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*Article*

# Modeling Sound Propagation Using the Corrective Smoothed Particle Method with an Acoustic Boundary Treatment Technique

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**Abstract:** The development of computational acoustics allows simulation of sound generation and propagation in complex environment. In particular, meshfree methods are widely used to solve acoustics problems through arbitrarily distributed field points and approximation smoothness flexibility. As a Lagrangian meshfree method, smoothed particle hydrodynamics (SPH) method reduce the difficulty in solving problems with deformable boundaries, complex topologies, or multiphase medium. The traditional SPH method has been applied in acoustic simulation. This study presents the corrective smoothed particle method (CSPM), which is a combination of SPH kernel estimate and Taylor series expansion. The CSPM is introduced as a Lagrangian approach to improve accuracy in solving acoustic wave equations in the time domain. Moreover, a boundary treatment technique based on the hybrid meshfree and finite difference time domain (FDTD) method is proposed to represent different acoustic boundaries with particles. To model sound propagation in pipes with different boundaries, soft, rigid, and absorbing boundary conditions are built with this technique. Numerical results show that the CSPM algorithm is consistent and demonstrates convergence with exact solutions. Main computational parameters are discussed, and different boundary conditions are validated to be effective for benchmark problems in computational acoustics.

**Keywords:** computational acoustics; meshfree method; Lagrangian approach; smoothed particle hydrodynamics; corrective smoothed particle method; boundary conditions; numerical method

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## 1. Introduction

Numerical methods have been implemented to model acoustic phenomena, and the development of computational acoustics allows simulation of sound generation and propagation in complex environment. Many classic numerical methods such as the finite difference method (FDM) [1], the finite element method (FEM) [2], and the boundary element method (BEM) [3], and other modified methods[4, 5] have been applied in spectral or temporal acoustic simulations. In particular, meshfree methods are widely applied to solve acoustics problems, because field points used in this method are arbitrarily distributed and the approximation smoothness order is chosen flexibility. The method of fundamental solutions (MFS) [6], the multiple-scale reproducing kernel particle method (RKPM) [7], the element-free Galerkin method (EFGM) [8], and other meshfree methods [9-12] are used to address certain acoustic problems.

The smoothed particle hydrodynamics (SPH) method, as a Lagrangian, meshfree particle method, was first independently pioneered by Lucy [13] and Gingold and Monaghan [14] for solving astrophysical problems. As a Lagrangian approach, SPH method has several advantages over standard grid-based numerical method: (i) numerical error generated by computing the advection is eliminated since the advection term is included in the Lagrangian derivative; (ii) complicated domain topologies and moving boundaries are easily represented due to its Lagrangian property as illustrated in recent reviews by Liu and Liu [15], Springel [16], and Monaghan [17]; (iii) interface between different medium can be naturally traced through particle density instead of using special algorithm such as the volume-of-fluid; (iv) it is easy to implement and has parallel processing ability for the approximation is implemented in the local support domain instead of the whole computational domain [18, 19]. Introducing the SPH method to acoustic computation can bring these advantages to this field.

Recently, the SPH method has gradually used in acoustic computation, and some researchers have attempted to obtain the acoustic field through direct numerical simulation. Wolfe [20] simulated room reverberation with sound generation and reception based on a SPH fluid mechanics algorithm, and Hahn [21] solved the fluid dynamic equations to obtain pressure perturbations during sound propagation. Both these works can be seen as direct numerical simulation (DNS) based on the SPH method. However, for various acoustic waves in engineering problems, acoustic variables such as the variation in pressure, density, and velocity, are generally small. On the contrary, the values of pressure, density, and velocity exist on a much larger scale than any variation in these variables, as shown in Chapter 1 in [22]. Acoustic wave equations are obtained based on the acoustic variables to avoid solving fluid dynamic variables. Consequently, solving acoustic wave equations requires a lower computational burden compared to solving the fluid dynamic equations directly, and this approach is widely used in modeling engineering problems.

In our recent work, we proposed the use of the SPH method to solve acoustic wave equations, and tests about the sound propagation and interference simulation were conducted [23, 24]. Some computational parameters were also discussed [25, 26]. Based on these tests, the SPH method solved acoustic wave equations accurately, and some parameters were investigated, but only the traditional SPH method was used.

