{d}x + \int_0^1 4\beta f_5\tilde{v}_5 \mathrm{d}x, \\ \int_0^1 kv_7\tilde{v}_7 \mathrm{d}x + \int_0^1 \sigma\partial_x v_3\tilde{v}_7 \mathrm{d}x - \int_0^1 \tau\partial_{xx}v_7\tilde{v}_7 \mathrm{d}x = \int_0^1 \sigma\partial_x f_3\tilde{v}_7 \mathrm{d}x + \int_0^1 kf_7\tilde{v}_7 \mathrm{d}x. \end{cases} \tag{2.8}$$

The sum of the equations in (2.8) gives the following variational formulation:

$$\begin{aligned} &a\left((v_1,v_3,v_5,v_7)^T, (\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right) = \tilde{a}\left((\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right), \\ &\forall\, (\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T \in H^1_*(0,1)\times \tilde{H}^1_*(0,1)\times \tilde{H}^1_*(0,1)\times L^2(0,1), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &a\left((v_1,v_3,v_5,v_7)^T, (\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right) \\ &= \int_0^1 G(-\partial_x v_1 - v_3 + 3v_5)(-\partial_x\tilde{v}_1 - \tilde{v}_3 + 3\tilde{v}_5)\mathrm{d}x + \int_0^1 \rho v_1\tilde{v}_1 \mathrm{d}x + \int_0^1 I_\rho v_3\tilde{v}_3 \mathrm{d}x \\ &\quad + \int_0^1 (3I_\rho + 4\gamma + 4\beta)v_5\tilde{v}_5 \mathrm{d}x + \int_0^1 kv_7\tilde{v}_7 \mathrm{d}x + \int_0^1 D\partial_x v_3\partial_x\tilde{v}_3 \mathrm{d}x + \int_0^1 3D\partial_x v_5\partial_x\tilde{v}_5 \mathrm{d}x \\ &\quad + \tau\int_0^1 \partial_x v_7\partial_x\tilde{v}_7 \mathrm{d}x + \sigma\int_0^1 (\partial_x v_7)\tilde{v}_3 \mathrm{d}x + \sigma\int_0^1 (\partial_x v_3)\tilde{v}_7 \mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} &\tilde{a}\left((\tilde{v}_1,\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)^T\right) \\ &= \int_0^1 \left(\rho(f_1+f_2)\tilde{v}_1 + I_\rho(f_3+f_4)\tilde{v}_3 + 3I_\rho(f_5+f_6)\tilde{v}_5 + 4\beta f_5\tilde{v}_5 + \sigma\partial_x f_3\tilde{v}_7 + kf_7\tilde{v}_7\right)\mathrm{d}x. \end{aligned}$$

Now, we introduce the Hilbert space $V = H^1_*(0,1)\times \tilde{H}^1_*(0,1)\times \tilde{H}^1_*(0,1)\times L^2(0,1)$ equipped with the norm

$$\|(v_1,v_3,v_5,v_7)\|_V^2 = \|-\partial_x v_1 - v_3 + 3v_5\|_2^2 + \|v_1\|_2^2 + \|\partial_x v_3\|_2^2 + \|\partial_x v_5\|_2^2 + \|\partial_x v_7\|_2^2.$$

It is clear that $a(\cdot,\cdot)$ and $\tilde{a}(\cdot)$ are bounded. Furthermore, we can obtain that there exists a positive constant m such that

$$\begin{aligned} &a\left((v_1,v_3,v_5,v_7)^T, (v_1,v_3,v_5,v_7)^T\right) \\ &= \int_0^1 G(-\partial_x v_1 - v_3 + 3v_5)^2\mathrm{d}x + \int_0^1 \rho v_1^2 \mathrm{d}x + \int_0^1 I_\rho v_3^2 \mathrm{d}x + \int_0^1 (3I_\rho + 4\gamma + 4\beta)v_5^2 \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 kv_7^2\mathrm{d}x+\int_0^1 D(\partial_x v_3)^2\mathrm{d}x+\int_0^1 3D(\partial_x v_5)^2\mathrm{d}x+\tau\int_0^1(\partial_x v_7)^2\mathrm{d}x\\
\geq& m\|(v_1,v_3,v_5,v_7)\|_V^2,
\end{aligned}
$$

which implies that $a(\cdot,\cdot)$ is coercive.

Hence, we assert that $a(\cdot,\cdot)$ is a bilinear continuous coercive form on $V\times V$, and $\tilde{a}(\cdot)$ is a linear continuous form on V. Applying the Lax-Milgram theorem [18], we obtain that (2.8) has a unique solution $(v_1,v_3,v_5,v_7)^T\in V$. Then, by substituting v_1,v_3,v_5 into (2.6), we obtain

$$v_2\in H_*^1(0,1), v_4\in \tilde{H}_*^1(0,1), v_6\in \tilde{H}_*^1(0,1).$$

Next, it remains to show that

$$v_1\in H_*^2(0,1), v_3\in \tilde{H}_*^2(0,1), v_5\in \tilde{H}_*^2(0,1), v_7\in H_*^1(0,1), \partial_x v_1(1)=\partial_x v_3(0)=\partial_x v_5(0).$$

Furthermore, if $(\tilde{v}_3,\tilde{v}_5,\tilde{v}_7)\equiv(0,0,0)\in \tilde{H}_*^1(0,1)\times\tilde{H}_*^1(0,1)\times L^2(0,1)$, then (2.9) reduces to

$$\int_0^1 G\partial_{xx}v_1\tilde{v}_1\mathrm{d}x=\int_0^1\rho v_1\tilde{v}_1\mathrm{d}x-\int_0^1 G\partial_x v_3\tilde{v}_1\mathrm{d}x+\int_0^1 3G\partial_x v_5\tilde{v}_1\mathrm{d}x-\int_0^1\rho(f_1+f_2)\tilde{v}_1\mathrm{d}x, \quad (2.10)$$

for all $\tilde{v}_1\in H_*^1(0,1)$, which implies

$$G\partial_{xx}v_1=\rho v_1-G\partial_x v_3+3G\partial_x v_5-\rho(f_1+f_2)\in L^2(0,1). \quad (2.11)$$

