 doi:10.20944/preprints201701.0087.v1

Well-Posedness and Asymptotic Stability of Solutions to the Bresse System under Cattaneo's Law with Infinite Memories and Time-Varying Delays

Gang Li * and Xiangyu Kong

College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China; 15651559885@163.com

* Correspondence: ligang@nuist.edu.cn

In this paper, we study a one-dimensional Bresse-Cattaneo system with infinite memories and time-dependent delay term (the coefficient of which is not necessarily positive) in the internal feedbacks. First, it is proved that the system is well-posed by means of the Hille-Yosida theorem under suitable assumptions on the relaxation functions. Then, without any restriction on the speeds of wave propagations, we establish the exponential or general decay result by introducing suitable energy and Lyapunov functionals.

Keywords: well-posedness; asymptotic stability; infinite memory; Cattaneo's law; time-varying delay

AMS Subject Classification (2010): 35B40, 93D15, 93D20.

1 Introduction

The Bresse system is known as the circular arch problem (see [16] for details) and is given by the following equations:

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + IN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1 w_{tt} = N_x - IQ + F_3, \end{cases} \tag{1.1}$$

where

$$N = k_3(w_x - I\varphi),\ Q = k_1(\varphi_x + Iw + \psi),\ M = k_2\psi_x$$

denote the axial force, the shear force and the bending moment, respectively, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_3 = EA$, $k_1 = k'GA$, $k_2 = EI$, $I = R^{-1}$. The functions w, φ and ψ are the longitudinal, vertical and shear angle displacements, respectively. We use ρ for the density, E for the elastic modulus, G for the shear modulus, k' for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section, R for the radius of curvature of the beam, F_i $(i = 1, 2, 3)$ for the external forces. The arch with elastic structure is widely used in the fields of engineering, architecture, ocean engineering, aviation and others. In particular, the free vibration of elastic structure is a function of its natural property, and it is an important research subject in engineering and Mathematics. In the field of mathematical analysis is interesting to know properties which relate the behavior of the energy associated with the respective dynamic model. For feedback laws, for example, We can ask what conditions of the kinetic model can be obtained from the decay of the energy of the solution. In this sense, the property of stabilization has been studied for dynamic problems in elastic structures translated in terms of partial differential equations.

In this paper, we investigate a Bresse-Cattaneo system with infinite memories and time-varying delays in the internal feedbacks

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + Iw)_x - Ik_3(w_x - I\varphi) + \sigma\theta_x + \mu_1\varphi_t \\ +\mu_2\varphi_t(x, t - \tau_1(t)) + \int_0^{+\infty} g_1(s)\varphi_{xx}(x, t - s)\mathrm{d}s = 0, & (x, t) \in (0, L) \times (0, +\infty), \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + Iw) + \tilde{\mu}_1\psi_t \\ +\tilde{\mu}_2\psi_t(x, t - \tau_2(t)) + \int_0^{+\infty} g_2(s)\psi_{xx}(x, t - s)\mathrm{d}s = 0, & (x, t) \in (0, L) \times (0, +\infty), \\ \rho_1 w_{tt} - k_3(w_x - I\varphi)_x + Ik_1(\varphi_x + \psi + Iw) + \tilde{\tilde{\mu}}_1 w_t \\ +\tilde{\tilde{\mu}}_2 w_t(x, t - \tau_3(t)) + \int_0^{+\infty} g_3(s)w_{xx}(x, t - s)\mathrm{d}s = 0, & (x, t) \in (0, L) \times (0, +\infty), \\ \rho_3\theta_t + q_x + \sigma\varphi_{xt} = 0, & (x, t) \in (0, L) \times (0, +\infty), \\ \tau q_t + q + \gamma\theta_x = 0, & (x, t) \in (0, L) \times (0, +\infty), \end{cases} \tag{1.2}$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2, 3)$ are given functions, $\tau_i(t)$ $(i = 1, 2, 3)$ are time-varying delays, $\mu_1, \tilde{\mu}_1, \tilde{\tilde{\mu}}_1, I, \sigma, \tau, \gamma, \rho_i, k_i$ $(i = 1, 2, 3)$ are positive constants, and the infinite integrals in (1.1) represent the infinite memories. This system is subject to the Dirichlet boundary conditions

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = w(0, t) = w(L, t) = \theta(0, t) = \theta(L, t) = 0, t \in [0, +\infty) \tag{1.3}$$

and to the initial conditions

$$\begin{cases} \varphi(x, -t) = \varphi_0(x, t), \psi(x, -t) = \psi_0(x, t), w(x, -t) = w_0(x, t), & (x, t) \in [0, L] \times [0, +\infty), \\ \varphi_t(x, 0) = \varphi_1(x), \psi_t(x, 0) = \psi_1(x), w_t(x, 0) = w_1(x), & x \in [0, L], \\ \theta(x, 0) = \theta_0(x), q(x, 0) = q_0(x), & x \in [0, L], \\ \varphi_t(x, t - \tau_1(0)) = f_0(x, t - \tau_1(0)), & (x, t) \in [0, L] \times [0, \tau_1(0)], \\ \psi_t(x, t - \tau_2(0)) = \tilde{f}_0(x, t - \tau_2(0)), & (x, t) \in [0, L] \times [0, \tau_2(0)], \\ w_t(x, t - \tau_3(0)) = \tilde{\tilde{f}}_0(x, t - \tau_3(0)), & (x, t) \in [0, L] \times [0, \tau_3(0)]. \end{cases} \tag{1.4}$$

Before stating our main result, let us first mention some other papers related to the problem we address. During the last few decades, there are many works treating about existence and stabilization of Bresse system. Alabau Boussouira et al. [1] considered a Bresse system with one frictional damping working only on the angle displacement. The authors proved that the exponential decay exists when the velocities of the wave propagations are the same. If the wave speeds are different, they showed that the energy of the system decays polynomially to zero with rates that can be improved by taking more regular initial data. Liu and Rao [18] studied the Bresse system with two different dissipative mechanism, given by two temperatures coupled to the system. The authors established exponential decay rate when the vertical and longitudinal waves have the same speeds of wave propagations. Otherwise, the solution decays polynomially to zero with rates $t^{-4+\epsilon}$ or $t^{-6+\epsilon}$ provided the boundary conditions is DirichletC NeumannCNeumann or DirichletCDirichletCDirichlet type respectively. This results was improved by Fatori and

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

Rivera [9], the authors considered the Bresse system with thermal dissipation effective only in one equation of the system and obtained exponential decay result when all the wave speeds are equal. In general, they showed the system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave propagations and the regularity of the initial data. Moreover, they introduced a necessary condition to dissipative the semigroup for the polynomial decay. With respect to asymptotic behavior of solutions for Bresse system with infinte memory, Guesmia and Kafini [14] studied a one-dimensional linear Bresse system with infinite memories acting in the three equations of the system of the form

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + Iw)_x - Ik_3(w_x - I\varphi) \\ \qquad\qquad + \int_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)\mathrm{d}s = 0, \ (x,t) \in (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + Iw) \\ \qquad\qquad + \int_0^{+\infty} g_2(s)\psi_{xx}(x,t-s)\mathrm{d}s = 0, \ (x,t) \in (0,L)\times(0,+\infty), \\ \rho_1 w_{tt} - k_3(w_x - I\varphi)_x + Ik_1(\varphi_x + \psi + Iw) \\ \qquad\qquad + \int_0^{+\infty} g_3(s)w_{xx}(x,t-s)\mathrm{d}s = 0, \ (x,t) \in (0,L)\times(0,+\infty). \end{cases}$$

The authors established the well-posedness and asymptotic stability results for the system under some conditions imposed into the relaxation functions regardless to the speeds of wave propagations. For more papers concerning infinite memory, we refer to [3, 12, 13, 27].

For stabilization of Timoshenko systems via heat effect, there are some results in recent time. Almeida Júnior [2] considered 1-D thermoelastic Timoshenko beam of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \sigma\theta_x = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) - \sigma\theta = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \rho_3\theta_t - \gamma\theta_{xx} + \sigma(\varphi_x + \psi)_t = 0, & (x,t) \in (0,L)\times(0,+\infty) \end{cases} \tag{1.5}$$

with two types of boundary (Dirichlet-Dirichlet-Dirichlet or Dirichlet-Neumann-Neumann) conditions and established both exponential and polynomial stability results depending on the wave speeds and the initial data. In the above system, the heat equation is governed by Fourier's law of heat conduction. It is well known that the model using the classic Fourier's law leads to the physical paradox of infinite speed of heat propagation. That is, any local thermal disturbance can have an instantaneous effect everywhere in the medium. However, experiments showed that heat conduction in some dielectric crystals at low temperatures propagates with a finite speed. This phenomenon in dielectric crystals is called second sound. To overcome this drawback, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made. One of which is the second sound effects observed experimentally in materials at a very low temperature. This theory suggests replacing the classic Fourier's law

$$q + \gamma\theta_x = 0,$$

where q is the heat flux and γ is the coefficient of thermal conductivity by a modified law of heat conduction called Cattaneo's law

$$\tau q_t + q + \gamma\theta_x = 0,$$

 doi:10.20944/preprints201701.0087.v1

where $\tau > 0$ represents the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature. On the basis of the above theory, Santos et al. [28] studied the Timoshenko beam model with second sound of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \tau q_t + \beta q + \theta_x = 0, & (x,t) \in (0,L) \times (0,+\infty). \end{cases}$$

The authors obtained exponential decay result when the stability number $\chi_\tau = 0$. Otherwise, the polynomial decay result is obtained. Moreover, they showed that the rate is optimal. For more papers related to the second sound, we refer the reader to [4, 6, 25] and the references therein.

In recent years, the PDEs with time delays have become an active area of research and arise in many practical problems. The presence of delay may act as a source of instability. In [8], the authors showed that a small delay in a boundary control can destabilize a system which is uniformly asymptotically stable in the absence delays. To stabilize a hyperbolic system involving input delay terms, additional control will be necessary [22, 29]. Kirane and Said-Houari [20] considered a viscoelastic wave equation with a linear damping and a delay of the form

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,t-s)\mathrm{d}s + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, (x,t) \in \Omega \times (0,\infty),$$

where μ_1 and μ_2 are positive constants. They established a general decay result under the condition that $\mu_2 \le \mu_1$. Later, Liu [15] improved this result by considering the equation with a time-varying delay term, with not necessarily positive coefficient μ_2 of the delay term. Moreover, some researchers considered the Timoshenko and Bresse systems with delay term. For instance, Said-Houari and Laskri [26] considered the following Timoshenko system with a constant time delay of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0, & (x,t) \in (0,1) \times (0,+\infty). \end{cases}$$

They established an exponential decay result for the case of equal-speed wave propagation under the assumption $\mu_2 < \mu_1$. More recently, Kirane el al. [19] considered the following Timoshenko system with a time-varying delay of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau(t)) = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases}$$

where $\tau(t)$ represents the time-varying delay, $0 < \tau_0 \le \tau(t) \le \bar{\tau}$ and μ_1, μ_2 are positive constants. Under the assumpyions $\mu_2 < \sqrt{1-d_1}\mu_1$ and $\tau'(t) \le 1$, they proved the exponential decay result.

Motivated by the above results, we investigate in this paper system (1.1) under suitable assumptions and prove the well-posedness and the asymptotic stability of system.

Using $(1.1)_5$ and the boundary condition, we can easily verify that

$$\frac{d}{dt}\int_0^L q(x,t)\mathrm{d}x + \frac{1}{\tau}\int_0^L q(x,t)\mathrm{d}x = 0.$$

 doi:10.20944/preprints201701.0087.v1

Consequently, we obtain

$$\int_0^L q(x,t)\mathrm{d}x = e^{-\frac{t}{\tau}} \int_0^L q_0(x)\mathrm{d}x.$$

If we set

$$\bar{q}(x,t)\mathrm{d}x = q(x,t) - \frac{e^{-\frac{t}{\tau}}}{L} \int_0^L q_0(x)\mathrm{d}x,$$

then simple substitution shows that $(\varphi, \psi, w, \theta, \bar{q})$ satisfies problem (1.1), and we have

$$\int_0^L \bar{q}(x,t)\mathrm{d}x = 0.$$

From now on, we use the new variables $(\varphi, \psi, w, \theta, \bar{q})$, but we denote them by $(\varphi, \psi, w, \theta, q)$ for simplicity.

The main difficulty in carrying out this paper is the simultaneous appearance of the infinite memories, heat effect and time-varying delay. To overcome this difficulty, we have two key points in the proofs. On the one hand, to create the negative counterparts of the terms in the energy, we combine the fireworks of [4], [14] and [19] with necessary modifications. On the other hand, to estimate the infinite integral terms in (4.20) below, we use the approach which was first proved by Guesmia [10] and used by many researchers (see [11, 14]).