With advance of the SPH method, the traditional SPH method is modified or improved to reduce numerical error. Chen et al. [27, 28] proposed the corrective smoothed particle method (CSPM) on the basis of Taylor series expansion in 1999. The CSPM is effective in reducing numerical error both inside the computational domain and around the boundary, so it has been used in different fields [29, 30]. Other notable modifications or corrections of the SPH method include the reproducing kernel particle method (RKPM) [31], the finite particle method (FPM) [32, 33], the moving least square particle hydrodynamics (MLSPH) [34, 35], and the modified smoothed particle hydrodynamics (MSPH) [36]. The present study focuses on using the CSPM to improve the simulation accuracy of the SPH method in solving acoustic wave equations.

The Implementation of boundary conditions in the SPH method is not as straightforward as in the grid-based numerical models. This characteristic has been regarded historically as a weak point of the particle method [37, 38]. Several approaches have been proposed to treat boundary conditions for computational fluid dynamics. Among them, it is feasible to use virtual particles [39] to implement the boundary conditions. These virtual particles are allocated on and outside the boundary as shown in several related works [40, 41]. So far, a limited implementation of acoustic boundaries is observed, and rigid acoustic boundary developed from the solid boundary in fluid dynamics is shown in [20, 21, 23-26]. The boundary treatment technique for different acoustic boundaries is important for acoustic numerical analysis by the Lagrangian meshfree method.

The present paper is organized as follows. In Section 2, the CSPM formulations are provided to solve the acoustic wave equations. In Section 3, a hybrid meshfree-FDTD (finite difference time domain) method is proposed for acoustic boundary treatment. In Section 4, sound propagation model is used to validate the CSPM algorithm, and the effects of different computational parameters are discussed. In Section 5, soft, rigid, and absorbing boundaries are used to simulate sound reflection and transmission, and numerical results are compared with theoretical solutions.

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## 2. CSPM Formulations for Sound Waves

### *2.1. Basic Concepts of SPH*

The kernel approximation of a function $f(\boldsymbol{r})$ at particle $i$ used in the SPH method can be written as a summation of neighboring particles as

$$< f(\boldsymbol{r}_i) >= \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\boldsymbol{r}_j) W_{ij} \tag{1}$$

where $f$ is a function of the position vector $\boldsymbol{r}$, $W_{ij} = W(\boldsymbol{r}_i - \boldsymbol{r}_j, h)$, $W$ is the smoothing kernel function, $h$ is the smoothing length defines the influence area of the smoothing function, $N$ indicates the number of particles in the support domain, $m_j$ is the mass of particle $j$, and $\rho_j$ is the density. In the SPH convention, the kernel approximation operator is marked by the angle bracket < >.

The particle approximation for the derivative can be described as

$$< \nabla \cdot f(\boldsymbol{r}) >= -\sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\boldsymbol{r}_j) \cdot \nabla_i W_{ij} \tag{2}$$

Present paper uses the cubic spline kernel function as the smoothing function. The cubic spline kernel is a widely used smoothing function which was originally used by Monaghan and Lattanzio [42]

$$W(r,h) = \alpha_D \begin{cases} 1 - \dfrac{3}{2}q^2 + \dfrac{3}{4}q^3 & 0 \le q \le 1 \\ \dfrac{1}{4}(2-q)^3 & 1 \le q \le 2 \\ 0 & q \ge 2 \end{cases} \tag{3}$$

where $q = r/h$, and in one-, two-, and three-dimensional space, $\alpha_D = 1/h$, $15/(7\pi h^2)$, and $3/(2\pi h^3)$ respectively.

### *2.2. Acoustic Wave Equations in Lagrangian Form*

In fluid dynamics, the governing equations for constructing SPH formulations are the laws of continuity, momentum and state which can be found in Chapter 4 of [43]. The Lagrangian form of the governing equations are

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{v} \tag{4}$$

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{F} + \boldsymbol{g} \tag{5}$$

$$\frac{\mathrm{D}P}{\mathrm{D}t} = c_0^2 \frac{\mathrm{D}\rho}{\mathrm{D}t} \tag{6}$$

where $\rho$ is the fluid density, $\boldsymbol{v}$ is the flow velocity, $P$ is the pressure, $t$ is time and $c$ is the speed of sound.