Thus, by the L^2 theory for the linear elliptic equations, we obtain that

$$v_1\in H_*^2(0,1).$$

Moreover, (2.10) is also true for any $\phi\in C^1([0,1])\subset H_*^1(0,1)$ $(\phi(0)=0)$. Hence, we get

$$\int_0^1 G\partial_x v_1\partial_x\phi\mathrm{d}x+\int_0^1\rho v_1\phi\mathrm{d}x-\int_0^1 G(\partial_x v_3)\phi\mathrm{d}x+\int_0^1 3G(\partial_x v_5)\phi\mathrm{d}x=\int_0^1\rho(f_1+f_2)\phi\mathrm{d}x.$$

By using the integration by parts, we have

$$\partial_x v_1(1)\phi(1)=0,\ \forall\phi\in C^1([0,1]),\ \phi(0)=0.$$

Therefore,

$$\partial_x v_1(1)=0.$$

In the same way, we get

$$v_3\in\tilde{H}_*^2(0,1), v_5\in\tilde{H}_*^2(0,1), v_7\in H_*^1(0,1), \partial_x v_3(0)=\partial_x v_5(0)=0.$$

Finally, the application of the classical regularity theory for linear elliptic equations guarantees the existence of unique solution $V\in D(\mathscr{A})$ which satisfies (2.4). Hence, the operator $Id-\mathscr{A}$ is surjective. Moreover, it is easy to see that $D(\mathscr{A})$ is dense in $\mathscr{H}$.

At last, by Hille-Yosida theorem (see [19, 20]) we have the well-posedness result stated in Theorem 2.1. □

3 Exponential stability

In this Section, we prove the exponential decay for problem (1.5)-(1.6). It will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) := E(u(t)) =& \frac{1}{2}\Big(\rho\int_0^1 \varphi_t^2 \mathrm{d}x + I_\rho\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + 3I_\rho\int_0^1 w_t^2\mathrm{d}x + G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x \\ &+ D\int_0^1(3w_x-\psi_x)^2\mathrm{d}x + 3D\int_0^1 w_x^2\mathrm{d}x + 4\gamma\int_0^1 w^2\mathrm{d}x + k\int_0^1\theta^2\mathrm{d}x\Big) \end{aligned} \tag{3.1}$$

If the wave speeds are equal, we have the following exponentially stable result.

Theorem 3.1 *Assume that* $\frac{\rho}{G}=\frac{I_\rho}{D}$ *hold. Let* $U^0\in\mathscr{H}$, *then there exists positive constants* c_0, c_1 *such that the energy* $E(t)$ *associated with problem* (1.5)-(1.6) *satisfies*

$$E(t)\le c_0 e^{-c_1 t},\ \ t\ge 0. \tag{3.2}$$

To prove our this result, we will state and prove some useful lemmas in advance.

Lemma 3.2 *Let* (φ,ψ,w,θ) *be the solution of* (1.5)-(1.6). *Then the energy functional satisfies*

$$\frac{d}{dt}E(t) = -4\beta\int_0^1 w_t^2\mathrm{d}x - \tau\int_0^1\theta_x^2\mathrm{d}x\le 0,\ \ \forall\, t\ge 0. \tag{3.3}$$

Proof. First, multiplying $(1.5)_1$ by φ_t, integrating over $(0,1)$, using integration by parts and the boundary conditions in (1.6), we have

$$\frac{d}{dt}\Big\{\frac{1}{2}\rho\int_0^1\varphi_t^2\mathrm{d}x\Big\} - G\int_0^1(\psi-\varphi_x)\varphi_{xt}\mathrm{d}x = 0. \tag{3.4}$$

Note that

$$\begin{aligned} G\int_0^1(\psi-\varphi_x)\varphi_{xt}\mathrm{d}x =& -G\int_0^1(\psi-\varphi_x)(\psi-\varphi_x-\psi)_t\mathrm{d}x \\ =&\frac{d}{dt}\Big\{-\frac{1}{2}G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x\Big\} + G\int_0^1(\psi-\varphi_x)\psi_t\mathrm{d}x. \end{aligned}$$

Hence, equation (3.4) becomes

$$\frac{d}{dt}\Big\{\frac{1}{2}\Big(\rho\int_0^1\varphi_t^2\mathrm{d}x + G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x\Big)\Big\} = G\int_0^1(\psi-\varphi_x)\psi_t\mathrm{d}x. \tag{3.5}$$

Similarly, multiplying $(1.5)_2$, $(1.5)_3$, $(1.5)_4$ by $3(w-\psi)_t$, $3w_t$, θ and integrating over $(0,1)$, using integration by parts the boundary conditions in (1.6), we can get

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{2}\Big(I_\rho\int_0^1(3w_t-\psi_t)^2\mathrm{d}x + D\int_0^1(3w_x-\psi_x)^2\mathrm{d}x\Big)\Big\} \\ =&G\int_0^1(\psi-\varphi_x)(3w-\psi)_t\mathrm{d}x - \sigma\int_0^1\theta_x(3w-\psi)_t\mathrm{d}x, \\ &\frac{d}{dt}\Big\{\frac{1}{2}\Big(3I_\rho\int_0^1 w_t^2\mathrm{d}x + 4\gamma\int_0^1 w^2\mathrm{d}x + 3D\int_0^1 w_x^2\mathrm{d}x\Big)\Big\} \end{aligned} \tag{3.6}$$

$$= -3G\int_0^1(\psi-\varphi_x)w_t\mathrm{d}x - 4\beta\int_0^1 w_t^2\mathrm{d}x, \tag{3.7}$$

$$\frac{d}{dt}\left\{\frac{1}{2}k\int_0^1\theta^2\mathrm{d}x\right\} = \sigma\int_0^1(3w-\psi)_t\theta_x\mathrm{d}x - \tau\int_0^1\theta_x^2\mathrm{d}x. \tag{3.8}$$