This paper is organized as follows. In section 2, we present some assumptions needed for our work and state the main results. In section 3, we prove the well-posedness of problem (1.1). In section 4, we prove the asymptotic stability of problem (1.1).

2 Preliminaries and main results

In this section, we shall introduce some notation, basic definitions and main results which will be needed in the course of this paper.

First, we assume the following hypotheses:

$$\tau_i(t) \in W^{2,\infty}([0,T]), \quad \forall T > 0, i = 1, 2, 3, \tag{2.1}$$

$$\begin{aligned} &0 < \tau_{01} < \tau_1(t) < \tilde{\tau}_1, \quad \forall t > 0, \\ &0 < \tau_{02} < \tau_2(t) < \tilde{\tau}_2, \quad \forall t > 0, \\ &0 < \tau_{03} < \tau_3(t) < \tilde{\tau}_3, \quad \forall t > 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\tau_1'(t) \le d_1 < 1, \quad \forall t > 0, \\ &\tau_2'(t) \le d_2 < 1, \quad \forall t > 0, \\ &\tau_3'(t) \le d_3 < 1, \quad \forall t > 0, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &|\mu_2| \le \sqrt{1-d_1}\mu_1, \\ &|\tilde{\mu}_2| \le \sqrt{1-d_2}\tilde{\mu}_1, \\ &|\tilde{\tilde{\mu}}_2| \le \sqrt{1-d_3}\tilde{\tilde{\mu}}_1, \end{aligned} \tag{2.4}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

where $\tau_{01}, \tau_{02}, \tau_{03}, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, d_1, d_2, d_3$ are positive constants.

Next, let us consider the following variables (see [7]):

$$\begin{cases} \eta_1(x,t,s) = \varphi(x,t) - \varphi(x,t-s), \ (x,t,s) \in (0,L) \times (0,+\infty) \times (0,+\infty), \\ \eta_2(x,t,s) = \psi(x,t) - \psi(x,t-s), \ (x,t,s) \in (0,L) \times (0,+\infty) \times (0,+\infty), \\ \eta_3(x,t,s) = w(x,t) - w(x,t-s), \ (x,t,s) \in (0,L) \times (0,+\infty) \times (0,+\infty), \end{cases} \tag{2.5}$$

$$\begin{cases} z_1(x,\rho,t) = \varphi_t(x,t-\tau_1(t)\rho), (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \\ z_2(x,\rho,t) = \psi_t(x,t-\tau_2(t)\rho), (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \\ z_3(x,\rho,t) = w_t(x,t-\tau_3(t)\rho), (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty). \end{cases} \tag{2.6}$$

Then, it is easy to check that

$$\begin{cases} \partial_t\eta_1 + \partial_s\eta_1 - \varphi_t = 0, & (x,t,s) \in (0,L) \times (0,+\infty) \times (0,+\infty), \\ \partial_t\eta_2 + \partial_s\eta_2 - \psi_t = 0, & (x,t,s) \in (0,L) \times (0,+\infty) \times (0,+\infty), \\ \partial_t\eta_3 + \partial_s\eta_3 - w_t = 0, & (x,t,s) \in (0,L) \times (0,+\infty) \times (0,+\infty), \\ \eta_i(0,t,s) = \eta_i(L,t,s) = 0, & (t,s) \in [0,+\infty) \times [0,+\infty), \ i = 1,2,3, \\ \eta_i(x,t,0) = 0, & (x,t) \in (0,L) \times (0,+\infty), \ i = 1,2,3, \end{cases} \tag{2.7}$$

$$\tau_i(t)z_{it}(x,\rho,t) + (1-\tau_i'(t)\rho)z_{i\rho}(x,\rho,t) = 0, \ (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \ i = 1,2,3. \tag{2.8}$$

Therefore, problem (1.1) takes the form:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + Iw)_x - Ik_3(w_x - I\varphi) + \mu_1\varphi_t + \mu_2 z_1(x,1,t) \\ + \int_0^{+\infty} g_1(s)\varphi_{xx}\mathrm{d}s - \int_0^{+\infty} g_1(s)\partial_{xx}\eta_1(x,t)\mathrm{d}s + \sigma\theta_x = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + Iw) + \tilde{\mu}_1\psi_t + \tilde{\mu}_2 z_2(x,1,t) \\ + \int_0^{+\infty} g_2(s)\psi_{xx}\mathrm{d}s - \int_0^{+\infty} g_2(s)\partial_{xx}\eta_2(x,t)\mathrm{d}s = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_1 w_{tt} - k_3(w_x - I\varphi)_x + Ik_1(\varphi_x + \psi + Iw) + \tilde{\tilde{\mu}}_1 w_t + \tilde{\tilde{\mu}}_2 z_3(x,1,t) \\ + \int_0^{+\infty} g_3(s)w_{xx}\mathrm{d}s - \int_0^{+\infty} g_3(s)\partial_{xx}\eta_3(x,t)\mathrm{d}s = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \rho_3\theta_t + q_x + \sigma\varphi_{xt} = 0, & (x,t) \in (0,L) \times (0,+\infty), \\ \tau q_t + q + \gamma\theta_x = 0, & (x,t) \in (0,L) \times (0,+\infty). \\ \tau_1(t)z_{1t}(x,\rho,t) + (1-\tau_1'(t)\rho)z_{1\rho}(x,\rho,t) = 0, & (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \\ \tau_2(t)z_{2t}(x,\rho,t) + (1-\tau_2'(t)\rho)z_{2\rho}(x,\rho,t) = 0, & (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \\ \tau_3(t)z_{3t}(x,\rho,t) + (1-\tau_3'(t)\rho)z_{3\rho}(x,\rho,t) = 0, & (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \end{cases} \tag{2.9}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

The above system is subject to the following initial and boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) \\ \qquad\qquad = w(0,t) = w(L,t) = \theta(0,t) = \theta(L,t) = 0, & t \in [0,+\infty), \\ \varphi(x,0) = \varphi_0(x), \psi(x,0) = \psi_0(x), w(x,0) = w_0(x), & x \in [0,L], \\ \varphi_t(x,0) = \varphi_1(x), \psi_t(x,0) = \psi_1(x), w_t(x,0) = w_1(x), & x \in [0,L], \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), & x \in [0,L], \\ z_1(x,0,t) = \varphi_t(x,t), z_2(x,0,t) = \psi_t(x,t), z_3(x,0,t) = w_t(x,t), \\ & (x,t) \in [0,L] \times [0,+\infty), \\ z_1(x,1,t) = f_0(x,t-\tau_1(t)), z_2(x,1,t) = \tilde{f}_0(x,t-\tau_2(t)),, \\ \qquad\qquad z_3(x,1,t) = \tilde{\tilde{f}}_0(x,t-\tau_3(t)) & (x,t) \in [0,L] \times [0,+\infty), \\ \eta_1(0,t,s) = \eta_1(L,t,s) = \eta_2(0,t,s) = \eta_2(L,t,s) \\ \qquad\qquad = \eta_3(0,t,s) = \eta_3(L,t,s) = 0, & (t,s) \in [0,+\infty) \times [0,+\infty), \\ \eta_1(x,t,0) = \eta_2(x,t,0) = \eta_3(x,t,0) = 0, & (x,t) \in [0,L] \times [0,+\infty), \\ \eta_1(x,0,s) = \eta_1^0(x,\cdot), \eta_2(x,0,s) = \eta_2^0(x,\cdot), \eta_3(x,0,s) = \eta_3^0(x,\cdot), & (x,s) \in [0,L] \times [0,+\infty). \end{cases} \tag{2.10}$$

Let

$$U = (\varphi, \psi, w, \theta, q, \varphi_t, \psi_t, w_t, z_1, z_2, z_3, \eta_1, \eta_2, \eta_3)^T$$

and

$$\begin{aligned} U^0(x) = (&\varphi_0(x,0), \psi_0(x,0), w_0(x,0), \theta(x,0), q(x,0), \varphi_1(x), \psi_1(x), w_1(x), \\ &f(\cdot, -\tau_1(0)\rho), f(\cdot, -\tau_2(0)\rho), f(\cdot, -\tau_3(0)\rho), \eta_1^0(x,\cdot), \eta_2^0(x,\cdot), \eta_3^0(x,\cdot))^T. \end{aligned}$$

Then problem (2.9)-(2.10) can be written as

$$\begin{cases} \partial_t U = \mathscr{A} U, \\ U(x,0) = U^0(x), \end{cases} \tag{2.11}$$

where time varying operator $\mathscr{A}$ is defined by

$$\mathscr{A}U = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ -\frac{1}{\rho_3}q_x - \frac{\sigma}{\rho_3}\varphi_{xt} \\ -\frac{1}{\tau}q - \frac{\gamma}{\tau}\theta_x \\ \frac{1}{\rho_1}\left(k_1 - \int_0^{+\infty} g_1(s)\mathrm{d}s\right)\varphi_{xx} - \frac{I^2k_3}{\rho_1}\varphi + \frac{k_1}{\rho_1}\psi_x + \frac{I}{\rho_1}(k_1+k_3)w_x - \frac{\mu_1}{\rho_1}\varphi_t - \frac{\mu_2}{\rho_1}z_1(\cdot,1) - \frac{\sigma}{\rho_1}\theta_x \\ +\frac{1}{\rho_1}\int_0^{+\infty} g_1(s)\partial_{xx}\eta_1\mathrm{d}s \\ -\frac{k_1}{\rho_2}\varphi_x + \frac{1}{\rho_2}\left(k_2 - \int_0^{+\infty} g_2(s)\mathrm{d}s\right)\psi_{xx} - \frac{k_1}{\rho_2}\psi - \frac{Ik_1}{\rho_2}w - \frac{\tilde{\mu}_1}{\rho_2}\psi_t - \frac{\tilde{\mu}_2}{\rho_2}z_2(\cdot,1) \\ +\frac{1}{\rho_2}\int_0^{+\infty} g_2(s)\partial_{xx}\eta_2\mathrm{d}s \\ -\frac{I}{\rho_1}(k_1+k_3)\varphi_x - \frac{Ik_1}{\rho_1}\psi + \frac{1}{\rho_1}\left(k_3 - \int_0^{\infty} g_3(s)\mathrm{d}s\right)w_{xx} - \frac{I^2k_1}{\rho_1}w - \frac{\tilde{\tilde{\mu}}_1}{\rho_1}w_t - \frac{\tilde{\tilde{\mu}}_2}{\rho_1}z_3(\cdot,1) \\ +\frac{1}{\rho_1}\int_0^{\infty} g_3(s)\partial_{xx}\eta_3\mathrm{d}s \\ -\frac{1-\tau_1'(t)\rho}{\tau_1(t)}z_{1\rho} \\ -\frac{1-\tau_2'(t)\rho}{\tau_2(t)}z_{2\rho} \\ -\frac{1-\tau_3'(t)\rho}{\tau_3(t)}z_{3\rho} \\ \varphi_t - \partial_s\eta_1 \\ \psi_t - \partial_s\eta_2 \\ w_t - \partial_s\eta_3 \end{pmatrix}.$$

Now, we consider the following space

$$L_*^2(0,L) = \left\{\omega \in L^2(0,L) : \int_0^L \omega(s)\mathrm{d}s = 0\right\}, \quad H_*^1 = H^1(0,L) \cap L_*^2(0,L)$$

and define the functional spaces of U as follow:

$$\mathscr{H} = \left(H_0^1(0,L)\right)^3 \times L^2(0,L) \times L_*^2(0,L) \times \left(L^2(0,L)\right)^3 \times \left(L^2((0,1)\times(0,L))\right)^3 \times H_1^* \times H_2^* \times H_3^*, \tag{2.12}$$

where

$$H_i^* = \left\{\omega : \mathbb{R}^+ \longrightarrow H_0^1(0,L), \int_0^L \int_0^{+\infty} g_i(s)\omega_x^2\mathrm{d}s\mathrm{d}x < +\infty\right\}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

Then the domain of $\mathscr{A}$ is defined by

$$D(\mathscr{A}) = \Big\{U \in \left(H^2(0,L) \cap H_0^1(0,L)\right)^3 \times H_0^1(0,L) \times H_*^1(0,L) \times \left(H_0^1(0,L)\right)^3 \times L^2\left((0,L), H^1(0,L)\right)^3 \times \mathscr{H}_1^* \times \mathscr{H}_2^* \times \mathscr{H}_3^*\Big\},$$

where

$$\mathscr{H}_i^* = \Big\{\omega_i \in H_i^*, \partial_s\omega_i \in H_i^*, \omega_i(x,t,0) = 0\Big\}.$$

For the relaxation functions g_i, motivated by [14], we have the following assumptions:

(**G1**) $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1,2,3)$ are non-increasing differentiable functions and integrate on $(0,+\infty)$ such that there exists a positive constant k_0 satisfying, for any $(\varphi,\psi,w) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$,

$$k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x \le \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + Iw)^2 + k_3(w_x - I\varphi)^2\right) \mathrm{d}x - \int_0^L \Big[\left(\int_0^{+\infty} g_1(s)\mathrm{d}s\right)\varphi_x^2 + \left(\int_0^{+\infty} g_2(s)\mathrm{d}s\right)\psi_x^2 + \left(\int_0^{+\infty} g_3(s)\mathrm{d}s\right) w_x^2\Big]\mathrm{d}x. \tag{2.13}$$

Remark 1 *It is easy to check that there exists a positive constant $\bar{k}_0$ such that, for any $(\varphi,\psi,w) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$,*

$$\bar{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x \le \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + Iw)^2 + k_3(w_x - I\varphi)^2\right) \mathrm{d}x. \tag{2.14}$$

Therefore, let

$$g_i^0 := \int_0^{+\infty} g_i(s)\mathrm{d}s < \bar{k}_0, \quad i = 1,2,3, \tag{2.15}$$

then (2.5) is satisfied with

$$k_0 = \bar{k}_0 - \max\left\{g_1^0,\ g_2^0,\ g_3^0\right\}.$$

On the other hand, due to Poincaré's inequality, there exists a positive constant $\tilde{k}_0$ such that, for $(\varphi,\psi,w) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$,

$$\int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + Iw)^2 + k_3(w_x - I\varphi)^2\right) \mathrm{d}x \le \tilde{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x. \tag{2.16}$$

Remark 2 *Under hypothesis (G1), H_i^* and $\mathscr{H}$ are Hilbert spaces, respectively, with the inner products that generate the norms*

$$\|\eta_i\|_{H_i^*}^2 = \int_0^L \int_0^{+\infty} g_i(s)(\partial_x\eta_i)^2 \mathrm{d}s\mathrm{d}x$$

and

$$\|U\|_{\mathscr{H}}^2 = \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + k_2\psi_x^2 + k_1(\varphi_x + \psi + Iw)^2 + k_3(w_x - I\varphi)^2 + \rho_3\theta^2\right.$$

$$+\frac{\tau}{\gamma}q^2\Big)\,\mathrm{d}x - \int_0^L \left(g_1^0\varphi_x^2 + g_2^0\psi_x^2 + g_3^0 w_x^2\right)\mathrm{d}x + \sum_{i=1}^{3}\zeta_i\tau_i(t)\int_0^L\int_0^1 z_i^2\,\mathrm{d}\rho\,\mathrm{d}x$$
$$+\|\eta_1\|_{H_1^*}^2 + \|\eta_2\|_{H_2^*}^2 + \|\eta_3\|_{H_3^*}^2,$$

where ζ_i $(i = 1, 2, 3)$ are positive constants such that

$$\begin{cases} \dfrac{|\mu_2|}{\sqrt{1-d_1}} \le \zeta_1 \le 2\mu_1 - \dfrac{|\mu_2|}{\sqrt{1-d_1}}, \\ \dfrac{|\tilde{\mu}_2|}{\sqrt{1-d_2}} \le \zeta_2 \le 2\tilde{\mu}_1 - \dfrac{|\tilde{\mu}_2|}{\sqrt{1-d_2}}, \\ \dfrac{|\tilde{\tilde{\mu}}_2|}{\sqrt{1-d_3}} \le \zeta_3 \le 2\tilde{\tilde{\mu}}_1 - \dfrac{|\tilde{\tilde{\mu}}_2|}{\sqrt{1-d_3}}. \end{cases} \tag{2.17}$$

(**G2**) For $i = 1, 2, 3$, there exist positive constants δ_i $(i = 1, 2, 3)$ such that

$$g_i'(s) \le -\delta_i g_i(s), \quad \forall s \in \mathbb{R}^+ \tag{2.18}$$

or there exists an increasing strictly convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(0, +\infty)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty \tag{2.19}$$

such that

$$\int_0^{+\infty} \frac{g_i(s)}{G^{-1}(-g_i'(s))}\mathrm{d}s + \sup_{s\in\mathbb{R}^+} \frac{g_i(s)}{G^{-1}(-g_i'(s))} < +\infty. \tag{2.20}$$

Now, we state the well-posedness result of problem (2.9)

Theorem 2.1 *Assume that (G1) holds. Let $U^0 \in \mathscr{H}$, then problem (2.9) has a unique weak solution*

$$U \in C(\mathbb{R}^+; \mathscr{H}).$$

Moreover, if $U^0 \in D(\mathscr{A})$, then

$$U \in C(\mathbb{R}^+; D(\mathscr{A})) \cap C^1(\mathbb{R}^+; \mathscr{H}).$$

The energy associated with problem (2.9) is defined by

$$\begin{aligned} E(t) =& \frac{1}{2}\Bigg(\int_0^L \Big(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + k_2\psi_x^2 + k_1(\varphi_x + \psi + Iw)^2 + k_3(w_x - I\varphi)^2 + \rho_3\theta^2 \\ &+ \frac{\tau}{\gamma}q^2\Big)\mathrm{d}x - \int_0^L \left(g_1^0\varphi_x^2 + g_2^0\psi_x^2 + g_3^0 w_x^2\right)\mathrm{d}x + \sum_{i=1}^{3}\zeta_i\tau_i(t)\int_0^L\int_0^1 z_i^2\,\mathrm{d}\rho\,\mathrm{d}x \\ &+ \|\eta_1\|_{H_1^*}^2 + \|\eta_2\|_{H_2^*}^2 + \|\eta_3\|_{H_3^*}^2\Bigg). \end{aligned} \tag{2.21}$$

Our decay results read as follows:

Theorem 2.2 *Let $U^0 \in \mathscr{H}$ be given. Assume that (G1) and (G2) hold,*

(i) If for all $i = 1, 2, 3$, (2.18) holds, then there exist positive constants c', c'' such that

$$E(t) \le c'' e^{-c't}, \tag{2.22}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

(ii) If for $i = 1,2,3$, either (2.18) *holds or* (2.19), (2.20) *and*

$$\exists\ M_i \geq 0 : \int_0^L (\partial_x \eta_i^0)^2 \mathrm{d}x \leq M_i,\ \ \forall\ s > 0 \text{ hold}, \tag{2.23}$$

then there exist positive constants c', c'' and ϵ_0 such that

$$E(t) \leq c'' H^{-1}(c't), \tag{2.24}$$

where

$$H(s) = \int_s^1 \frac{1}{\tau G'(\epsilon_0 \tau)} \mathrm{d}\tau,\ \ \forall\ s \in (0,1]. \tag{2.25}$$

3 Proof of the well-posedness

In this section, we prove the well-posedness of the solution of problem (2.11). For this purpose, we will follow the method used in [4],[14] and [23] with the necessary modification imposed by the nature of our problem.

Proof of Theorem 2.1. In order to prove result stated in Theorem 2.1, first, we prove that the operator $\mathscr{A}$ is dissipative. A simple computation implies that, for any $U \in D(\mathscr{A})$,

$$\begin{aligned}
\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = &- \left(\mu_1 - \frac{\zeta_1}{2}\right) \int_0^L \varphi_t^2 \mathrm{d}x - \left(\tilde{\mu}_1 - \frac{\zeta_2}{2}\right) \int_0^L \psi_t^2 \mathrm{d}x - \left(\tilde{\tilde{\mu}}_1 - \frac{\zeta_3}{2}\right) \int_0^L w_t^2 \mathrm{d}x \\
&- \mu_2 \int_0^L z_1(x,1,t)\varphi_t \mathrm{d}x - \tilde{\mu}_2 \int_0^L z_2(x,1,t)\psi_t \mathrm{d}x - \tilde{\tilde{\mu}}_2 \int_0^L z_3(x,1,t) w_t \mathrm{d}x \\
&- \frac{1}{2}\int_0^L \int_0^{+\infty} g_1(s)\partial_s(\partial_x \eta_1)^2 \mathrm{d}s\mathrm{d}x - \frac{1}{2}\int_0^L \int_0^{+\infty} g_2(s)\partial_s(\partial_x \eta_2)^2 \mathrm{d}s\mathrm{d}x \\
&- \frac{1}{2}\int_0^L \int_0^{+\infty} g_3(s)\partial_s(\partial_x \eta_3)^2 \mathrm{d}s\mathrm{d}x - \int_0^L \frac{1}{\gamma} q^2 \mathrm{d}x \\
&- \sum_{i=1}^{3} \frac{\zeta_i (1-\tau_i'(t))}{2} \int_0^L z_i^2(x,1,t)\mathrm{d}x.
\end{aligned}$$

Integrating by parts, using (G1) and the boundary conditions in (2.10), we get

$$\begin{aligned}
\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = &- \left(\mu_1 - \frac{\zeta_1}{2}\right) \int_0^L \varphi_t^2 \mathrm{d}x - \left(\tilde{\mu}_1 - \frac{\zeta_2}{2}\right) \int_0^L \psi_t^2 \mathrm{d}x - \left(\tilde{\tilde{\mu}}_1 - \frac{\zeta_3}{2}\right) \int_0^L w_t^2 \mathrm{d}x \\
&- \mu_2 \int_0^L z_1(x,1,t)\varphi_t \mathrm{d}x - \tilde{\mu}_2 \int_0^L z_2(x,1,t)\psi_t \mathrm{d}x - \tilde{\tilde{\mu}}_2 \int_0^L z_3(x,1,t) w_t \mathrm{d}x \\
&+ \frac{1}{2}\int_0^L \int_0^{+\infty} \left(g_1'(s)(\partial_x \eta_1)^2 + g_2'(s)(\partial_x \eta_2)^2 + g_3'(s)(\partial_x \eta_3)^2\right) \mathrm{d}s\mathrm{d}x - \int_0^L \frac{1}{\gamma} q^2 \mathrm{d}x \\
&- \sum_{i=1}^{3} \frac{\zeta_i (1-\tau_i'(t))}{2} \int_0^L z_i^2(x,1,t)\mathrm{d}x.
\end{aligned}$$

Then, using Young's inequality, we obtain

$$\langle \mathscr{A}U, U\rangle_{\mathscr{H}} \leq - \left(\mu_1 - \frac{\zeta_1}{2} - \frac{|\mu_2|}{2\sqrt{1-d}}\right) \int_0^L \varphi_t^2 \mathrm{d}x - \left(\tilde{\mu}_1 - \frac{\zeta_2}{2} - \frac{|\tilde{\mu}_2|}{2\sqrt{1-d}}\right) \int_0^L \psi_t^2 \mathrm{d}x$$

$$
\begin{aligned}
&-\left(\tilde{\tilde{\mu}}_1-\frac{\zeta_3}{2}-\frac{|\tilde{\tilde{\mu}}_2|}{2\sqrt{1-d}}\right)\int_0^L w_t^2 \mathrm{d}x \\
&-\left(\frac{\zeta_1(1-\tau_1'(t))}{2}-\frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_0^L z_1^2(x,1,t)\mathrm{d}x \\
&-\left(\frac{\zeta_2(1-\tau_2'(t))}{2}-\frac{|\tilde{\mu}_2|\sqrt{1-d}}{2}\right)\int_0^L z_2^2(x,1,t)\mathrm{d}x \\
&-\left(\frac{\zeta_3(1-\tau_3'(t))}{2}-\frac{|\tilde{\tilde{\mu}}_2|\sqrt{1-d}}{2}\right)\int_0^L z_3^2(x,1,t)\mathrm{d}x \\
&+\frac{1}{2}\int_0^L\int_0^{+\infty}\left(g_1'(s)(\partial_x\eta_1)^2+g_2'(s)(\partial_x\eta_2)^2+g_3'(s)(\partial_x\eta_3)^2\right)\mathrm{d}s\mathrm{d}x-\int_0^L\frac{1}{\gamma}q^2\mathrm{d}x.
\end{aligned}
\tag{3.1}
$$

Notice the fact that, for $i = 1, 2, 3$, the kernel g_i is non-increasing and using (2.3), (2.17), we obtain

$$\langle \mathscr{A}U, U\rangle_{\mathscr{H}} \leq 0. \tag{3.2}$$

Hence, $\mathscr{A}$ is a dissipative operator.