For most common acoustical problems, there are two assumptions are always used to simplify the question. On one hand, the medium is lossless and at rest. On the other hand, a small departure from quiet conditions occurs. In this work, the medium for sound propagation and reflection is ideal fluid, and the process is adiabatic. In addition, sound pressure $\delta p$, the density change of particle $\delta\rho$ and the particle velocity $\boldsymbol{v}$ are supposed small, which can be expressed as

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$$\begin{cases} P = p_0 + \delta p, & |\delta p| << p_0 \\ \rho = \rho_0 + \delta\rho, & |\delta\rho| << \rho_0 \\ \boldsymbol{v} = 0 + \boldsymbol{v}, & |\boldsymbol{v}| << c_0 \end{cases} \tag{7}$$

Substitute these expressions into the continuity and momentum equations (Eq. (4) and Eq. (5)) with ignoring the viscosity and force:

$$\frac{\mathrm{D}(\rho_0 + \delta\rho)}{\mathrm{D}t} = -(\rho_0 + \delta\rho)\nabla\cdot\boldsymbol{v} \tag{8}$$

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{(\rho_0 + \delta\rho)}\nabla(p_0 + \delta p) \tag{9}$$

Considering $\rho_0$ and $p_0$ remains the same during the computation, they can also be expressed as

$$\frac{\mathrm{D}\delta\rho}{\mathrm{D}t} = -(\rho_0 + \delta\rho)\nabla\cdot\boldsymbol{v} \tag{10}$$

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{(\rho_0 + \delta\rho)}\nabla\delta p \tag{11}$$

Ignoring small quantities of first order, Lagrangian form of the continuity and momentum equations governing sound waves can be written as

$$\frac{\mathrm{D}\delta\rho}{\mathrm{D}t} = -\rho_0\nabla\cdot\boldsymbol{v} \tag{12}$$

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho_0}\nabla\delta p \tag{13}$$

The state equation for ideal gas is written as

$$\frac{\mathrm{D}\delta p}{\mathrm{D}t} = c_0^2\frac{\mathrm{D}\delta\rho}{\mathrm{D}t} \tag{14}$$

*2.3. CSPM Formulations for Acoustic Waves*

2.3.1. Particle Approximation of the Continuity Equation

Applying the particle approximation equation (Eq. (2)) to the continuity equation (Eq. (10)) yields

$$\frac{\mathrm{D}\delta\rho_i}{\mathrm{D}t} = -(\rho_0 + \delta\rho_i)\sum_{j=1}^{N}\frac{m_j}{(\rho_0 + \delta\rho_j)}\boldsymbol{v}_j\nabla_i W_{ij} \tag{15}$$

where the subscript $i$ and $j$ stand for variables associated with particles $i$ and $j$.

By adding the gradient of the unity [43], another SPH formulation of the continuity equation can be obtained as

$$\frac{\mathrm{D}\delta\rho_i}{\mathrm{D}t} = -(\rho_0 + \delta\rho_i)\sum_{j=1}^{N}\frac{m_j}{(\rho_0 + \delta\rho_j)}\boldsymbol{v}_{ij}\nabla_i W_{ij} \tag{16}$$

where $\boldsymbol{v}_{ij} = \boldsymbol{v}_i - \boldsymbol{v}_j$. Considering Eq. (7), the continuity equation can also be written as

$$\frac{\mathrm{D}\delta\rho_i}{\mathrm{D}t} = -\sum_{j=1}^{N} m_j\boldsymbol{v}_{ij}\nabla_i W_{ij} \tag{17}$$

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2.3.2. Particle Approximation of the Momentum Equation

Applying the particle approximation equation (Eq. (2)) to the momentum equation (Eq. (11)), it appears as

$$\frac{\mathrm{D}\boldsymbol{v}_i}{\mathrm{D}t} = -\frac{1}{(\rho_0+\delta\rho_i)}\sum_{j=1}^{N}\frac{m_j}{(\rho_0+\delta\rho_j)}\delta p_j \nabla_i W_{ij} \tag{18}$$

Other forms of the momentum equation can be written as

$$\frac{\mathrm{D}\boldsymbol{v}_i}{\mathrm{D}t} = -\sum_{j=1}^{N} m_j \frac{\delta p_i+\delta p_j}{(\rho_0+\delta\rho_i)(\rho_0+\delta\rho_j)}\nabla_i W_{ij} \tag{19}$$

$$\frac{\mathrm{D}\boldsymbol{v}_i}{\mathrm{D}t} = -\sum_{j=1}^{N} m_j \left[\frac{\delta p_i}{(\rho_0+\delta\rho_i)^2}+\frac{\delta p_j}{(\rho_0+\delta\rho_j)^2}\right]\nabla_i W_{ij} \tag{20}$$