Finally, adding (3.5)-(3.8), we obtain (3.3), which completes the proof. □

Next, in order to construct a Lyapunov functional equivalent to the energy, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.3 *Let* (φ,ψ,w,θ) *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_1(t) = -\rho\int_0^1\varphi\varphi_t\mathrm{d}x$$

satisfies the estimate

$$I_1'(t) \le -\rho\int_0^1\varphi_t^2\mathrm{d}x + (G+\varepsilon_1)\int_0^1(\psi-\varphi_x)^2\mathrm{d}x + c(\varepsilon_1)\int_0^1(3w_x-\psi_x)^2\mathrm{d}x + c(\varepsilon_1)\int_0^1 w_x^2\mathrm{d}x, \tag{3.9}$$

for any $\varepsilon_1 > 0$.

Proof. By differentiating $I_1(t)$ with respect to t, using $(1.5)_1$ and integrating by parts, we obtain

$$I_1'(t) = -\rho\int_0^1\varphi_t^2\mathrm{d}x - G\int_0^1\varphi_x(\psi-\varphi_x)\mathrm{d}x.$$

Note that

$$-G\int_0^1(\psi-\varphi_x)\varphi_x\mathrm{d}x = G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x - G\int_0^1\psi(\psi-\varphi_x)\mathrm{d}x.$$

Then, we deduce that

$$I_1'(t) = -\rho\int_0^1\varphi_t^2\mathrm{d}x + G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x - G\int_0^1\psi(\psi-\varphi_x)\mathrm{d}x.$$

Making use of Young's inequality with $\varepsilon_1 > 0$, we obtain

$$I_1'(t) \le -\rho\int_0^1\varphi_t^2\mathrm{d} + (G+\varepsilon_1)\int_0^1(\psi-\varphi_x)^2\mathrm{d}x + c(\varepsilon_1)\int_0^1\psi_x^2\mathrm{d}x.$$

Note that

$$\int_0^1\psi_x^2\mathrm{d}x = \int_0^1(\psi_x-3w_x+3w_x)^2\mathrm{d}x \le 2\int_0^1(3w_x-\psi_x)^2\mathrm{d}x + 18\int_0^1 w_x^2\mathrm{d}x.$$

Then the estimate (3.9) is established. □

Lemma 3.4 *Let* (φ,ψ,w,θ) *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_2(t) = I_\rho\int_0^1(3w-\psi)(3w-\psi)_t\mathrm{d}x$$

satisfies the estimate

$$I_2'(t) \leq -(D-\varepsilon_2)\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + I_\rho\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x \\ + c(\varepsilon_2)\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x + c(\varepsilon_2)\int_0^1 \theta^2\mathrm{d}x, \tag{3.10}$$

for any $\varepsilon_2 > 0$.

Proof. Taking the derivative of $I_5(t)$ with respect to t, using $(1.5)_2$ and integrating by parts, we get

$$I_2'(t) = -D\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + I_\rho\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x \\ + G\int_0^1 (\psi-\varphi_x)(3w-\psi)\mathrm{d}x + \sigma\int_0^1 (3w-\psi)_x\theta\mathrm{d}x.$$

Then, using Young's inequality, we arrive at (3.10). □

Lemma 3.5 *Let* (φ,ψ,w,θ) *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_3(t) = I_\rho\int_0^1 ww_t\mathrm{d}x$$

satisfies the estimate

$$I_3'(t) \leq -\left(\frac{4\gamma}{3}-\varepsilon_3\right)\int_0^1 w^2\mathrm{d}x - D\int_0^1 w_x^2\mathrm{d}x + (I_\rho + c(\varepsilon_3))\int_0^1 w_t^2\mathrm{d}x \\ + c(\varepsilon_3)\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x, \tag{3.11}$$

for any $\varepsilon_3 > 0$.

Proof. By differentiating $I_1(t)$ with respect to t, using $(1.5)_3$ and integrating by parts, we obtain

$$I_3'(t) = I_\rho\int_0^1 w_t^2\mathrm{d}x - G\int_0^1 w(\psi-\varphi_x)\mathrm{d}x - \frac{4\gamma}{3}\int_0^1 w^2\mathrm{d}x - \frac{4\beta}{3}\int_0^1 ww_t\mathrm{d}x - D\int_0^1 w_x^2\mathrm{d}x.$$

We then use Young's inequality with $\varepsilon_3 > 0$ to obtain (3.11). □

Lemma 3.6 *Let* (φ,ψ,w,θ) *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_4(t) = \frac{kI_\rho}{\sigma}\int_0^1 (3w-\psi)_t\int_0^x \theta\mathrm{d}y\mathrm{d}x$$

satisfies the estimate

$$I_4'(t) \leq -(I_\rho-\varepsilon_5)\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + (k+c(\varepsilon_4))\int_0^1 \theta^2\mathrm{d}x + \varepsilon_4\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x \\ + \varepsilon_4\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + c(\varepsilon_5)\int_0^1 \theta_x^2\mathrm{d}x, \tag{3.12}$$

for any $\varepsilon_4,\ \varepsilon_5 > 0$.