Next, we prove that the operator $Id - \mathscr{A}$ is surjective. For this purpose, Given

$$F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}) \in \mathscr{H},$$

we prove that there exists

$$V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, z_1, z_2, z_3, v_9, v_{10}, v_{11}) \in D(\mathscr{A})$$

satisfying

$$(Id - \mathscr{A})V = F. \tag{3.3}$$

Equation (3.3) is equivalent to

$$\begin{cases} v_1 - v_6 = f_1, \\ v_2 - v_7 = f_2, \\ v_3 - v_8 = f_3, \\ \rho_3 v_4 + \partial_x v_5 + \sigma \partial_x v_6 = \rho_3 f_4, \\ (1+\tau) v_5 + \gamma \partial_x v_4 = \tau f_5, \\ \rho_1 v_6 - \left(k_1 - \int_0^{+\infty} g_1(s) \mathrm{d}s\right) \partial_{xx} v_1 + I^2 k_3 v_1 - k_1 \partial_x v_2 - I(k_1+k_3)\partial_x v_3 + \sigma \partial_x v_4 \\ \qquad\qquad + \mu_1 v_6 + \mu_2 z_1(.,1) - \int_0^{+\infty} g_1(s) \partial_{xx} v_9 \mathrm{d}s = \rho_1 f_6, \\ \rho_2 v_7 + k_1 \partial_x v_1 - \left(k_2 - \int_0^{+\infty} g_2(s) \mathrm{d}s\right) \partial_{xx} v_2 + k_1 v_2 + I k_1 v_3 \\ \qquad\qquad + \tilde{\mu}_1 v_7 + \tilde{\mu}_2 z_2(.,1) - \int_0^{+\infty} g_2(s) \partial_{xx} v_{10} \mathrm{d}s = \rho_2 f_7, \\ \rho_1 v_8 + I(k_1+k_3)\partial_x v_1 + I k_1 v_2 - \left(k_3 - \int_0^{\infty} g_3(s) \mathrm{d}s\right) \partial_{xx} v_3 + I^2 k_1 v_3 \\ \qquad\qquad + \tilde{\tilde{\mu}}_1 v_8 + \tilde{\tilde{\mu}}_2 z_3(.,1) - \int_0^{\infty} g_3(s) \partial_{xx} v_{11} \mathrm{d}s = \rho_1 f_8, \\ z_1 + \dfrac{1-\tau_1'(t)\rho}{\tau_1(t)} z_{1\rho} = f_9, \\ z_2 + \dfrac{1-\tau_2'(t)\rho}{\tau_2(t)} z_{1\rho} = f_{10}, \\ z_3 + \dfrac{1-\tau_3'(t)\rho}{\tau_3(t)} z_{1\rho} = f_{11}, \\ v_9 - v_6 + \partial_s v_9 = f_{12}, \\ v_{10} - v_7 + \partial_s v_{10} = f_{13}, \\ v_{11} - v_8 + \partial_s v_{11} = f_{14}. \end{cases} \tag{3.4}$$

Suppose v_1, v_2, v_3, v_5 are found with the appropriate regularity, then $(3.4)_1$-$(3.4)_3$ and $(3.4)_5$ give

$$\begin{aligned} v_6 &= v_1 - f_1 \in H_0^1(0,L), \\ v_7 &= v_2 - f_2 \in H_0^1(0,L), \\ v_8 &= v_3 - f_3 \in H_0^1(0,L), \\ \partial_x v_4 &= -\frac{(1+\tau)}{\gamma} v_5 + \frac{\tau}{\gamma} f_5 \in L_*^2(0,L). \end{aligned} \tag{3.5}$$

The last equation of (3.5) yield

$$v_4 = -\frac{(1+\tau)}{\gamma} \int_0^x v_5 \mathrm{d}y + \frac{\tau}{\gamma} \int_0^x f_5 \mathrm{d}y, \tag{3.6}$$

then

$$v_4(0,t) = v_4(L,t) = 0.$$

 doi:10.20944/preprints201701.0087.v1

On the other hand, by using (2.6), we can find z_1, z_2, z_3 as

$$z_1(x,0) = v_6,\ z_2(x,0) = v_7,\ z_3(x,0) = v_8,\ \text{for } x \in (0,L). \tag{3.7}$$

Following the same approach in [23] and using $(3.4)_9$-$(3.4)_{11}$, we obtain

$$z_1(x,\rho) = v_6 e^{-\rho\tau_1(t)} + \tau_1(t)e^{-\rho\tau_1(t)} \int_0^\rho f_9(x,\sigma)e^{\sigma\tau_1(t)}\mathrm{d}\sigma,\ \text{if } \tau_1'(t) = 0$$

and

$$z_1(x,\rho) = v_6 e^{\vartheta_{1\rho}(t)} + e^{\vartheta_{1\rho}(t)} \int_0^\rho \frac{f_9(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\vartheta_{1\sigma}(t)}\mathrm{d}\sigma,\ \text{if } \tau_1'(t) \neq 0,$$

Hence, from (3.5), we obtain

$$z_1(x,\rho) = v_1 e^{-\rho\tau_1(t)} - f_1 e^{-\rho\tau_1(t)} + \tau_1(t)e^{-\rho\tau_1(t)} \int_0^\rho f_9(x,\sigma)e^{\sigma\tau_1(t)}\mathrm{d}\sigma,\ \text{if } \tau_1'(t) = 0 \tag{3.8}$$

and

$$z_1(x,\rho) = v_1 e^{\vartheta_{1\rho}(t)} - f_1 e^{\vartheta_{1\rho}(t)} + e^{\vartheta_{1\rho}(t)} \int_0^\rho \frac{f_9(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\vartheta_{1\sigma}(t)}\mathrm{d}\sigma,\ \text{if } \tau_1'(t) \neq 0. \tag{3.9}$$

Similarly, we get

$$z_2(x,\rho) = v_2 e^{-\rho\tau_2(t)} - f_2 e^{-\rho\tau_2(t)} + \tau_2(t)e^{-\rho\tau_2(t)} \int_0^\rho f_{10}(x,\sigma)e^{\sigma\tau_2(t)}\mathrm{d}\sigma,\ \text{if } \tau_2'(t) = 0, \tag{3.10}$$

$$z_2(x,\rho) = v_2 e^{\vartheta_{2\rho}(t)} - f_2 e^{\vartheta_{2\rho}(t)} + e^{\vartheta_{2\rho}(t)} \int_0^\rho \frac{f_{10}(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma} e^{-\vartheta_{2\sigma}(t)}\mathrm{d}\sigma,\ \text{if } \tau_2'(t) \neq 0. \tag{3.11}$$

$$z_3(x,\rho) = v_3 e^{-\rho\tau_3(t)} - f_3 e^{-\rho\tau_3(t)} + \tau_3(t)e^{-\rho\tau_3(t)} \int_0^\rho f_{11}(x,\sigma)e^{\sigma\tau_3(t)}\mathrm{d}\sigma,\ \text{if } \tau_3'(t) = 0 \tag{3.12}$$

and

$$z_3(x,\rho) = v_3 e^{\vartheta_{3\rho}(t)} - f_3 e^{\vartheta_{3\rho}(t)} + e^{\vartheta_{3\rho}(t)} \int_0^\rho \frac{f_{11}(x,\sigma)\tau_3(t)}{1-\tau_3'(t)\sigma} e^{-\vartheta_{3\sigma}(t)}\mathrm{d}\sigma,\ \text{if } \tau_3'(t) \neq 0, \tag{3.13}$$

where

$$\vartheta_{i\rho}(t) = \frac{\tau_i(t)}{\tau_i'(t)} \ln(1-\tau_i'(t)\rho),\ i = 1,2,3.$$

From (3.8)-(3.13), we have

$$z_1(x,1) = \begin{cases} v_1 e^{-\tau_1(t)} + z_{10}(x), & \text{if } \tau_1'(t) = 0, \\ v_1 e^{\vartheta_{1\rho}(t)} + z_{10}(x), & \text{if } \tau_1'(t) \neq 0. \end{cases} \tag{3.14}$$

$$z_2(x,1) = \begin{cases} v_2 e^{-\tau_2(t)} + z_{20}(x), & \text{if } \tau_2'(t) = 0, \\ v_2 e^{\vartheta_{2\rho}(t)} + z_{20}(x), & \text{if } \tau_2'(t) \neq 0. \end{cases} \tag{3.15}$$

$$z_3(x,1) = \begin{cases} v_3 e^{-\tau_3(t)} + z_{30}(x), & \text{if } \tau_3'(t) = 0, \\ v_3 e^{\vartheta_{3\rho}(t)} + z_{30}(x), & \text{if } \tau_3'(t) \neq 0, \end{cases} \tag{3.16}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

where

$$z_{10}(x) = \begin{cases} -f_1 e^{-\tau_1(t)} + \tau_1(t) e^{-\tau_1(t)} \displaystyle\int_0^1 f_9(x,\sigma) e^{\sigma\tau_1(t)} \mathrm{d}\sigma, \text{ if } \tau_1'(t) = 0, \\ -f_1 e^{\vartheta_{1\rho}(t)} + e^{\vartheta_{1\rho}(t)} \displaystyle\int_0^\rho \frac{f_9(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\vartheta_{1\sigma}(t)} \mathrm{d}\sigma, \text{ if } \tau_1'(t) \neq 0. \end{cases} \tag{3.17}$$

$$z_{20}(x) = \begin{cases} -f_2 e^{-\tau_2(t)} + \tau_2(t) e^{-\tau_2(t)} \displaystyle\int_0^1 f_{10}(x,\sigma) e^{\sigma\tau_2(t)} \mathrm{d}\sigma, \text{ if } \tau_2'(t) = 0, \\ -f_2 e^{\vartheta_{2\rho}(t)} + e^{\vartheta_{2\rho}(t)} \displaystyle\int_0^\rho \frac{f_{10}(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma} e^{-\vartheta_{2\sigma}(t)} \mathrm{d}\sigma, \text{ if } \tau_2'(t) \neq 0. \end{cases} \tag{3.18}$$

$$z_{30}(x) = \begin{cases} -f_3 e^{-\tau_3(t)} + \tau_3(t) e^{-\tau_3(t)} \displaystyle\int_0^1 f_{11}(x,\sigma) e^{\sigma\tau_3(t)} \mathrm{d}\sigma, \text{ if } \tau_3'(t) = 0, \\ -f_3 e^{\vartheta_{3\rho}(t)} + e^{\vartheta_{3\rho}(t)} \displaystyle\int_0^\rho \frac{f_{11}(x,\sigma)\tau_3(t)}{1-\tau_3'(t)\sigma} e^{-\vartheta_{3\sigma}(t)} \mathrm{d}\sigma, \text{ if } \tau_3'(t) \neq 0. \end{cases} \tag{3.19}$$

It is clear from the above formula that z_{10}, z_{20} and z_{30} depend only on f_i, $i = 1, 2, 3, 9, 10, 11$.

Next, $(3.4)_{12}$-$(3.4)_{14}$ and (3.5) imply

$$\partial_s v_9 + v_9 = v_1 - f_1 + f_{12},\ \partial_s v_{10} + v_{10} = v_2 - f_2 + f_{13},\ \partial_s v_{11} + v_{11} = v_3 - f_3 + f_{14}.$$

By solving above three differential equations and noticing that $v_9(0) = v_{10}(0) = v_{11}(0) = 0$, we get

$$\begin{aligned} v_9 &= (1-e^{-s})(v_1 - f_1) + \int_0^s e^{\tau - s} f_{12}(\tau)\mathrm{d}\tau \in H_1^*. \\ v_{10} &= (1-e^{-s})(v_2 - f_2) + \int_0^s e^{\tau - s} f_{13}(\tau)\mathrm{d}\tau \in H_2^*, \\ v_{11} &= (1-e^{-s})(v_3 - f_3) + \int_0^s e^{\tau - s} f_{14}(\tau)\mathrm{d}\tau \in H_3^*. \end{aligned} \tag{3.20}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