2.3.3. Particle Approximation of the Equation of State

Particle approximation of the equation of state for ideal gas is

$$\frac{\mathrm{D}\delta p_i}{\mathrm{D}t} = c_0^2\frac{\mathrm{D}\delta\rho_i}{\mathrm{D}t} \tag{21}$$

2.3.4. Corrective Smoothed Particle Method

The Taylor series expansion is used to improve the accuracy of the SPH method, which is named as CSPM. If a function $f(\boldsymbol{r})$ is assumed to be sufficiently smooth in a domain that contains $\boldsymbol{r}$, the Taylor series expansion for $f(\boldsymbol{r})$ in the vicinity of $\xi$ can be written as

$$f(\boldsymbol{r}) = f(\xi)+(\boldsymbol{r}^\alpha-\xi^\alpha)f_\alpha(\xi)+\frac{1}{2!}(\boldsymbol{r}^\alpha-\xi^\alpha)(\boldsymbol{r}^\beta-\xi^\beta)f_{\alpha\beta}(\xi)+\ldots \tag{22}$$

where $\alpha$, $\beta$ = 1, 2, 3, represent different dimensional space and

$$f_\alpha(\xi) = \frac{\partial f(\xi)}{\partial \boldsymbol{r}^\alpha} \tag{23}$$

$$f_{\alpha\beta}(\xi) = \frac{\partial^2 f(\xi)}{\partial \boldsymbol{r}^\alpha \partial \boldsymbol{r}^\beta} \tag{24}$$

Multiplying both sides of Equation (22) by a smoothing function $W$ defined in the local support domain $\Omega$ of $\xi$, and integrating over the support domain, the following formulation can be obtained

$$\begin{aligned}\int_\Omega f(\boldsymbol{r})W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r} &= \int_\Omega f(\xi)W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r}+\int_\Omega(\boldsymbol{r}^\alpha-\xi^\alpha)f_\alpha(\xi)W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r}\\ &+\int_\Omega\frac{1}{2!}(\boldsymbol{r}^\alpha-\xi^\alpha)(\boldsymbol{r}^\beta-\xi^\beta)f_{\alpha\beta}(\xi)W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r}+\ldots\end{aligned} \tag{25}$$

From the above equation, a corrective kernel approximation for function $f(\boldsymbol{r})$ at $\xi$ can be written as

$$< f(\xi) >= \frac{\int_\Omega f(\boldsymbol{r})W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r}}{\int_\Omega W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r}} \tag{26}$$

Replacing $W(\boldsymbol{r}-\xi, h)$ with $\nabla W(\boldsymbol{r}-\xi, h)$, a corrective kernel approximation for the first derivative of $f(\boldsymbol{r})$ at $\xi$ can be written as

$$< f_{\alpha}(\xi) >= \frac{\int_{\Omega}[f(\boldsymbol{r}) - f(\xi)]\nabla W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r}}{\int_{\Omega}(\boldsymbol{r}^{\alpha} - \xi^{\alpha})\nabla W(\boldsymbol{r}-\xi,h)\mathrm{d}\boldsymbol{r}} \tag{27}$$

The particle formulations of Equations (26) and (27) are given as

$$f(\boldsymbol{r}_i) = \frac{\sum_{j=1}^{N}\frac{m_j}{\rho_j}f(\boldsymbol{r}_j)W_{ij}}{\sum_{j=1}^{N}\frac{m_j}{\rho_j}W_{ij}} \tag{28}$$

$$f_{\alpha}(\boldsymbol{r}_i) = \frac{\sum_{j=1}^{N}\frac{m_j}{\rho_j}\left[f(\boldsymbol{r}_j) - f(\boldsymbol{r}_i)\right]\nabla_i W_{ij}}{\sum_{j=1}^{N}\frac{m_j}{\rho_j}(\boldsymbol{r}_j^{\alpha} - \boldsymbol{r}_i^{\alpha})\nabla_i W_{ij}} \tag{29}$$

The second order leap-frog integration [44] is used in the paper to update parameters, and all-pair searching approach [43] is used to realize the neighbour particle searching. The code is developed from a SPH algorithm used in [45].

## 3. Hybrid Meshfree-FDTD Method for Boundary Treatment

Meshfree method suffers from the problem that not enough particles in the support domain can be used for computation at the boundary. In the present paper, the FDTD method is introduced to combine with the virtual particle technique, and thus a technique based on the meshfree-FDTD hybrid method for acoustic boundary treatment is accordingly constructed. The feasibility and validity of the meshfree-FDTD hybrid method is verified by simulating sound propagation in pipes with boundaries.