Proof. Taking the derivative of $I_4(t)$ with respect to t, using $(1.5)_2$, $(1.5)_4$ and integrating by parts, we get

$$\begin{aligned}I_4'(t) =& - I_\rho \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + \frac{kG}{\sigma} \int_0^1 (\psi - \varphi_x) \int_0^x \theta \mathrm{d}y \mathrm{d}x - \frac{kD}{\sigma} \int_0^1 (3w - \psi)_x \theta \mathrm{d}x \\ &+ k \int_0^1 \theta^2 \mathrm{d}x + \frac{\tau I_\rho}{\sigma} \int_0^1 (3w - \psi)_t \theta_x \mathrm{d}x.\end{aligned}$$

Using Young's inequality with $\varepsilon_4,\ \varepsilon_5 > 0$, we establish the (3.12). □

Lemma 3.7 *Let* $(\varphi, \psi, w, \theta)$ *be the solution of* (1.5)-(1.6). *Then the functional*

$$I_5(t) = I_\rho \int_0^1 w_t(\psi - \varphi_x)\mathrm{d}x + I_\rho \int_0^1 w_t \varphi_x \mathrm{d}x - \frac{D\rho}{G} \int_0^1 (w_x \varphi_t - w_{xt}\varphi)\mathrm{d}x$$

satisfies the estimate

$$\begin{aligned}I_5'(t) \leq& - (G - \varepsilon_6) \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + \varepsilon_6 \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ &+ c(\varepsilon_6) \int_0^1 w^2 \mathrm{d}x + c(\varepsilon_6) \int_0^1 w_t^2 \mathrm{d}x,\end{aligned} \tag{3.13}$$

for any $\varepsilon_6 > 0$.

Proof. By $(1.5)_1$, $(1.5)_3$ and integrating by parts, we get

$$\begin{aligned}&\frac{d}{dt}\left\{ I_\rho \int_0^1 w_t(\psi - \varphi_x)\mathrm{d}x \right\} \\ =& - D \int_0^1 w_x(\psi - \varphi_x)_x \mathrm{d}x - G \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x - \frac{4\gamma}{3} \int_0^1 w(\psi - \varphi_x)\mathrm{d}x - \frac{4\beta}{3} \int_0^1 w_t(\psi - \varphi_x)\mathrm{d}x \\ &+ I_\rho \int_0^1 w_t \psi_t \mathrm{d}x - I_\rho \int_0^1 w_t \varphi_{xt} \mathrm{d}x \\ =& \frac{D\rho}{G} \int_0^1 w_x \varphi_{tt} \mathrm{d}x - G \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x - \frac{4\gamma}{3} \int_0^1 w(\psi - \varphi_x)\mathrm{d}x - \frac{4\beta}{3} \int_0^1 w_t(\psi - \varphi_x)\mathrm{d}x \\ &+ I_\rho \int_0^1 w_t \psi_t \mathrm{d}x - \frac{d}{dt}\left\{ I_\rho \int_0^1 w_t \varphi_x \mathrm{d}x \right\} + I_\rho \int_0^1 w_{tt} \varphi_x \mathrm{d}x \\ =& \frac{D\rho}{G} \left\{ \frac{d}{dt} \int_0^1 (w_x \varphi_t - w_{xt}\varphi)\mathrm{d}x - \int_0^1 w_{tt} \varphi_x \mathrm{d}x \right\} - G \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x - \frac{4\gamma}{3} \int_0^1 w(\psi - \varphi_x)\mathrm{d}x \\ &- \frac{4\beta}{3} \int_0^1 w_t(\psi - \varphi_x)\mathrm{d}x + I_\rho \int_0^1 w_t \psi_t \mathrm{d}x - \frac{d}{dt}\left\{ I_\rho \int_0^1 w_t \varphi_x \mathrm{d}x \right\} + I_\rho \int_0^1 w_{tt} \varphi_x \mathrm{d}x.\end{aligned}$$

We conclude for

$$\begin{aligned}I_5'(t) =& D\left(\frac{I_\rho}{D} - \frac{\rho}{G} \right) \int_0^1 w_{tt} \varphi_x \mathrm{d}x - G \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x - \frac{4\gamma}{3} \int_0^1 w(\psi - \varphi_x)\mathrm{d}x \\ &- \frac{4\beta}{3} \int_0^1 w_t(\psi - \varphi_x)\mathrm{d}x + I_\rho \int_0^1 w_t \psi_t \mathrm{d}x.\end{aligned}$$

Using Young's inequality and $\frac{\rho}{G} = \frac{I_\rho}{D}$, we get (3.13). □

Now, we turn to prove our main result in this section.

Proof of Theorem 3.1. Let $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5 > 0$ and $\frac{\rho}{G} = \frac{I_\rho}{D}$, we define

$$L_1(t) = E(t) + \delta_1 I_1(t) + \delta_2 I_2(t) + \delta_3 I_3(t) + \delta_4 I_4(t) + \delta_5 I_5(t). \tag{3.14}$$

Using Cauchy-Schwarz inequality and Poincaré inequality, one can easily see that all the $I_i(t), i = 1, 2, 3, 4, 5$ are bounded by an expression containing the existing terms in the energy $E(t)$. This leads to the equivalence of $L_1(t)$ and $E(t)$.

Gathering the estimates in the previous lemmas, we obtain

$$\begin{aligned} L_1'(t) \le & -\delta_1 \rho \int_0^1 \varphi_t^2 \mathrm{d}x - (\delta_4 I_\rho - \delta_4 \varepsilon_5 - \delta_2 I_\rho - \delta_5 \varepsilon_6) \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ & - (4\beta - \delta_3 I_\rho - \delta_3 c(\varepsilon_3) - \delta_5 c(\varepsilon_6)) \int_0^1 w_t^2 \mathrm{d}x \\ & - (\delta_5 G - \delta_5 \varepsilon_6 - \delta_1 G - \delta_1 \varepsilon_1 - \delta_2 c(\varepsilon_2) - \delta_3 c(\varepsilon_3) - \delta_4 \varepsilon_4) \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ & - (\delta_2 D - \delta_1 c(\varepsilon_1) - \delta_2 \varepsilon_2 - \delta_4 \varepsilon_4) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x \\ & - (\delta_3 D - \delta_1 c(\varepsilon_1)) \int_0^1 w_x^2 \mathrm{d}x - \left(\frac{4\gamma}{3} \delta_3 - \delta_3 \varepsilon_3 - \delta_5 c(\varepsilon_6) \right) \int_0^1 w^2 \mathrm{d}x \\ & - (\tau - \delta_4 c(\varepsilon_5)) \int_0^1 \theta_x^2 \mathrm{d}x + (\delta_2 c(\varepsilon_2) + \delta_4 k + \delta_4 c(\varepsilon_4)) \int_0^1 \theta^2 \mathrm{d}x. \end{aligned} \tag{3.15}$$

