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The Dynamics of the Wave Packet in the Majorana Equation

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Abstract: In the Majorana equation for particles with arbitrary spin, the wave packet is due not only to the uncertainty affecting position and momentum but also to the infinite components with decreasing mass forming the Majorana spinor. In this paper we prove that such components contribute to increase the spreading of the wave packet. Moreover, as occurs in the time propagation of the Dirac wave packet, also in that of Majorana the Zitterbewegung takes place, but it shows a peculiar fine structure. Finally, the group velocity always remains subluminal and the contributions due to the infinite components decrease progressively increasing the spin.

Keywords: wave packet; infinite components wavefunctions; Zitterbewegung

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1 Introduction

The study of the relativistic wave packet of a free particle is one of the most investigated topics in quantum mechanics, both for pedagogical reasons [1-3] and for its implication in some fields of the theoretical research [4-5]. In fact, the relativistic wave packet shows particularities and peculiarities that do not occur for the non-relativistic one, obtained from the Schrödinger equation. The latter spreads during the time evolution maintaining its Gaussian profile [6], while the relativistic wave packet spreads losing its characteristic initial bell-shaped [2,6]. Moreover, the negative energy solutions of Dirac equation lead to the Zitterbewegung [7], a hypothetical wiggling motion of the position of an elementary particle around its mean value. This motion has been studied by Schrödinger in 1930 and has been interpreted as a rapid motion due to the overlapping of the positive and negative energy solutions of Dirac equation in the position space [8]. Recently, has been proven that this motion not necessarily have to be attributed to the interference between positive and negative energy solutions, but that can be explained on the basis of the complex phase factor of the Dirac wave function [9]. This opens the way to introduce the Zitterbewegung also in quantum equation whose solutions have only one sign of energies.

In this study the propagation of the wave packet of a fermionic free particle is investigated in the Majorana equation with infinite components [10]. The study proves that the wave packet is given by the contribution of all infinite components with decreasing mass spectrum, to which correspond increasing spin. Each of these components propagates over time just as occurs for a single Dirac wave packet. Moreover, although the bradyonic solutions of Majorana equation have only positive energies, the Zitterbewegung still occurs and can be explained just on the basis of the opposite sign of their complex phase factors. Because of the contribution of all the infinite components of the spinor, the spreading of the wave packet is increased compare to that of Dirac. The same infinite components lead to an anomalous *fine structure* of the Zitterbewegung.

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Concerning the group velocity, we prove that it remains always subluminal. Particularly, the group velocities associated to the solutions with increasing spin progressively decrease in proportion to the mass spectrum of the bradyonic tower.

2 The Wave Packet in the Dirac Equation

In literature there are many papers concerning the study of the relativistic wave packet in the Dirac equation [1-3, 11-13]. In this section we retrace in detail the work of Theller [6], which will be used as methodological approach for the investigation of the relativistic Majorana wave packet [10]. In the Dirac equation the wave packet in one space dimension is given by the Fourier integral of the overlapping between the positive and negative energy solutions:

$$\Psi(x,t) = \int_{-\infty}^{\infty}[a(p)\varphi_+(p,x,t) + b(p)\varphi_-(p,x,t)]dp \qquad \textbf{(1)}$$

The coefficients $a(p)$ and $b(p)$ are functions of the impulse while φ_+ and φ_- are the solutions with positive and negative energy:

$$\begin{cases} \varphi_+(p,x,t) = \frac{1}{\sqrt{2\pi}} u_+(p) exp\left\{-i\frac{(px-Et)}{\hbar}\right\} \\ \varphi_-(p,x,t) = \frac{1}{\sqrt{2\pi}} u_-(p) exp\left\{i\frac{(px-Et)}{\hbar}\right\} \end{cases} \qquad \textbf{(2)}$$

where u_+ and u_- are the Dirac spinor of the particle and antiparticle. We remark that for the one dimensional motion there is not spin-flip (up-down inversion of the spin): particle and antiparticle have the same spin component (either up or down) and then u_+ and u_- are two component vectors. The functions $a(p)$ and $b(p)$ are the Fourier transform of the wave packet at the time $t = 0$:

$$\begin{cases} a(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ipx/\hbar} dx \left[\frac{1}{2}\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_0(p)}{E_+}\right)\right] \\ b(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ipx/\hbar} dx \left[\frac{1}{2}\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_0(p)}{E_-}\right)\right] \end{cases} \qquad \textbf{(3)}$$

where $\mathbb{1}$ is the 2x2 unitary matrix and $\mathbb{H}_0(p)$ is the 2x2 Hamiltonian matrix:

$$\mathbb{H}_0(p) = \begin{pmatrix} m_0c^2 & cp \\ cp & -m_0c^2 \end{pmatrix} \qquad \textbf{(4)}$$