By using (3.5)-(3.6) and (3.20), it can be shown that v_1, v_2, v_3 and v_5 satisfy

$$\begin{cases} (\rho_1+\mu_1)v_1 - \left(k_1 - \int_0^{+\infty} e^{-s} g_1(s)\mathrm{d}s\right)\partial_{xx}v_1 + I^2k_3v_1 - k_1\partial_x v_2 - I(k_1+k_3)\partial_x v_3 \\ -\frac{\sigma(1+\tau)}{\gamma}v_5 + \mu_2 z_1(\cdot,1) = \rho_1(f_1+f_6) + \mu_1 f_1 - \int_0^{+\infty} g_1(s)(1-e^{-s})\partial_{xx}f_1\mathrm{d}s \\ -\frac{\tau\sigma}{\gamma}f_5 + \int_0^{+\infty} g_1(s)\partial_{xx}\left(\int_0^s e^{\tau-s}f_{12}(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \\ (\rho_2+\tilde{\mu}_1)v_2 + k_1\partial_x v_1 - \left(k_2 - \int_0^{+\infty} e^{-s}g_2(s)\mathrm{d}s\right)\partial_{xx}v_2 + k_1v_2 + Ik_1v_3 + \tilde{\mu}_2 z_2(\cdot,1) \\ = \rho_2(f_2+f_7) + \tilde{\mu}_1 f_2 - \int_0^{+\infty} g_2(s)(1-e^{-s})\partial_{xx}f_2\mathrm{d}s + \int_0^{+\infty} g_2(s)\partial_{xx}\left(\int_0^s e^{\tau-s}f_{13}(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \\ (\rho_1+\tilde{\tilde{\mu}}_1)v_3 + I(k_1+k_3)\partial_x v_1 + Ik_1v_2 - \left(k_3 - \int_0^{\infty} e^{-s}g_3(s)\mathrm{d}s\right)\partial_{xx}v_3 + I^2k_1v_3 + \tilde{\tilde{\mu}}_2 z_3(\cdot,1) \\ = \rho_1(f_3+f_8) + \tilde{\tilde{\mu}}_1 f_3 - \int_0^{+\infty} g_3(s)(1-e^{-s})\partial_{xx}f_3\mathrm{d}s + \int_0^{+\infty} g_3(s)\partial_{xx}\left(\int_0^s e^{\tau-s}f_{14}(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \\ \frac{(1+\tau)}{\gamma}\rho_3\int_0^x v_5\mathrm{d}y - \partial_x v_5 - \sigma\partial_x v_1 = \frac{\tau\rho_3}{\gamma}\int_0^x f_5\mathrm{d}y - \sigma\partial_x f_1 - \rho_3 f_4. \end{cases} \tag{3.21}$$

Then, multiplying $(3.21)_1$-$(3.21)_4$ by $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{v}_3$ and $\frac{1+\tau}{\gamma}\int_0^x \tilde{v}_5$, respectively, integrating their sum over $(0,L)$ and using (3.14)-(3.16), we get

$$a\left((v_1,v_2,v_3,v_5)^T, (\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\right) = \tilde{a}\left((\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\right),$$
$$\forall\ (\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T \in H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\times L_*^2(0,L), \tag{3.22}$$

where

$$\begin{aligned} &a\left((v_1,v_2,v_3,v_5)^T, (\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\right) \\ &= \int_0^L (k_1(\partial_x v_1 + v_2 + Iv_3)(\partial_x\tilde{v}_1 + \tilde{v}_2 + I\tilde{v}_3)\mathrm{d}x + \int_0^L k_3(\partial_x v_3 - Iv_1)(\partial_x\tilde{v}_3 - I\tilde{v}_1)\mathrm{d}x \\ &+ \int_0^L k_2\partial_x v_2\partial_x\tilde{v}_2\mathrm{d}x + \int_0^L \left(\left(\rho_1+\mu_1+e^{-\tau_1(t)}\right)v_1\tilde{v}_1 + \left(\rho_2+\tilde{\mu}_1+e^{-\tau_2(t)}\right)v_2\tilde{v}_2\right. \\ &+ \left.\left(\rho_1+\tilde{\tilde{\mu}}_1+e^{-\tau_1(t)}\right)v_3\tilde{v}_3 + \frac{1+\tau}{\gamma}v_5\tilde{v}_5\right)\mathrm{d}x + \frac{\rho_3(1+\tau)^2}{\gamma^2}\int_0^L\left(\int_0^x v_5\mathrm{d}y\int_0^x\tilde{v}_5\mathrm{d}y\right)\mathrm{d}x \\ &+ \int_0^L\left(-\tilde{g}_1^0\partial_x v_1\partial_x\tilde{v}_1 - \tilde{g}_2^0\partial_x v_2\partial_x\tilde{v}_2 - \tilde{g}_3^0\partial_x v_3\partial_x\tilde{v}_3\right)\mathrm{d}x + \frac{\sigma(1+\tau)}{\gamma}\int_0^L v_1\tilde{v}_5\mathrm{d}x \\ &- \frac{\sigma(1+\tau)}{\gamma}\int_0^L v_5\tilde{v}_1\mathrm{d}x, \end{aligned} \tag{3.23}$$

$$\tilde{g}_1^0 = \int_0^{+\infty} e^{-s}g_i(s)\mathrm{d}s$$

and

$$\tilde{a}\left((\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\right) = \int_0^L (\rho_1(f_1+f_6)\tilde{v}_1 + \rho_2(f_2+f_7)\tilde{v}_2 + \rho_1(f_3+f_8)\tilde{v}_3)\mathrm{d}x$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

$$
\begin{aligned}
&+\int_0^L \left(\left(g_1^0-\tilde{g}_1^0\right)\partial_x f_1 \partial_x \tilde{v}_1+\left(g_2^0-\tilde{g}_2^0\right)\partial_x f_2 \partial_x \tilde{v}_2+\left(g_3^0-\tilde{g}_3^0\right)\partial_x f_3 \partial_x \tilde{v}_3\right)\mathrm{d}x \\
&-\int_0^L \left(\int_0^{+\infty} g_1(s)\int_0^s e^{\tau-s}\partial_x f_9(\tau)\mathrm{d}\tau \mathrm{d}s\right)\partial_x \tilde{v}_1 \mathrm{d}x+\frac{\tau\sigma}{\gamma}\int_0^L f_5 \tilde{v}_1 \mathrm{d}x \\
&-\int_0^L \left(\int_0^{+\infty} g_2(s)\int_0^s e^{\tau-s}\partial_x f_{10}(\tau)\mathrm{d}\tau \mathrm{d}s\right)\partial_x \tilde{v}_2 \mathrm{d}x+\frac{\sigma(1+\tau)}{\gamma}\int_0^L f_1 \tilde{v}_5 \mathrm{d}x \\
&-\int_0^L \left(\int_0^{+\infty} g_3(s)\int_0^s e^{\tau-s}\partial_x f_{11}(\tau)\mathrm{d}\tau \mathrm{d}s\right)\partial_x \tilde{v}_3 \mathrm{d}x \\
&-\mu_2\int_0^L z_{10}(x)\tilde{v}_1 \mathrm{d}x-\tilde{\mu}_2\int_0^L z_{20}(x)\tilde{v}_2 \mathrm{d}x-\tilde{\tilde{\mu}}_2\int_0^L z_{30}(x)\tilde{v}_3 \mathrm{d}x \\
&+\frac{\rho_3(1+\tau)}{\gamma}\int_0^L \left(\frac{\tau}{\gamma}\int_0^x f_5 \mathrm{d}y-f_4\right)\int_0^x \tilde{v}_5 \mathrm{d}y \mathrm{d}x, \qquad (3.24)
\end{aligned}
$$

if $\tau_i'(t)=0$.

If $\tau_i'(t)\neq 0$, we get

$$
\begin{aligned}
&a\left((v_1,v_2,v_3,v_5)^T,(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\right)=\tilde{a}\left((\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\right), \\
&\forall\ (\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\in H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\times L_*^2(0,L), \qquad (3.25)
\end{aligned}
$$

where

$$
\begin{aligned}
&a\left((v_1,v_2,v_3,v_5)^T,(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_5)^T\right) \\
&=\int_0^L (k_1(\partial_x v_1+v_2+Iv_3)(\partial_x\tilde{v}_1+\tilde{v}_2+I\tilde{v}_3)\mathrm{d}x+\int_0^L k_3(\partial_x v_3-Iv_1)(\partial_x\tilde{v}_3-I\tilde{v}_1)\mathrm{d}x \\
&+\int_0^L k_2\partial_x v_2\partial_x\tilde{v}_2\mathrm{d}x+\int_0^L \left(\left(\rho_1+\mu_1+e^{\vartheta_{1\rho}(t)}\right)v_1\tilde{v}_1+\left(\rho_2+\tilde{\mu}_1+e^{\vartheta_{2\rho}(t)}\right)v_2\tilde{v}_2\right. \\
&\left.+\left(\rho_1+\tilde{\tilde{\mu}}_1+e^{\vartheta_{3\rho}(t)}\right)v_3\tilde{v}_3+\frac{\tau+1}{\gamma}v_5\tilde{v}_5\right)\mathrm{d}x+\frac{\rho_3(1+\tau)^2}{\gamma^2}\int_0^L\left(\int_0^x v_5\mathrm{d}y\int_0^x\tilde{v}_5\mathrm{d}y\right)\mathrm{d}x \\
&+\int_0^L\left(-\tilde{g}_1^0\partial_x v_1\partial_x\tilde{v}_1-\tilde{g}_2^0\partial_x v_2\partial_x\tilde{v}_2-\tilde{g}_3^0\partial_x v_3\partial_x\tilde{v}_3\right)\mathrm{d}x+\frac{\sigma(1+\tau)}{\gamma}\int_0^L v_1\tilde{v}_5\mathrm{d}x \\
&-\frac{\sigma(1+\tau)}{\gamma}\int_0^L v_5\tilde{v}_1\mathrm{d}x, \qquad (3.26)
\end{aligned}
$$

the operator $\tilde{a}$ is defined by the same formula (3.23).

Now, we introduce the Hilbert space $V=H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)\times L_*^2(0,L)$ equipped with the norm

$$
\|(v_1,v_2,v_3,v_5)\|_V^2=\|(\partial_x v_1+v_2+Iv_3\|_2^2+\|\partial_x v_3-Iv_1\|_2^2+\|\partial_x v_2\|_2^2+\|v_5\|_2^2.
$$

It is clear that a and $\tilde{a}$ are bounded. Furthermore, from (2.13), we find that there exists a positive constant c such that

$$
a\left((v_1,v_2,v_3,v_5)^T,(v_1,v_2,v_3,v_5)^T\right)\geq c\|(v_1,v_2,v_3,v_5)\|_V^2,
$$

which implies that a is coercive.

From the above, we obtain that a is a bilinear continuous coercive form on $V\times V$, and $\tilde{a}$ is a linear continuous form on V. Therefore, using the Lax-Milgram theorem [24], we obtain that

(3.22) and (3.25) have a unique solution $(v_1, v_2, v_3, v_5)^T \in V$. By substituting v_1, v_2, v_3, v_5 into (3.5), we obtain

$$v_4 \in H_0^1(0, L), v_6 \in H_0^1(0, L), v_7 \in H_0^1(0, L), v_8 \in H_0^1(0, L).$$

Next, it remains to show that

$$v_1, v_2, v_3 \in H^2(0, L) \cap H_0^1(0, L), v_5 \in H_*^1(0, L).$$

Recalling (3.5) and using (3.21), we have

$$\begin{aligned}\left(k_1 - \tilde{g}_1^0\right) \partial_{xx} v_1 - I^2 k_3 v_1 =& - k_1 \partial_x v_2 - I(k_1 + k_3)\partial_x v_3 + \sigma \partial_x v_4 - \mu_1 z_1(x, 1) + (\rho_1 + \mu_1) v_6 - \rho_1 f_6 \\ &+ \left(g_1^0 - \tilde{g}_1^0\right) \partial_{xx} f_1 + \int_0^{+\infty} g_1(s) \partial_{xx} \left(\int_0^s e^{\tau - s} f_9(\tau) \mathrm{d}\tau \right) \mathrm{d}s \in L^2(0, L).\end{aligned}$$

Then, by the L^2 theory for the linear elliptic equations, we obtain that

$$v_1 \in H^2(0, L) \cap H_0^1(0, L).$$

In the same way, we obtain

$$v_2 \in H^2(0, L) \cap H_0^1(0, L), v_3 \in H^2(0, L) \cap H_0^1(0, L).$$

Similarly, recalling (3.6) and using (3.21), we have

$$\partial_x v_5 = -\sigma \partial_x v_6 - \rho_3 v_4 + \rho_3 f_4 \in L^2(0, L),$$

consequently, we get $v_5 \in H_*^1(0, L)$.

Then, using the classical regularity theory of linear elliptic equations, we obtain a unique solution $V \in D(\mathscr{A})$ which satisfies (3.3). Hence, the operator $Id - \mathscr{A}$ is surjective.

Finally, from above, we get $\mathscr{A}$ is a maximal monotone operator. Then, by using the Hille-Yosida theorem [5], we obtain that if $U^0 \in D(\mathscr{A})$, then $U \in C(\mathbb{R}^+; D(\mathscr{A})) \cap C^1(\mathbb{R}^+; \mathscr{H})$. Moreover, it is easy to see that $D(\mathscr{A})$ is dense in $\mathscr{H}$. At last, basing on the above analysis, the well-posedness result stated in Theorem 2.1 follows from the Hille-Yosida [5, 17].

4 Proof the Stability

In this section, we prove Theorem 2.2. Our method builds on a suitable Lyapunov functional that can be obtained by the energy method.

Before proving our main results, we will state and prove some useful lemmas in advance.