At this point, we need to choose our constants very carefully. First, we choose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$ small enough so that

$$\begin{aligned} L_1'(t) \le & -\delta_1 \rho \int_0^1 \varphi_t^2 \mathrm{d}x - \left(\frac{\delta_4 I_\rho}{2} - \delta_2 I_\rho \right) \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ & - (4\beta - \delta_3 I_\rho - \delta_3 c(\varepsilon_3) - \delta_5 c(\varepsilon_6)) \int_0^1 w_t^2 \mathrm{d}x \\ & - \left(\frac{\delta_5 G}{2} - \delta_1 G - \delta_2 c(\varepsilon_2) - \delta_3 c(\varepsilon_3) \right) \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ & - \left(\frac{\delta_2 D}{2} - \delta_1 c(\varepsilon_1) \right) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - (\delta_3 D - \delta_1 c(\varepsilon_1)) \int_0^1 w_x^2 \mathrm{d}x \\ & - \left(\frac{2\gamma}{3} \delta_3 - \delta_5 c(\varepsilon_6) \right) \int_0^1 w^2 \mathrm{d}x - (\tau - \delta_4 c(\varepsilon_5)) \int_0^1 \theta_x^2 \mathrm{d}x \\ & + (\delta_2 c(\varepsilon_2) + \delta_4 k + \delta_4 c(\varepsilon_4)) \int_0^1 \theta^2 \mathrm{d}x. \end{aligned} \tag{3.16}$$

Then, we select δ_4 small enough so that

$$\tau - \delta_4 c(\varepsilon_5) > 0.$$

Next, we choose δ_2 small enough so that

$$\frac{\delta_4 I_\rho}{2} - \delta_2 I_\rho > 0.$$

Furthermore, we select δ_3 and δ_5 small enough so that

$$4\beta - \delta_3 I_\rho - \delta_3 c\,(\varepsilon_3) - \delta_5 c\,(\varepsilon_6) > 0,\ \ \delta_3 D - \delta_1 c(\varepsilon_1) > 0,\ \ \frac{\delta_5 G}{2} - \delta_3 c(\varepsilon_3) > 0.$$

Finally, we select δ_3 even smaller (if needed) and δ_1 small enough so that

$$\frac{\delta_2 D}{2} - \delta_1 c(\varepsilon_1) > 0,\ \ \delta_3 D - \delta_1 c(\varepsilon_1) > 0,\ \ \frac{\delta_5 G}{2} - \delta_1 G - \delta_2 c(\varepsilon_2) - \delta_3 c(\varepsilon_3) > 0.$$

From the above, we deduce that there exist positive constants C_1 and C_2 such that (3.16) becomes

$$\begin{aligned} L_1'(t) \le& - C_1 E(t) - (\tau - \delta_4 c(\varepsilon_5)) \int_0^1 \theta_x^2 \mathrm{d}x + C_2 \int_0^1 \theta^2 \mathrm{d}x \\ \le& - C_1 E(t) + C_2 \int_0^1 \theta_x^2 \mathrm{d}x. \end{aligned} \tag{3.17}$$

By (3.3), we get

$$L_1'(t) \le - C_1 E(t) - C_3 E'(t), \tag{3.18}$$

for some positive constant C_3. It is obvious that

$$\mathscr{L}_1(t) = L(t) + C_3 E(t) \sim E(t).$$

Recalling (3.18), we obtain

$$\mathscr{L}_1'(t) = L'(t) + C_3 E'(t) \le -C_1 E(t) \le -c\mathscr{L}_1(t), \tag{3.19}$$

for some positive constant c_1. Then, a simple integration of (3.19) over $(0, t)$ yields

$$\mathscr{L}_1(t) \le \mathscr{L}_1(0) e^{-c_1 t},\ \ \forall\, t \ge 0. \tag{3.20}$$

At last, estimate (3.20) gives the desired result (3.2) when combined with the equivalence of $L(t)$ and $E(t)$. $\square$

4 The lack of exponential stability

This Section is concerning the lack of exponential stability. Our result is achieved by Gearhart-Herbst-Prüss-Huang theorem to dissipative systems, see Prüss [21] and Huang [22].

Theorem 4.1 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supset \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of the differential operator $\mathcal{A}$.

Next, we state and prove the main result of this section.

Theorem 4.2 *Assume that* $\frac{\rho}{G} \neq \frac{I_\rho}{D}$ *hold. Then the semigroup associated to problem* (1.5)-(1.6) *is not exponentially stable.*