$E_\pm$ are the energies of the positive and negative solutions given by the relativistic relation $\pm\sqrt{p^2c^2 + m_0^2c^4}$. In this study we suppose that the initial plan wave is of Gaussian type:

$$\Psi(x,0)_\pm = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{x^2}{2\sigma^2}\right\} u_\pm \qquad \textbf{(5)}$$

At $t = 0$ the particle position is within the range $\pm\sigma$ with a zero mean value. Substituting equation **(5)** in **(3)** and performing the Fourier integral we get:

$$\begin{cases} a(p) = \frac{\sigma}{2\sqrt{\pi\hbar}}\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_0(p)}{E_+}\right) e^{-\sigma^2p^2} \\ b(p) = \frac{\sigma}{2\sqrt{\pi\hbar}}\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_0(p)}{E_-}\right) e^{-\sigma^2p^2} \end{cases} \qquad \textbf{(6)}$$

Let us consider the two matrices in brackets; by easy calculation we get their explicit form:

$$\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_0(p)}{E_\pm}\right) = \begin{pmatrix} 1 \mp \frac{1}{\gamma} & \mp 1 \\ \mp 1 & 1 \pm \frac{1}{\gamma} \end{pmatrix} \qquad \textbf{(7)}$$

where γ is the Lorentz relativistic factor. Since we are interested in the dynamic of the particle close to the speed of light, the **(7)** can be simplified as:

$$\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_0(p)}{E_\pm}\right)_{v\to c} = \begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix} \qquad \textbf{(8)}$$

Replacing **(2)**, **(6)** and **(8)** in the **(1)** and considering the column vectors $u_+ = (1,0)^t$, $u_- = (0,1)^t$ we get the wave packet spinor:

$$\Psi(x,t) = \frac{\sigma}{2\sqrt{2\pi^2\hbar}}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-iEt/\hbar} \int_{-\infty}^{\infty} e^{-\sigma^2p^2} e^{-ipx/\hbar} dp + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{iEt/\hbar} \int_{-\infty}^{\infty} e^{-\sigma^2p^2} e^{ipx/\hbar} dp\right] \qquad \textbf{(9)}$$

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The first and the latter integral are respectively the Fourier transform and anti-transform of the function $e^{-\sigma^2 p^2}$:

$$T\left(e^{-\sigma^2 p^2}\right) = \frac{1}{\sqrt{8}\sigma} e^{-x^2/\sigma^2}\ , \quad T^{-1}\left(e^{-\sigma^2 p^2}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \qquad \textbf{(10)}$$

Substituting **(10)** in **(9)** we obtain the final form of the wave packet in one dimensional space, the components of which are:

$$\Psi(x,t) = \begin{cases} \frac{1}{8\pi\sqrt{\hbar}} e^{-iEt/\hbar} e^{-(x^2/\sigma^2)} + \frac{1}{4\pi\sqrt{\pi\hbar}} e^{iEt/\hbar} e^{-(x^2/2\sigma^2)} \\ -\frac{1}{8\pi\sqrt{\hbar}} e^{-iEt/\hbar} e^{-(x^2/\sigma^2)} + \frac{1}{4\pi\sqrt{\pi\hbar}} e^{iEt/\hbar} e^{-(x^2/2\sigma^2)} \end{cases} \qquad \textbf{(11)}$$

The spinor **(11)** shows that the two components are linear combinations of Gaussians with different width, modulated by a time oscillating function. As soon as $t \neq 0$ the Gaussians $e^{-(x^2/\sigma^2)}$ and $e^{-(x^2/2\sigma^2)}$ begin translating in opposite directions by an amount equal to $Et/\hbar$ losing their symmetrical shape. Replacing in the term $Et/\hbar$ the explicit form of the relativistic energy and considering that $x = ct$, we get the translation T:

$$T = \pm\gamma \frac{m_0 c}{\hbar} \qquad \textbf{(12)}$$

where the positive and negative signs refer respectively to the first and second component of the wave packet **(11)**. Depending on the Lorentz factor, the translation **(12)** increases with the velocity of the particle.

As mentioned in the introduction [7-8], the interference of the two solutions with positive and negative energy leads to the Zitterbewegung. This motion occurs at the speed of light and is sustained as long as the solutions with positive and negative energy overlap in the position space. Therefore, the ripples characterizing the motion of the particle position around its mean value fades away as soon as the particle begins to evolve over the time.