Lemma 4.1 *Let $(\varphi, \psi, w, \theta, q)$ be the solution of problem* (2.9). *Then the energy functional $E(t)$ defined by* (2.21) *satisfies*

$$\begin{aligned}E'(t) \leq& - c \int_0^L \left(\varphi_t^2 + \psi_t^2 + w_t^2\right) \mathrm{d}x - c \int_0^L \left(z_1^2(x, 1, t) + z_2^2(x, 1, t) + z_3^2(x, 1, t)\right) \mathrm{d}x \\ &+ \frac{1}{2} \int_0^L \int_0^{+\infty} \left(g_1'(s)(\partial_x \eta_1)^2 + g_2'(s)(\partial_x \eta_2)^2 + g_3'(s)(\partial_x \eta_3)^2\right) \mathrm{d}s\mathrm{d}x - \int_0^L \frac{1}{\gamma} q^2 \mathrm{d}x \leq 0.\end{aligned} \tag{4.1}$$

Proof. Multiplying the first five equations in (2.9) by φ_t, ψ_t, w_t, θ and q, respectively, integrating over $(0, L)$, using integration by parts and the boundary conditions, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\int_0^L \Big(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + k_2\psi_x^2 + k_1(\varphi_x + \psi + Iw)^2 + k_3(w_x - I\varphi)^2 + \rho_3\theta^2 \\
&+ \frac{\tau}{\gamma}q^2\Big)\mathrm{d}x - \int_0^L \left(g_1^0\varphi_x^2 + g_2^0\psi_x^2 + g_3^0 w_x^2\right)\mathrm{d}x + \|\eta_1\|^2_{H_1^*} + \|\eta_2\|^2_{H_2^*} + \|\eta_3\|^2_{H_3^*}\Big] \\
=&\frac{1}{2}\int_0^L\int_0^{+\infty}\left(g_1'(s)(\partial_x\eta_1)^2 + g_2'(s)(\partial_x\eta_2)^2 + g_3'(s)(\partial_x\eta_3)^2\right)\mathrm{d}s\mathrm{d}x - \frac{1}{\gamma}\int_0^L q^2\mathrm{d}x. \qquad (4.2)
\end{aligned}$$

Then, multiplying the last three equations in (2.9) by $\zeta_1 z_1$, $\zeta_2 z_2$ and $\zeta_3 z_3$ respectively, integrating the resulting equations over $(0, L) \times (0, 1)$ with respect to x and ρ, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1 \left(\zeta_1\tau_1(t)z_1^2(x,\rho,t) + \zeta_2\tau_2(t)z_2^2(x,\rho,t) + \zeta_3\tau_3(t)z_3^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x \\
=&\frac{1}{2}\int_0^L\int_0^1 \left(\zeta_1\tau_1'(t)z_1^2(x,\rho,t) + \zeta_2\tau_2'(t)z_2^2(x,\rho,t) + \zeta_3\tau_3'(t)z_3^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x \\
&-\int_0^L\int_0^1 \big(\zeta_1(1-\tau_1'(t)\rho)z_1(x,\rho,t)z_{1\rho}(x,\rho,t) + \zeta_2(1-\tau_2'(t)\rho)z_2(x,\rho,t)z_{2\rho}(x,\rho,t) \\
&+\zeta_3(1-\tau_3'(t)\rho)z_3(x,\rho,t)z_{3\rho}(x,\rho,t)\big)\mathrm{d}\rho\mathrm{d}x. \\
=&-\frac{1}{2}\int_0^L\int_0^1 \frac{\partial}{\partial_\rho}\left((1-\tau_1'(t)\rho)z_1^2(x,\rho,t) + (1-\tau_2'(t)\rho)z_2^2(x,\rho,t) + (1-\tau_3'(t)\rho)z_3^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x \\
=&\frac{1}{2}\int_0^L \left(\zeta_1\varphi_t^2(x,t) + \zeta_2\psi_t^2(x,t) + \zeta_3 w_t^2(x,t)\right)\mathrm{d}x \\
&-\frac{1}{2}\int_0^L \left(\zeta_1(1-\tau_1'(t))z_1^2(x,1,t) + \zeta_2(1-\tau_2'(t))z_2^2(x,1,t) + \zeta_3(1-\tau_3'(t))z_3^2(x,1,t)\right)\mathrm{d}x. \qquad (4.3)
\end{aligned}$$

From (4.2) and (4.3), we have

$$\begin{aligned}
\frac{dE(t)}{dt} =& -\left(\mu_1 - \frac{\zeta_1}{2}\right)\int_0^L \varphi_t^2\mathrm{d}x - \left(\tilde{\mu}_1 - \frac{\zeta_2}{2}\right)\int_0^L \psi_t^2\mathrm{d}x - \left(\tilde{\tilde{\mu}}_1 - \frac{\zeta_3}{2}\right)\int_0^L w_t^2\mathrm{d}x \\
&- \mu_2\int_0^L z_1(x,1,t)\varphi_t\mathrm{d}x - \tilde{\mu}_2\int_0^L z_2(x,1,t)\psi_t\mathrm{d}x - \tilde{\tilde{\mu}}_2\int_0^L z_3(x,1,t)w_t\mathrm{d}x \\
&- \frac{1}{2}\int_0^L\int_0^{+\infty} g_1(s)\partial_s(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x - \frac{1}{2}\int_0^L\int_0^{+\infty} g_2(s)\partial_s(\partial_x\eta_2)^2\mathrm{d}s\mathrm{d}x \\
&- \frac{1}{2}\int_0^L\int_0^{+\infty} g_3(s)\partial_s(\partial_x\eta_3)^2\mathrm{d}s\mathrm{d}x - \int_0^L \frac{1}{\gamma}q^2\mathrm{d}x \\
&- \sum_{i=1}^{3}\frac{\zeta_i(1-\tau_i'(t))}{2}\int_0^L z_i^2(x,1,t)\mathrm{d}x.
\end{aligned}$$

Finally, by using (2.3), (2.17) and Young's inequality, we obtain (4.1) (note that g_i is non-increasing).

As in [14], let us define the functionals:

$$I_1(t) = -\rho_1\int_0^L \varphi_t\int_0^{+\infty} g_1(s)\eta_1\mathrm{d}s\mathrm{d}x,$$

$$I_2(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g_2(s)\eta_2 \mathrm{d}s\mathrm{d}x$$

and

$$I_3(t) = -\rho_3 \int_0^L w_t \int_0^{+\infty} g_3(s)\eta_3 \mathrm{d}s\mathrm{d}x.$$

Lemma 4.2 *The functionals $I_i (i = 1, 2, 3)$ satisfy, for any $\delta > 0$,*

$$\begin{aligned} I_1'(t) \leq & -\rho_1 \left(g_1^0 - \left(1 + \frac{1}{\rho_1}\right)\delta \right) \int_0^L \varphi_t^2 \mathrm{d}x \\ & + \delta \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + Iw)^2 + (w_x - I\varphi)^2 + \theta^2 + z_1^2(x,1,t)\right) \mathrm{d}x \\ & + c_\delta \int_0^L \int_0^{+\infty} g_1(s)(\partial_x \eta_1)^2 \mathrm{d}s\mathrm{d}x - c_\delta \int_0^L \int_0^{+\infty} g_1'(s)(\partial_x \eta_1)^2 \mathrm{d}s\mathrm{d}x, \end{aligned} \tag{4.4}$$

$$\begin{aligned} I_2'(t) \leq & -\rho_2 \left(g_2^0 - \left(1 + \frac{1}{\rho_2}\right)\delta \right) \int_0^L \psi_t^2 \mathrm{d}x \\ & + \delta \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + Iw)^2 + (w_x - I\varphi)^2 + z_2^2(x,1,t)\right) \mathrm{d}x \\ & + c_\delta \int_0^L \int_0^{+\infty} g_2(s)(\partial_x \eta_2)^2 \mathrm{d}s\mathrm{d}x - c_\delta \int_0^L \int_0^{+\infty} g_2'(s)(\partial_x \eta_2)^2 \mathrm{d}s\mathrm{d}x \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} I_3'(t) \leq & -\rho_1 \left(g_3^0 - \left(1 + \frac{1}{\rho_1}\right)\delta \right) \int_0^L w_t^2 \mathrm{d}x \\ & + \delta \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + Iw)^2 + (w_x - I\varphi)^2 + z_2^3(x,1,t)\right) \mathrm{d}x \\ & + c_\delta \int_0^L \int_0^{+\infty} g_3(s)(\partial_x \eta_3)^2 \mathrm{d}s\mathrm{d}x - c_\delta \int_0^L \int_0^{+\infty} g_3'(s)(\partial_x \eta_3)^2 \mathrm{d}s\mathrm{d}x, \end{aligned} \tag{4.6}$$

where g_i^0 is defined by (2.15) *and c_δ is a positive constant depending on δ.*

Proof. Differentiating I_1 with respect to t, using the first equation of problem (2.9), integrating by parts and using the fact that

$$\begin{aligned} \partial_t \int_0^{+\infty} g_1(s)\eta_1 \mathrm{d}s = & \partial_t \int_0^{+\infty} g_1(t-s)(\varphi(t) - \varphi(s))\mathrm{d}s \\ = & \int_0^{+\infty} g_1'(t-s)(\varphi(t) - \varphi(s))\mathrm{d}s + \left(\int_0^{+\infty} g_1(t-s)\mathrm{d}s \right) \varphi_t \\ = & \int_0^{+\infty} g_1'(s)\eta_1 \mathrm{d}s + g_1^0 \varphi_t, \end{aligned}$$

we obtain

$$\begin{aligned} I_1'(t) = & -\rho_1 g_1^0 \int_0^L \varphi_t^2 \mathrm{d}x - \rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1'(s)\eta_1 \mathrm{d}s\mathrm{d}x - \sigma \int_0^L \theta \int_0^{+\infty} g_1(s)\partial_x \eta_1 \mathrm{d}s\mathrm{d}x \\ & + k_1 \int_0^L (\varphi_x + \psi + Iw) \int_0^{+\infty} g_1(s)\partial_x \eta_1 \mathrm{d}s\mathrm{d}x - Ik_3 \int_0^L (w_x - I\varphi) \int_0^{+\infty} g_1(s)\eta_1 \mathrm{d}s\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&- g_1^0 \int_0^L \varphi_x\Big(\int_0^{+\infty} g_1(s)\partial_x\eta_1 \mathrm{d}s\Big)\mathrm{d}x + \int_0^L \Big(\int_0^{+\infty} g_1(s)\partial_x\eta_1 \mathrm{d}s\Big)^2\mathrm{d}x \\
&+ \mu_1 \int_0^L \varphi_t \int_0^{+\infty} g_1(s)\eta_1 \mathrm{d}s\mathrm{d}x + \mu_2 \int_0^L z_1(x,1,t)\int_0^{+\infty} g_1(s)\eta_1 \mathrm{d}s\mathrm{d}x.
\end{aligned}
$$

Using Young's, Poincaré's and Hölder's inequalities for the last eight terms of this equality, we get

$$
\begin{aligned}
I_1'(t) \le &- \rho_1 g_1^0 \int_0^L \varphi_t^2\mathrm{d}x + \varepsilon\rho_1 \int_0^L \varphi_t^2\mathrm{d}x \\
&- \frac{\rho_1 c}{4\varepsilon}\int_0^L\int_0^{+\infty} g_1'(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x + \varepsilon k_1 \int_0^L (\varphi_x+\psi+Iw)^2\mathrm{d}x \\
&+ \frac{k_1 g_1^0}{4\varepsilon}\int_0^L\int_0^{+\infty} g_1(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x + \varepsilon I k_3 \int_0^L (w_x - I\varphi)^2\mathrm{d}x \\
&+ \frac{Ik_3 c g_1^0}{4\varepsilon}\int_0^L\int_0^{+\infty} g_1(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x + \varepsilon g_1^0 \int_0^L \varphi_x^2\mathrm{d}x \\
&+ \frac{(g_1^0)^2}{4\varepsilon}\int_0^L\int_0^{+\infty} g_1(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x + \varepsilon\sigma \int_0^L \theta^2\mathrm{d}x \\
&+ \frac{\sigma g_1^0}{4\varepsilon}\int_0^L\int_0^{+\infty} g_1(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x + g_1^0 \int_0^L\int_0^{+\infty} g_1(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x \\
&+ \varepsilon\mu_1 \int_0^L \varphi_t^2\mathrm{d}x + \frac{\mu_1 c g_1^0}{4\varepsilon}\int_0^L\int_0^{+\infty} g_1(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x \\
&+ \varepsilon|\mu_2| \int_0^L z_1^2(x,1,t)\mathrm{d}x + \frac{|\mu_2| c g_1^0}{4\varepsilon}\int_0^L\int_0^{+\infty} g_1(s)(\partial_x\eta_1)^2\mathrm{d}s\mathrm{d}x.
\end{aligned}
$$

By using (2.14) to estimate $\int_0^L \varphi_x^2\mathrm{d}x$ and choosing

$$
\delta = \max\left\{\varepsilon\left(\frac{k_1 g_1^0}{\bar{k}_0} + k_1\right), \frac{\varepsilon g_1^0 k_2}{\bar{k}_0}, \varepsilon\left(\frac{g_1^0 k_3}{\bar{k}_0} + Ik_3\right), \varepsilon\sigma, \varepsilon, \varepsilon\mu_1, \varepsilon|\mu_2|\right\},
$$

$$
c_\delta = \max\left\{\frac{\rho_1 c}{4\varepsilon}, \frac{k_1 g_1^0}{4\varepsilon} + \frac{Ik_3 g_1^0 c}{4\varepsilon} + \frac{(g_1^0)^2}{4\varepsilon} + \frac{\sigma g_1^0}{4\varepsilon} + g_1^0 + \frac{\mu_1 c g_1^0}{4\varepsilon} + \frac{|\mu_2| c g_1^0}{4\varepsilon}\right\},
$$

(4.4) is established. Similarly, using the second and the third equations of problem (2.9), we obtain (4.5) and (4.6).