Proof. We will prove that there exists a sequence of imaginary number λ_μ and function $F_\mu \in \mathcal{H}$ with $\|F_\mu\|_{\mathcal{H}} \leq 1$ such that $\|(\lambda_\mu I - \mathcal{A})^{-1} F_\mu\|_{\mathcal{H}} = \|U_\mu\|_{\mathcal{H}} \to \infty$, where

$$\lambda_\mu U_\mu - \mathcal{A} U_\mu = F_\mu, \tag{4.1}$$

with $U_\mu = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)^T$ not bounded. Rewrite spectral equation (4.1) in term of its components, we have for $\lambda_\mu = \lambda$

$$\begin{cases} \lambda v_1 - v_2 = g_1, \\ \rho\lambda v_2 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2, \\ \lambda v_3 - v_4 = g_3, \\ I_\rho \lambda v_4 + G\partial_x v_1 + G v_3 - D\partial_{xx} v_3 - 3G v_5 + \sigma\partial_x v_7 = I_\rho g_4, \\ \lambda v_5 - v_6 = g_5, \\ I_\rho \lambda v_6 + \dfrac{4\beta}{3} v_6 - G\partial_x v_1 - G v_3 + \left(3G + \dfrac{4\gamma}{3}\right) v_5 - D\partial_{xx} v_5 = I_\rho g_6, \\ k\lambda v_7 - \tau\partial_{xx} v_7 + \sigma\partial_x v_4 = k g_7, \end{cases} \tag{4.2}$$

where $\lambda \in \mathbb{R}$ and $F = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T \in \mathcal{H}$. Taking $g_1 = g_3 = g_5 = 0$, then the above system becomes

$$\begin{cases} \rho\lambda^2 v_1 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2, \\ I_\rho \lambda^2 v_3 + G\partial_x v_1 + G v_3 - D\partial_{xx} v_3 - 3G v_5 + \sigma\partial_x v_7 = I_\rho g_4, \\ I_\rho \lambda^2 v_5 + \dfrac{4\beta}{3}\lambda v_5 - G\partial_x v_1 - G v_3 + \left(3G + \dfrac{4\gamma}{3}\right) v_5 - D\partial_{xx} v_5 = I_\rho g_6, \\ k\lambda v_7 - \tau\partial_{xx} v_7 + \lambda\sigma\partial_x v_3 = k g_7. \end{cases} \tag{4.3}$$

Because of the boundary conditions in (1.6), we can suppose that

$$v_1 = A\sin\left(\frac{\mu\pi}{2}x\right), \; v_3 = B\cos\left(\frac{\mu\pi}{2}x\right), \; v_5 = C\cos\left(\frac{\mu\pi}{2}x\right), \; v_7 = E\sin\left(\frac{\mu\pi}{2}x\right).$$

Now, choosing

$$g_2 = \frac{1}{\rho}\sin\left(\frac{\mu\pi}{2}x\right), \; g_4 = g_6 = g_7 = 0,$$

we arrive at

$$\begin{cases} \left(\rho\lambda^2 + G\left(\dfrac{\mu\pi}{2}\right)^2\right) A + G\left(\dfrac{\mu\pi}{2}\right) B - 3G\left(\dfrac{\mu\pi}{2}\right) C = 1, \\ G\left(\dfrac{\mu\pi}{2}\right) A + \left(I_\rho\lambda^2 + G + D\left(\dfrac{\mu\pi}{2}\right)^2\right) B - 3GC + \sigma\left(\dfrac{\mu\pi}{2}\right) E = 0, \\ -G\left(\dfrac{\mu\pi}{2}\right) A - GB + \left(I_\rho\lambda^2 + \dfrac{4\beta}{3}\lambda + 3G + \dfrac{4\gamma}{3} + D\left(\dfrac{\mu\pi}{2}\right)^2\right) C = 0, \\ -\lambda\sigma(\dfrac{\mu\pi}{2})B + \left(k\lambda + \tau\left(\dfrac{\mu\pi}{2}\right)^2\right) E = 0. \end{cases} \tag{4.4}$$

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Now, we take $\lambda = \lambda_\mu$ such that

$$\rho\lambda^2 + G\left(\frac{\mu\pi}{2}\right)^2 = 0,$$

then the above system can be written as

$$\begin{cases} G\left(\frac{\mu\pi}{2}\right)B - 3G\left(\frac{\mu\pi}{2}\right)C = 1, \\ G\left(\frac{\mu\pi}{2}\right)A + \left(G + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2\right)B - 3GC + \sigma\left(\frac{\mu\pi}{2}\right)E = 0, \\ -G\left(\frac{\mu\pi}{2}\right)A - GB + \left(\frac{4\beta}{3}\lambda + 3G + \frac{4\gamma}{3} + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2\right)C = 0, \\ -i\sigma\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right)^2 B + \left(ik\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right) + \tau\left(\frac{\mu\pi}{2}\right)^2\right)E = 0. \end{cases} \tag{4.5}$$

Adding $(4.5)_2$ to $(4.5)_3$, we get

$$I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 B + \sigma\left(\frac{\mu\pi}{2}\right)E + \left(\frac{4\beta}{3}\lambda + \frac{4\gamma}{3} + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2\right)C = 0. \tag{4.6}$$

From $(4.5)_4$, we get

$$E = \frac{i\sigma\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right)}{ik\sqrt{\frac{G}{\rho}} + \tau\left(\frac{\mu\pi}{2}\right)}B.$$

Substituting E into (4.6), we get

$$C = -\frac{\Lambda_\mu}{\Gamma_\mu}B,$$

where

$$\Lambda_\mu = I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 + \frac{i\sigma^2\sqrt{\frac{G}{\rho}}\left(\frac{\mu\pi}{2}\right)^2}{ik\sqrt{\frac{G}{\rho}} + \tau\left(\frac{\mu\pi}{2}\right)},$$

$$\Gamma_\mu = \frac{4\beta}{3}\lambda + \frac{4\gamma}{3} + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2.$$

Substituting C into $(4.5)_3$, we get

$$A = -\frac{G\Gamma_\mu + \Lambda_\mu\Gamma_\mu + 3G\Lambda_\mu}{G\left(\frac{\mu\pi}{2}\right)\Gamma_\mu}B.$$

Similarly, substituting C into $(4.5)_1$, we get

$$B = \frac{\Gamma_\mu}{G\left(\frac{\mu\pi}{2}\right)(\Gamma_\mu + 3\Lambda_\mu)}.$$

Let $\mu \to \infty$, we get

$$\left(\frac{\mu\pi}{2}\right)B \to \frac{1}{4G}.$$

Substituting this expression into A, C and E, we obtain for $\mu \to \infty$,

$$A \to \frac{D}{4\rho G}\left(\frac{\rho}{G} - \frac{I_\rho}{D}\right), \quad C \to O\left(\frac{1}{\mu}\right), \quad E \to O\left(\frac{1}{\mu}\right).$$

Thus

$$\begin{aligned}\|U_\mu\|^2_{\mathcal{H}} \geq G\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x &= G\left(3C - B - \left(\frac{\mu\pi}{2}\right)A\right)^2 \int_0^1 \cos^2\left(\frac{\mu\pi}{2}x\right)\mathrm{d}x \\ &= \frac{1}{2}G\left(3C - B - \left(\frac{\mu\pi}{2}\right)A\right)^2 \to \infty,\ as\ \mu\to\infty.\end{aligned}$$

Therefore, there is no exponential stability. This completes the proof. □

5 Polynomial stability

In this section, we consider the situation when the wave propagations are not the same.