During the time evolution, the wave packet spreads more and more in accordance with the functional relationship between the variance and the propagation time [14]:

$$\sigma = \sigma_0 \sqrt{1 + \frac{2\hbar^2 t^2}{m_0^2 \sigma_0^2}} \qquad \textbf{(13)}$$

Finally, the group velocity of the relativistic Dirac wave packet is given by:

$$v_g = \frac{\partial E}{\partial p} = \frac{\partial}{\partial p}\sqrt{p^2c^2 + m_0^2 c^4} = \frac{c}{\sqrt{2-\beta^2}} \leq c \qquad \textbf{(14)}$$

where $\beta = v/c$. The group velocity always remains subluminal and tends to c as the particle velocity approach the speed of light.

3 The Wave Packet in the Majorana Equation

Let us now to study the wave packet in the Majorana equation using the same approach adopted in the previous section. We focus the attention on a free particle with half-integer spin s_0 and rest mass m_0. As it is known, the Majorana equation returns a solution with infinite components given by the linear combination of the ground state and of all the infinite excited states with increasing intrinsic angular momentum [10,15]:

$$\varphi_{\pm}(n,p,x,t) = \frac{1}{\sqrt{2\pi}} u_{\pm}(n,p) exp\left\{\mp i \frac{(p_n x - E_n t)}{\hbar}\right\} \qquad \textbf{(15)}$$

n is the positive integer number corresponding to the excited state with mass and spin given by [16]:

$$\begin{cases} m(n) = m_0 / \left(\frac{1}{2} + J_n\right) \\ J_n = s_0 + n \end{cases} \qquad \textbf{(16)}$$

Momentum and energy are [16]:

$$\begin{cases} p(n) = \gamma \frac{m_0}{(n+1)} v \\ E(n) = \gamma \frac{m_0}{(n+1)} c^2 \end{cases} \qquad \textbf{(17)}$$

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The probability of occupation of the nth excited state is [16]:

$$P_n = \sqrt{(v/c)^n - (v/c)^{n+1}} \tag{18}$$

Equations **(17)** and **(18)** prove that when the velocity of the particle approaches the speed of light, the excited states with high spin J_n are stabilized. Another peculiarity of the Majorana equation is that all the bradyonic solutions have positive energies [10]. Therefore, in the picture of Majorana theory of particles with arbitrary spin the wave function **(1)** becomes:

$$\Psi(x,t) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} P_n [a(n,p)\varphi_+(n,p,x,t) + b(n,p)\varphi_-(n,p,x,t)]dp \tag{19}$$

where every excited state has been weighted for its occupation probability. Since all the solutions (particle and antiparticles) have positive energies, the functional coefficients $a(n,p)$ and $b(n,p)$ are equal:

$$a(n,p) = b(n,p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ip_n x/\hbar} dx \left[J_n\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_n(p)}{E(n)}\right)\right] \tag{20}$$

Proceeding in a similar way to what done in the picture of Dirac, the matrix $\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_n(p)}{E(n)}\right)$ becomes:

$$\left(\mathbb{1}m_0c^2 - \frac{\mathbb{H}_n(p)}{E(n)}\right) = \begin{pmatrix} 1 - \frac{n+1}{\gamma} & -1 \\ -1 & 1 - \frac{n+1}{\gamma} \end{pmatrix} \tag{21}$$

Unlike what has been done in the previous section, in this case is not possible simplify the **(21)** because as the particle approaches the speed of light, both γ and n increase. Substituting **(20)** and **(21)** in the **(19)** and using the spin vectors $u_+ = (1,0)^t$, $u_- = (0,1)^t$ we get:

$$\Psi(x,t) = \begin{cases} \sum_{n=0}^{\infty} P_n J_n \left[\frac{1-(n+1)/\gamma}{8\pi\sqrt{\hbar}} e^{-iE_n t/\hbar} e^{-(x^2/\sigma^2)} - \frac{1}{4\pi\sqrt{\pi\hbar}} e^{iE_n t/\hbar} e^{-(x^2/2\sigma^2)}\right] \\ \sum_{n=0}^{\infty} P_n J_n \left[\frac{1}{4\pi\sqrt{\pi\hbar}} e^{iE_n t/\hbar} e^{-(x^2/\sigma^2)} - \frac{1-(n+1)/\gamma}{8\pi\sqrt{\hbar}} e^{-iE_n t/\hbar} e^{-(x^2/2\sigma^2)}\right] \end{cases} \tag{22}$$