Lemma 4.3 *There exist positive constants c_1 and c_2 such that the functional*

$$
I_4(t) = \int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1 w w_t)\mathrm{d}x
$$

satisfies

$$
\begin{aligned}
I_4'(t) \le &\int_0^L \left((\rho_1+c_2)\varphi_t^2 + (\rho_2+c_2)\psi_t^2 + (\rho_1+c_2)w_t^2\right)\mathrm{d}x \\
&- c_1 \int_0^L \left(\psi_x^2 + (\varphi_x+\psi+Iw)^2 + (w_x - I\varphi)^2\right)\mathrm{d}x + c_2 \int_0^L \theta^2\mathrm{d}x \\
&+ c_2 \int_0^L\int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right)\mathrm{d}s\mathrm{d}x \\
&+ c_2 \int_0^L \left(z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right)\mathrm{d}x. \qquad (4.7)
\end{aligned}
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

Proof. By exploiting first three equations of problem (2.9) and integrating by parts, we have

$$\begin{aligned} I_4'(t) =& \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2\right) \mathrm{d}x - k_1 \int_0^L (\varphi_x + \psi + Iw)^2 \mathrm{d}x \\ &- k_3 \int_0^L (w_x - I\varphi)^2 \mathrm{d}x + g_1^0 \int_0^L \varphi_x^2 \mathrm{d}x - \left(k_2 - g_2^0\right) \int_0^L \psi_x^2 \mathrm{d}x + g_3^0 \int_0^L w_x^2 \mathrm{d}x \\ &- \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\partial_x\eta_1 \mathrm{d}s\mathrm{d}x - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\partial_x\eta_2 \mathrm{d}s\mathrm{d}x \\ &- \int_0^L w_x \int_0^{+\infty} g_3(s)\partial_x\eta_3 \mathrm{d}s\mathrm{d}x + \sigma \int_0^L \varphi_x\theta \mathrm{d}x \\ &- \mu_1 \int_0^L \varphi\varphi_t \mathrm{d}x - \tilde{\mu}_1 \int_0^L \psi\psi_t \mathrm{d}x - \tilde{\tilde{\mu}}_1 \int_0^L ww_t \mathrm{d}x \\ &- \mu_2 \int_0^L z_1(x,1,t)\varphi \mathrm{d}x - \tilde{\mu}_2 \int_0^L z_2(x,1,t)\psi \mathrm{d}x - \tilde{\tilde{\mu}}_2 \int_0^L z_3(x,1,t) w \mathrm{d}x. \end{aligned} \tag{4.8}$$

Using Young's Poincaré's and Hölder's inequalities for the last ten terms in the right hand side of (4.8), we get, for any $\epsilon > 0$, there exists a positive constant c_ϵ such that

$$\begin{aligned} &- \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\partial_x\eta_1 \mathrm{d}s\mathrm{d}x - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\partial_x\eta_2 \mathrm{d}s\mathrm{d}x - \int_0^L w_x \int_0^{+\infty} g_3(s)\partial_x\eta_3 \mathrm{d}s\mathrm{d}x \\ &+ \sigma \int_0^L \varphi_x\theta \mathrm{d}x - \mu_1 \int_0^L \varphi\varphi_t \mathrm{d}x - \tilde{\mu}_1 \int_0^L \psi\psi_t \mathrm{d}x - \tilde{\tilde{\mu}}_1 \int_0^L ww_t \mathrm{d}x \\ &- \mu_2 \int_0^L z_1(x,1,t)\varphi \mathrm{d}x - \tilde{\mu}_2 \int_0^L z_2(x,1,t)\psi \mathrm{d}x - \tilde{\tilde{\mu}}_2 \int_0^L z_3(x,1,t) w \mathrm{d}x \\ \leq& \epsilon \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x + c_\epsilon \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right) \mathrm{d}s\mathrm{d}x \\ &+ c_\epsilon \int_0^L \theta^2 \mathrm{d}x + c_\epsilon \int_0^L \left(\varphi_t^2 + \psi_t^2 + w_t^2\right) \mathrm{d}x + c_\epsilon \int_0^L \left(z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right) \mathrm{d}x. \end{aligned}$$

Inserting this inequality into (4.8) and using (2.13), we get

$$\begin{aligned} I_4'(t) \leq& \int_0^L \left((p_1 + c_\epsilon)\varphi_t^2 + (p_2 + c_\epsilon)\psi_t^2 + (p_1 + c_\epsilon) w_t^2\right) \mathrm{d}x - (k_0 - \epsilon) \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) \mathrm{d}x \\ &+ c_\epsilon \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right) \mathrm{d}s\mathrm{d}x + c_\epsilon \int_0^L \theta^2 \mathrm{d}x \\ &+ c_\epsilon \int_0^L \left(z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right) \mathrm{d}x. \end{aligned} \tag{4.9}$$

Then, choosing $0 < \epsilon < k_0$ and inserting (2.16) in (4.9), we get (4.7) with $c_1 = \dfrac{(k_0 - \epsilon)}{\bar{k}_0}$ and $c_2 = c_\epsilon$.

Lemma 4.4 *([4]) For any $\varepsilon_3 > 0$, the functional*

$$I_5(t) = \tau\rho_3 \int_0^L \theta \int_0^x q(y) \mathrm{d}y \mathrm{d}x$$

satisfies

$$I_5'(t) \leq -\frac{\gamma\rho_3}{2} \int_0^L \theta^2 \mathrm{d}x + \varepsilon_3 \int_0^L \varphi_t^2 \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_3}\right) \int_0^L q^2 \mathrm{d}x. \tag{4.10}$$

 doi:10.20944/preprints201701.0087.v1

Proof. Taking the derivative of I_5 with respect to t, using the fourth and fifth equations of problem (2.9) and integrating by parts, we get

$$I_5'(t) = -\gamma p_3 \int_0^L \theta^2 \mathrm{d}x + \tau \int_0^L q^2 \mathrm{d}x + \tau\sigma \int_0^L q\varphi_t \mathrm{d}x - p_3 \int_0^L \theta \int_0^x q(y)\mathrm{d}y\mathrm{d}x.$$

Using Cauchy-Schwarz's and Young's inequalities with $\varepsilon_3 > 0$, we get (4.10).

Lemma 4.5 *The functionals*

$$I_6 = \zeta_1\tau_1(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho} z_1^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x,$$

$$I_7 = \zeta_2\tau_2(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho} z_2^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x,$$

$$I_8 = \zeta_3\tau_3(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho} z_3^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x$$

satisfies

$$I_6'(t) \le -2I_6(t) + \zeta_1 \int_0^l \varphi_t^2 \mathrm{d}x, \tag{4.11}$$

$$I_7'(t) \le -2I_7(t) + \zeta_2 \int_0^l \psi_t^2 \mathrm{d}x, \tag{4.12}$$

$$I_8'(t) \le -2I_8(t) + \zeta_3 \int_0^l w_t^2 \mathrm{d}x. \tag{4.13}$$

Proof. Taking the derivative of I_6 with respect to t, we get

$$\begin{aligned} I_6'(t) =& \zeta_1\tau_1'(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho} z_1^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - 2\zeta_1\tau_1(t)\tau_1'(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho}\rho z_1^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \\ &+ 2\zeta_1\tau_1(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho} z_1 z_{1t}(x,\rho,t)\mathrm{d}\rho\mathrm{d}x. \end{aligned} \tag{4.14}$$

By using (2.8), the last term in (4.14) can be rewritten as follows

$$2\zeta_1\tau_1(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho} z_1 z_{1t}(x,\rho,t)\mathrm{d}\rho\mathrm{d}x = -2\zeta_1\int_0^L\int_0^1 e^{-2\tau_1(t)\rho}(1-\tau_1'(t)\rho) z_1 z_{1\rho}(x,\rho,t))\mathrm{d}\rho\mathrm{d}x. \tag{4.15}$$

Also, one can see that

$$\begin{aligned} &-2\zeta_1\int_0^L\int_0^1 e^{-2\tau_1(t)\rho}(1-\tau_1'(t)\rho) z_1 z_{1\rho}(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \\ =& -\zeta_1\int_0^L\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau_1(t)\rho}(1-\tau_1'(t)\rho) z_1^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x \\ &-2\zeta_1\tau_1(t)\int_0^L\int_0^1 e^{-2\tau_1(t)\rho}(1-\tau_1'(t)\rho) z_1^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \end{aligned}$$

 doi:10.20944/preprints201701.0087.v1

$$- \zeta_1 \tau_1'(t) \int_0^L \int_0^1 e^{-2\tau_1(t)\rho} z_1^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x. \tag{4.16}$$

Using (4.15) and (4.16), equation (4.14) takes the form

$$\begin{aligned} I_6'(t) = & - 2\zeta_1 \tau_1(t) \int_0^L \int_0^1 e^{-2\tau_1(t)\rho} z_1^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x + \zeta_1 \int_0^L \varphi_t^2 \mathrm{d}x \\ & - \zeta_1 (1 - \tau_1'(t)) e^{-2\tau_1(t)} \int_0^1 z_1^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x, \end{aligned} \tag{4.17}$$

from which immediately follows (4.11). Similarly, we get (4.12) and (4.13) in the same way.

Now, let $N_1, N_2, N_3 > 0$, and

$$L(t) = N_1 E + N_2(I_1 + I_2 + I_3) + I_4 + N_3 I_5 + I_6 + I_7 + I_8.$$

First, taking the derivative of $L(t)$ with respect to t, using (4.4)-(4.6) with $\delta = \frac{1}{N_2^2}$, (4.7), (4.10), (4.11), (4.12) and (4.13), we get

$$\begin{aligned} L'(t) \le & - \left(c_1 - \frac{3}{N_2} \right) \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + Iw)^2 + (w_x - I\varphi)^2 \right) \mathrm{d}x \\ & - \left(N_1 c + \rho_1 \left(N_2 g_1^0 - \left(1 + \frac{1}{\rho_1}\right) \frac{1}{N_2} - 1 \right) - c_2 - N_3 \varepsilon_3 - \zeta_1 \right) \int_0^L \varphi_t^2 \mathrm{d}x \\ & - \left(N_2 c + \rho_2 \left(N_2 g_2^0 - \left(1 + \frac{1}{\rho_2}\right) \frac{1}{N_2} - 1 \right) - c_2 - \zeta_2 \right) \int_0^L \psi_t^2 \mathrm{d}x \\ & - \left(N_3 c + \rho_1 \left(N_2 g_3^0 - \left(1 + \frac{1}{\rho_1}\right) \frac{1}{N_2} - 1 \right) - c_2 - \zeta_3 \right) \int_0^L w_t^2 \mathrm{d}x \\ & - \left(\frac{N_3 \gamma \rho_3}{2} - \frac{1}{N_2} - c_2 \right) \int_0^L \theta^2 \mathrm{d}x - \left(\frac{N_1}{\gamma} - N_3 c \left(1 + \frac{1}{\varepsilon_3} \right) \right) \int_0^L q^2 \mathrm{d}x \\ & - \left(N_1 c - \frac{1}{N_2} - c_2 \right) \int_0^L z_1^2(x,1,t) \mathrm{d}x - \left(N_1 c - \frac{1}{N_2} - c_2 \right) \int_0^L z_2^2(x,1,t) \mathrm{d}x \\ & - \left(N_1 c - \frac{1}{N_2} - c_2 \right) \int_0^L z_3^2(x,1,t) \mathrm{d}x - 2I_6(t) - 2I_7(t) - 2I_8(t) \\ & + \left(\frac{N_1}{2} - c_{N_2} \right) \int_0^L \int_0^{+\infty} \left(g_1'(s)(\partial_x \eta_1)^2 + g_2'(s)(\partial_x \eta_2)^2 + g_3'(s)(\partial_x \eta_3)^2 \right) \mathrm{d}s \mathrm{d}x \\ & + c_{N_2} \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x \eta_1)^2 + g_2(s)(\partial_x \eta_2)^2 + g_3(s)(\partial_x \eta_3)^2 \right) \mathrm{d}s \mathrm{d}x, \end{aligned} \tag{4.18}$$

where $c_{N_2} = N_2 c_\delta + c_2$. At this point, we choose N_3 large enough so that

$$\frac{N_3 \gamma \rho_3}{2} - c_2 > 0.$$

Then, we choose ε small enough and N_2 large enough (note that g_i is continuous non-negative and $g_i(0) > 0$) so that

$$\begin{aligned} & c_1 - \frac{3}{N_2} > 0, \frac{N_3 \gamma \rho_3}{2} - \frac{1}{N_2} - c_2 > 0, \rho_1 \left(N_2 g_1^0 - \left(1 + \frac{1}{\rho_1}\right) \frac{1}{N_2} - 1 \right) - c_2 - N_3 \varepsilon_3 - \zeta_1 > 0, \\ & \rho_2 \left(N_2 g_2^0 - \left(1 + \frac{1}{\rho_2}\right) \frac{1}{N_2} - 1 \right) - c_2 - \zeta_2 > 0, \rho_1 \left(N_2 g_3^0 - \left(1 + \frac{1}{\rho_1}\right) \frac{1}{N_2} - 1 \right) - c_2 - \zeta_3 > 0. \end{aligned}$$