Theorem 5.1 *Assume that $\frac{\rho}{G} \neq \frac{I_\rho}{D}$ hold. Let $U^0 \in \mathscr{H}$, then there exists a positive constant c_2 such that the energy $E(t)$ associated with problem* (1.5)-(1.6) *satisfies*

$$E(t) \leq \frac{c_2}{t},\ t>0. \tag{5.1}$$

Proof. In this regard, we establish a polynomial decay result. As we will see, due to the presence of the $\int_0^1 w_{tt}\varphi_x \mathrm{d}x$, we cannot directly perform the same proof as for the case where $\frac{\rho}{G} \neq \frac{I_\rho}{D}$. To overcome this difficulty, the second-order energy method is needed. The second-order energy is defined by

$$\begin{aligned}\mathcal{E}(t) =&\frac{1}{2}\Big(\rho\int_0^1 \varphi_{tt}^2\mathrm{d}x + I_\rho\int_0^1 (3w_{tt}-\psi_{tt})^2\mathrm{d}x + 3I_\rho\int_0^1 w_{tt}^2\mathrm{d}x + G\int_0^1(\psi_t-\varphi_{xt})^2\mathrm{d}x \\ &+ D\int_0^1 (3w_{xt}-\psi_{xt})^2\mathrm{d}x + 3D\int_0^1 w_{xt}^2\mathrm{d}x + 4\gamma\int_0^1 w_t^2\mathrm{d}x + k\int_0^1\theta_t^2\mathrm{d}x\Big).\end{aligned}$$

A simple calculation (Similar to (3.3)) implies that

$$\mathcal{E}'(t) = -4\beta\int_0^1 w_{tt}^2\mathrm{d}x - \tau\int_0^1 \theta_{xt}^2\mathrm{d}x. \tag{5.2}$$

As in (3.14), we also define a Lyapunov functional $L_2(t)$ as follows:

$$L_2(t) = E(t) + \mathcal{E}(t) + \delta_1 I_1(t) + \delta_2 I_2(t) + \delta_3 I_3(t) + \delta_4 I_4(t) + \delta_5 I_5(t), \tag{5.3}$$

where $I_i(t), i=1,2,3,4$ remain as defined in Lemma 3.3-Lemma 3.4 with derivatives of $I_1(t)$-$I_4(t)$ remain the same while the derivative of $I_5(t)$ is given as

$$\begin{aligned}I_5'(t) =&D\left(\frac{I_\rho}{D}-\frac{\rho}{G}\right)\int_0^1 w_{tt}\varphi_x\mathrm{d}x - G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x - \frac{4\gamma}{3}\int_0^1 w(\psi-\varphi_x)\mathrm{d}x \\ &-\frac{4\beta}{3}\int_0^1 w_t(\psi-\varphi_x)\mathrm{d}x + \int_0^1 I_\rho w_t\psi_t\mathrm{d}x \\ \leq& -(G-\varepsilon_6)\int_0^1(\psi-\varphi_x)^2\mathrm{d}x + \varepsilon_6\int_0^1(3w_t-\psi_t)^2\mathrm{d}x + c(\varepsilon_6)\int_0^1 w^2\mathrm{d}x \\ &+ c(\varepsilon_6)\int_0^1 w_t^2\mathrm{d}x + c(\varepsilon_7)\int_0^1 w_{tt}^2\mathrm{d}x + \varepsilon_7\int_0^1\varphi_x^2\mathrm{d}x,\end{aligned} \tag{5.4}$$

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for any $\varepsilon_6, \varepsilon_7 > 0$. Observing

$$\begin{aligned}\int_0^1 \varphi_x^2 \mathrm{d}x \leq& 2\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + 2\int_0^1 \psi^2 \mathrm{d}x \\ \leq& 2\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + 4\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + 36\int_0^1 w_x^2 \mathrm{d}x. \end{aligned} \tag{5.5}$$

Then, combining (5.4)-(5.5), we get

$$\begin{aligned} I_5'(t) \leq& -(G - \varepsilon_6 - \varepsilon_7)\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + \varepsilon_6 \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + c(\varepsilon_6)\int_0^1 w^2 \mathrm{d}x \\ &+ c(\varepsilon_6)\int_0^1 w_t^2 \mathrm{d}x + c(\varepsilon_7)\int_0^1 w_{tt}^2 \mathrm{d}x + \varepsilon_7 \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + \varepsilon_7 \int_0^1 w_x^2 \mathrm{d}x. \end{aligned} \tag{5.6}$$