For $J_n = 1/2$ and for slow motion, only the ground state is occupied and **(22)** becomes:

$$\Psi(x,t) = \begin{cases} \frac{1}{4\pi\sqrt{\pi\hbar}} e^{iE_0 t/\hbar} e^{-(x^2/2\sigma^2)} \\ -\frac{1}{4\pi\sqrt{\pi\hbar}} e^{iE_0 t/\hbar} e^{-(x^2/2\sigma^2)} \end{cases} \tag{23}$$

i.e, the two components of the Majorana wave packet are opposite Gaussian functions translating with the same velocity $m_0c/\hbar$. When $v \gg 0$ then $J_n > 1/2$ and $n > 1$ and each component of the spinor **(22)** is a linear combination of infinite Gaussian functions weighted for the coefficient P_n. In particular, the first component is always negative while the latter is always positive. In the space-time each Gaussian of the linear combination **(22)** translates of a quantity given by:

$$T_n = \pm\gamma \frac{m_0 c}{\hbar(n+1)} \tag{24}$$

and the Majorana wave packet fades away in all its infinite components, spreading faster than that of Dirac. We note that each component of the spinor propagates with its own velocity. In fact, using the **(16)** we obtain:

$$v_T(n) = \frac{m_0 c}{\left(\frac{1}{2} + s_0 + n\right)} \tag{25}$$

showing that the propagation velocity decreases with the order n of the excited state. Since the Majorana solutions have only positive energies, the Zitterbewegung occurs on the basis of the different sign of the complex phase factor. Moreover, the fact that all the infinite components translate with different velocities, as proved by the **(25)**, the Zitterbewegung shows a *fine structure* formed by small ripples due to the progressive decreases of the overlapping of the infinite Gaussian functions. This behaviour does not appear in the Dirac wave packet. Since the occupation probability of the excited states increases with the particle velocity, the *fine structure* fades away as faster as slower is the motion of the particle. If the Zitterbewegung is a peculiarity of the relativistic dynamics of the wave packet, its *fine structure* is even more.

In the picture of Majorana, the relationship between variance and propagation time is very similar to the **(13)** but with an additional term representing the states with increasing spin:

$$\sigma = \sigma_0\sqrt{1 + \frac{2(n+1)\hbar^2 t^2}{m_0^2\sigma_0^2}} \quad \textbf{(26)}$$

From the **(26)** we conclude that the components of the wave packet associated to the excited states spread as faster as higher is their order n. This confirms the comment made above about the **(24)**.

Let us calculate now the group velocity of the Majorana wave packet. To such a purpose we have to take in consideration the fact that it will depend on the occupation probability of the nth excited state. For that reason we can write the group velocity as:

$$v_g^M(n) = v_g\sqrt{(v/c)^n - (v/c)^{n+1}} = v_g\sqrt{1-(v/c)}(v/c)^{n/2} \quad \textbf{(27)}$$

where v_g refers to the **(14)** and subscript M refers to Majorana theory. Replacing the explicit form of **(14)** in the **(27)** we obtain:

$$v_g^M(n) = \frac{c}{\sqrt{2-\beta^2}}\sqrt{1-\beta}(\beta)^{n/2} \quad \textbf{(28)}$$

The term $\sqrt{1-\beta}(\beta)^{n/2}$ is always lower than one and tends to zero as the velocity of the particle approaches the speed of light. That means the states with high spin have a group velocity tending to zero and this is a further confirmation that the Majorana wave packet spreads faster than that of Dirac. The overall group velocity of the Majorana wave packet is given by the relativistic composition of all the single terms **(28)** and always remains subluminal. This composition can be easily performed in the reference frame of the center of mass of the particle, where the occupied state is the fundamental one [10]. From the **(28)** we note also that for slow motion both $\beta$and n tend to zero and the group velocity becomes equal to that of the Dirac wave packet.

4 Conclusion

This study shows that when free particles with arbitrary spin are taken in account in the picture of Majorana theory, the relativistic wave packet spreads over time not only because of the uncertainty affecting position and momentum but also due to the infinite components of the spinor with decreasing masses. During the propagation these components translate with different velocities and their overlapping in the position space gradually decreases to zero leading to the formation of a *fine structure* in the Zitterbewegung. Since the occupation probabilities of the spinor components decreases with the increasing of their order n, the *fine structure* fade away very quickly and it is considerable only when the particle velocity approaches the speed of light. This result is completely unexpected and, even if is not physically observable, represents a further breaking point between the theories of Dirac and Majorana. We proved also that the overall group velocity is given by the contribute of the single group velocities of the infinite components, whose value decreases with increasing their order n, and remains always subluminal.

The results obtained in this theoretical work could facilitate the comprehension of the relativistic motion of a fermionic particle with any value of spin, especially in cases where the material wave interacts with a finite barrier of potential. This last topic will be the subject of a new study whose purpose is to investigate the behaviour of a Majorana fermion in tunnelling phenomena.

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