Finally, we choose N_1 large enough so that

$$N_1 c - \frac{1}{N_2} - c_2 > 0, \frac{N_1}{\gamma} - N_3 c\left(1 + \frac{1}{\varepsilon_3}\right) > 0.$$

From the above, we deduce that there exist positive constants c_3 and c_4 such that (4.18) becomes

$$\begin{aligned} L'(t) \leq & - c_3 E(t) + \left(\frac{N_1}{2} - c_4\right) \int_0^L \int_0^{+\infty} \left(g_1'(s)(\partial_x \eta_1)^2 + g_2'(s)(\partial_x \eta_2)^2 + g_3'(s)(\partial_x \eta_3)^2\right) \mathrm{d}s\mathrm{d}x \\ & + c_4 \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x \eta_1)^2 + g_2(s)(\partial_x \eta_2)^2 + g_3(s)(\partial_x \eta_3)^2\right) \mathrm{d}s\mathrm{d}x. \end{aligned}$$

On the other hand, using (2.13) and definition of E, I_i and L, there exists a positive constant N_4 (not depending on N_1) such that

$$(N_1 - N_4)E \leq L \leq (N_1 + N_4)E, \tag{4.19}$$

then, choosing $N_1 > \max\{2c_4, N_4\}$ and using the fact that $g_i' \leq 0$, we get

$$L'(t) \leq -c_3 E(t) + c_4 \int_0^L \left[\int_0^{+\infty} \left(g_1(s)(\partial_x \eta_1)^2 + g_2(s)(\partial_x \eta_2)^2 + g_3(s)(\partial_x \eta_3)^2\right) \mathrm{d}s\right] \mathrm{d}x. \tag{4.20}$$

In order to finish the proof of the stability, we need to estimate the last three terms in the right hand of (4.20). Inspired by [14], we have the following lemma.

Lemma 4.6 *([14]) For $i = 1, 2, 3$, there exist positive constants d_i and $\tilde{d}_i$ such that, for any $\epsilon_0 > 0$, then we have the following inequalities:*

$$\int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 \mathrm{d}s\mathrm{d}x \leq -d_i E'(t), \text{ if (2.18) holds} \tag{4.21}$$

and

$$G'(\epsilon_0 E(t)) \int_0^L \left(\int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 \mathrm{d}s\right) \mathrm{d}x \leq -\tilde{d}_i E'(t) + \tilde{d}_i \epsilon_0 E(t) G'(\epsilon_0 E(t)), \\ \text{if (2.20) holds and (2.18) does not hold.} \tag{4.22}$$

Proof. The proof of this lemma is similar to the proof of Lemma 3.3 in [14] and is omitted.

Now, going back to the proof of Theorem 2.2, if (2.18) holds for all $i \in \{1, 2, 3\}$, then (4.20) and (4.21) imply that

$$L'(t) \leq -c_3 E(t) - c_4(d_1 + d_2 + d_3)E'(t). \tag{4.23}$$

Let $F(t) = L(t) + c_4(d_1 + d_2 + d_3)E(t)$, combining (4.19) with (4.23), we have $F'(t) \leq -c' F(t)$, where

$$c' = \frac{c_3}{N_1 + N_4 + c_4(d_1 + d_2 + d_3)}.$$

Integrating over $(0, t)$ and using (4.19), we obtain (2.22) with

$$c'' = \frac{F(0)}{N_1 - N_4 + c_4(d_1 + d_2 + d_3)}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 January 2017 doi:10.20944/preprints201701.0087.v1

Next, if (2.18) does not hold at least for one $i \in \{1,2,3\}$, then thanks to (4.21) and (4.22), we get that

$$G'(\epsilon_0 E(t)) \int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 \mathrm{d}s\mathrm{d}x \leq -\alpha_i G'(\epsilon_0 E(t))E'(t) - \beta_i E'(t) + \epsilon_0 \beta_i G'(\epsilon_0 E(t))E'(t), \tag{4.24}$$

where

$$\alpha_i = \begin{cases} d_i, & \text{if (2.18) holds,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\beta_i = \begin{cases} 0, & \text{if (2.20) holds,} \\ \tilde{d}_i, & \text{otherwise.} \end{cases}$$

Multiplying (4.20) by $G'(\epsilon_0 E(t))$, using (4.24) and let $0 < \epsilon_0 < \frac{c_3}{c_4(\beta_1+\beta_2+\beta_3)}$, we obtain

$$G'(\epsilon_0 E(t))L(t) + c_4(\beta_1 + \beta_2 + \beta_3 + (\alpha_1 + \alpha_2 + \alpha_3)G'(\epsilon_0 E(t)))E'(t) \leq -c_5 E(t)G'(\epsilon_0 E(t)), \tag{4.25}$$

where

$$c_5 = c_3 - c_4 \epsilon_0 (\beta_1 + \beta_2 + \beta_3).$$

Let

$$F(t) = \kappa \left[G'(\epsilon_0 E(t))L(t) + c_4(\beta_1 + \beta_2 + \beta_3 + (\alpha_1 + \alpha_2 + \alpha_3)G'(\epsilon_0 E(t)))E'(t) \right], \tag{4.26}$$

where $\kappa > 0$. Using the fact that $G'(\epsilon_0 E(t))$ is non-increasing and (4.25), we get

$$F'(t) \leq -c_5 \tau E(t) G'(\epsilon_0 E(t)).$$

Combining (4.19) with the fact that $G'(\epsilon_0 E(t)) \leq G'(\epsilon_0 E(0))$, we choose $\tau > 0$ small enough such that

$$F(t) \leq E(t) \quad \text{and} \quad F(0) \leq 1.$$

Then, choosing $c' = c_5 \tau$ (note that $s \mapsto sG'(\epsilon_0 s)$ is non-decreasing), we arrive at

$$F'(t) \leq -c' F(t) G'(\epsilon_0 F(t)). \tag{4.27}$$

This implies that $(H(F(t)))' \geq c'$, where $H(t)$ is defined in (2.25). Then, integrating (4.27) over $(0,t)$, using $F(0) \leq 1, H(1) = 0$ and $H(t)$ is decreasing, we get

$$H(F(t)) \geq c' t,$$

which implies $F(t) \leq H^{-1}(c't)$. Finally, the fact that $F(t) \sim E(t)$ gives (2.24) with

$$C'' = \frac{1}{\tau c_4(\beta_1 + \beta_2 + \beta_3)},$$

which completes the proof Theorem 2.2.

References

[1] F. Alabau Boussouira, J. E. Muñoz Rivera and D. S. Almeida Júnior, Stability to weak dissipative Bresse system, J. Math. Anal. Appl. **374** (2011), no. 2, 481–498.

[2] D. S. Almeida Júnior, M. L. Santos and J. E. Muñoz Rivera, Stability to 1-D thermoelastic Timoshenko beam acting on shear force, Z. Angew. Math. Phys. **65** (2014), no. 6, 1233–1249.

[3] F. Ammar-Khodja et al., Energy decay for Timoshenko systems of memory type, J. Differential Equations **194** (2003), no. 1, 82–115.

[4] T. A. Apalara, S. A. Messaoudi and A. A. Keddi, On the decay rates of Timoshenko system with second sound, Math. Methods Appl. Sci. **39** (2016), no. 10, 2671–2684.

[5] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York (2011).

[6] B. D. Coleman, W. J. Hrusa and D. R. Owen, Stability of equilibrium for a nonlinear hyperbolic system describing heat propagation by second sound in solids, Arch. Rational Mech. Anal. **94** (1986), no. 3, 267–289.

[7] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Rational Mech. Anal. **37** (1970).

[8] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, SIAM J. Control Optim. **24** (1986), no. 1, 152–156.

[9] L. H. Fatori and J. E. M. Rivera, Rates of decay to weak thermoelastic Bresse system, IMA J. Appl. Math. **75** (2010), no. 6, 881–904.

[10] A. Guesmia, Asymptotic stability of abstract dissipative systems with infinite memory, J. Math. Anal. Appl. **382** (2011), no. 2, 748–760.

[11] A. Guesmia, On the stabilization for Timoshenko system with past history and frictional damping controls, Palest. J. Math. **2** (2013), no. 2, 187–214.

[12] A. Guesmia and S. A. Messaoudi, On the control of a viscoelastic damped Timoshenko-type system, Appl. Math. Comput. **206** (2008), no. 2, 589–597.

[13] A. Guesmia and S. A. Messaoudi, General energy decay estimates of Timoshenko systems with frictional versus viscoelastic damping, Math. Methods Appl. Sci. **32** (2009), no. 16, 2102–2122.

[14] A. Guesmia and M. Kafini, Bresse system with infinite memories, Math. Methods Appl. Sci. **38**(2014),no. 11, 2389–2402.

[15] W. Liu, General decay of the solution for a viscoelastic wave equation with a time-varying delay term in the internal feedback, J. Math. Phys. **54** (2013), no. 4, 043504, 9 pp.

[16] J. E. Lagnese, G. Leugering and E. J. P. G. Schmidt, Modeling, analysis and control of dynamic elastic multi-link structures, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA (1994).

[17] V. Komornik, *Exact controllability and stabilization*, RAM: Research in Applied Mathematics, Masson, Paris, 1994.

[18] Z. Liu and B. Rao, Energy decay rate of the thermoelastic Bresse system, Z. Angew. Math. Phys. **60** (2009), no. 1, 54–69.

[19] M. Kirane, B. Said-Houari and M. N. Anwar, Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks, Commun. Pure Appl. Anal. **10** (2011), no. 2, 667–686.

[20] M. Kirane and B. Said-Houari, Existence and asymptotic stability of a viscoelastic wave equation with a delay, Z. Angew. Math. Phys. **62** (2011), no. 6, 1065–1082.

[21] J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems—global existence and exponential stability, J. Math. Anal. Appl. **276** (2002), no. 1, 248–278.

[22] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim. **45** (2006), no. 5, 1561–1585.

[23] S. Nicaise, C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), no. 3, 693–722.

[24] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, 44, Springer, New York, 1983.

[25] R. Racke, Thermoelasticity with second sound—exponential stability in linear and non-linear 1-d, Math. Methods Appl. Sci. **25** (2002), no. 5, 409–441.

[26] B. Said-Houari and Y. Laskri, A stability result of a Timoshenko system with a delay term in the internal feedback, Appl. Math. Comput. **217** (2010), no. 6, 2857–2869.

[27] B. Said-Houari, S. A. Messaoudi and A. Guesmia, General decay of solutions of a nonlinear system of viscoelastic wave equations, NoDEA Nonlinear Differential Equations Appl. **18** (2011), no. 6, 659–684.

[28] M. L. Santos, D. S. Almeida Júnior and J. E. Muñoz Rivera, The stability number of the Timoshenko system with second sound, J. Differential Equations **253** (2012), no. 9, 2715–2733.

[29] G. Q. Xu, S. P. Yung and L. K. Li, Stabilization of wave systems with input delay in the boundary control, ESAIM Control Optim. Calc. Var. **12** (2006), no. 4, 770–785.