Next, differentiating $L_2(t)$, we obtain

$$\begin{aligned} L_2'(t) \leq& -\delta_1 \rho \int_0^1 \varphi_t^2 \mathrm{d}x - (\delta_4 I_\rho - \delta_4 \varepsilon_5 - \delta_2 I_\rho - \delta_5 \varepsilon_6)\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ &- (4\beta - \delta_3 I_\rho - \delta_3 c(\varepsilon_3) - \delta_5 c(\varepsilon_6))\int_0^1 w_t^2 \mathrm{d}x \\ &- (\delta_5 G - \delta_5 \varepsilon_6 - \delta_1 G - \delta_1 \varepsilon_1 - \delta_2 c(\varepsilon_2) - \delta_3 c(\varepsilon_3) - \delta_4 \varepsilon_4 - \delta_5 \varepsilon_7)\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ &- (\delta_2 D - \delta_1 c(\varepsilon_1) - \delta_2 \varepsilon_2 - \delta_4 \varepsilon_4 - \delta_5 \varepsilon_7)\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x \\ &- (\delta_3 D - \delta_1 c(\varepsilon_1) - \delta_5 \varepsilon_7)\int_0^1 w_x^2 \mathrm{d}x - \left(\frac{4\gamma}{3}\delta_3 - \delta_3 \varepsilon_3 - \delta_5 c(\varepsilon_6)\right)\int_0^1 w^2 \mathrm{d}x \\ &- (\tau - \delta_4 c(\varepsilon_5))\int_0^1 \theta_x^2 \mathrm{d}x + (\delta_2 c(\varepsilon_2) + \delta_4 k + \delta_4 c(\varepsilon_4))\int_0^1 \theta^2 \mathrm{d}x + \delta_5 c(\varepsilon_7)\int_0^1 w_{tt}^2 \mathrm{d}x. \end{aligned} \tag{5.7}$$

At this point, we need to choose our constants very carefully. First, we choose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ small enough so that

$$\begin{aligned} L_2'(t) \leq& -\delta_1 \rho \int_0^1 \varphi_t^2 \mathrm{d}x - \left(\frac{\delta_4 I_\rho}{2} - \delta_2 I_\rho\right)\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ &- (4\beta - \delta_3 I_\rho - \delta_3 c(\varepsilon_3) - \delta_5 c(\varepsilon_6))\int_0^1 w_t^2 \mathrm{d}x \\ &- \left(\frac{\delta_5 G}{2} - \delta_1 G - -\delta_2 c(\varepsilon_2) - \delta_3 c(\varepsilon_3)\right)\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ &- \left(\frac{\delta_2 D}{2} - \delta_1 c(\varepsilon_1)\right)\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \left(\frac{\delta_3 D}{2} - \delta_1 c(\varepsilon_1)\right)\int_0^1 w_x^2 \mathrm{d}x \\ &- \left(\frac{2\gamma}{3}\delta_3 - \delta_5 c(\varepsilon_6)\right)\int_0^1 w^2 \mathrm{d}x - (\tau - \delta_4 c(\varepsilon_5))\int_0^1 \theta_x^2 \mathrm{d}x \\ &+ (\delta_2 c(\varepsilon_2) + \delta_4 k + \delta_4 c(\varepsilon_4))\int_0^1 \theta^2 \mathrm{d}x + \delta_5 c(\varepsilon_7)\int_0^1 w_{tt}^2 \mathrm{d}x. \end{aligned} \tag{5.8}$$

Then, we select δ_4 small enough so that

$$\tau - \delta_4 c(\varepsilon_5) > 0.$$

Next, we choose δ_2 small enough so that

$$\frac{\delta_4 I_\rho}{2} - \delta_2 I_\rho > 0.$$

Furthermore, we select δ_3 and δ_5 small enough so that

$$4\beta - \delta_3 I_\rho - \delta_3 c(\varepsilon_3) - \delta_5 c(\varepsilon_6) > 0,\ \delta_3 D - \delta_1 c(\varepsilon_1) > 0,\ \frac{\delta_5 G}{2} - \delta_3 c(\varepsilon_3) > 0.$$

Finally, we select δ_3 even smaller (if needed) and δ_1 small enough so that

$$\frac{\delta_2 D}{2} - \delta_1 c(\varepsilon_1) > 0,\ \delta_3 D - \delta_1 c(\varepsilon_1) > 0,\ \frac{\delta_5 G}{2} - \delta_1 G - \delta_2 c(\varepsilon_2) - \delta_3 c(\varepsilon_3) > 0.$$

Thus, we deduce that there exist positive constants C_4, C_5 and C_6 such that (5.8) becomes

$$\begin{aligned} L_2'(t) \leq & - C_4 E(t) - (\tau - \delta_4 c(\varepsilon_5)) \int_0^1 \theta_x^2 \mathrm{d}x + C_5 \int_0^1 \theta^2 \mathrm{d}x + C_6 \int_0^1 w_{tt}^2 \mathrm{d}x \\ \leq & - C_4 E(t) + C_5 \int_0^1 \theta_x^2 \mathrm{d}x + C_6 \int_0^1 w_{tt}^2 \mathrm{d}x. \end{aligned} \tag{5.9}$$

By (3.3) and (5.2), we get

$$L_2'(t) \leq - C_4 E(t) - C_7 E'(t) - C_8 \mathcal{E}'(t), \tag{5.10}$$

for some positive constant C_7 and C_8. It is obvious that

$$\mathscr{L}_2(t) = L_2(t) + C_7 E(t) + C_8 \mathcal{E}(t) \sim L_2(t). \tag{5.11}$$

Next, recalling (5.10), we obtain

$$\mathscr{L}_2'(t) = L_2'(t) + C_7 E'(t) + C_8 \mathcal{E}'(t) \leq -C_4 E(t). \tag{5.12}$$

From the above, we deduce that there exist positive constant c such that (5.12) becomes

$$L_2'(t) \leq - cE(t). \tag{5.13}$$

A simple integration of (5.13) over $(0, t)$, recalling that E is non-increasing, yields

$$tE(t) \leq \int_0^t E(s)\mathrm{d}s \leq \frac{1}{c}(L_2(0) - L_2(t)) \leq \frac{L_2(0)}{c}. \tag{5.14}$$

Finally, for a positive constant c_2, we have

$$E(t) \leq \frac{c_2}{t},\ \forall\ t > 0,$$

which completes the proof. $\square$

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