Since the FDTD method is proposed by Yee [46] in 1966, it has received widely concern, and used to solve problems in many different research fields. The FDTD method can solve fundamental equations in the time domain. In this paper, for building the hybrid method, the FDTD method proposed by Wang [47] that used to simulate underwater acoustic boundary and the virtual particle technique are combined.

In the hybrid meshfree-FDTD boundary treatment, three types of particles need to be built before computation, namely the fluid particle, the boundary particle, and the virtual particle. During the computation, the numerical method chosen for these three kinds of particles are shown as below

$$\begin{array}{c}\text{boundary treatment}\\ \text{for particle } i\end{array} = \begin{cases}\text{meshfree method (SPH/CSPM)} & \text{if } i = \text{fluid particles}\\ \text{meshfree method (SPH/CSPM)} & \text{if } i = \text{boundary particles}\\ \text{FDTD method} & \text{if } i = \text{virtual particles}\end{cases}$$

The hybrid meshfree-FDTD boundary treatment technique means the meshfree method to obtain the parameter value of fluid and boundary paticles, and the FDTD method to solve the parameter value of virtual particles.

Figure 1 is the sketch of treatment of particles on wall boundary by using meshfree-FDTD method. $i$ represents the number of particles. Virtual particles can be obtained through extending boundary particles to the outside of the computation region, and the distribution of virtual particles are regular. The number of layer can be chosen according to the scale of the support domain.

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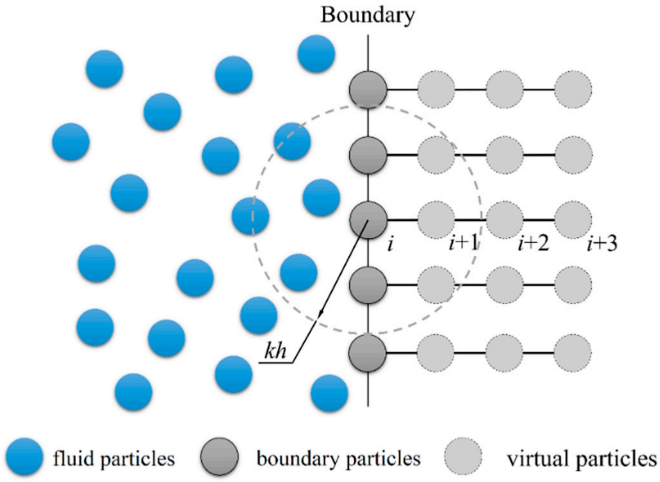


**Figure 1.** The sketch of simulating acoustic boundary using hybrid meshfree-FDTD method.

For the soft boundary, boundary conditions are

$$\delta p = 0 \tag{30}$$

The formulation for sound pressure of virtual particles is written as

$$\delta p_{i+1} = \delta p_{i+2} = \delta p_{i+3} = 0 \tag{31}$$

Assuming that the velocity perpendicular to the surface is $u$, the velocity of virtual particles (e.g. particle $i$ + 1) according to the central difference scheme should be calculated from the momentum equation as

$$\frac{u_{i+1}^{(n)} - u_{i+1}^{(n-1)}}{\Delta t} = -\frac{1}{\rho_0}\frac{(\delta p_{i+2}^{(n-1)} - \delta p_{i}^{(n-1)})}{2\Delta x} \tag{32}$$

which can be written as

$$u_{i+1}^{(n)} = u_{i+1}^{(n-1)} - \frac{(\delta p_{i+2}^{(n-1)} - \delta p_{i}^{(n-1)})\Delta t}{2\rho_0 \Delta x} \tag{33}$$

where the superscript $n$ represents the temporal index, $\Delta t$ is the time step, and $\Delta x$ is the particle spacing.

For the rigid boundary, the normal component of the pressure gradient on the surface equals zero when the wave vertically incident the boundary. Therefore, for the rigid case, the sound pressure $\delta p$ satisfies the following

$$\frac{\partial \delta p}{\partial \boldsymbol{n}} = 0,\ \delta v = 0 \tag{34}$$

where $\boldsymbol{n}$ represents the normal direction of surface. According to the finite difference scheme we have

$$\frac{\delta p_{i+1}^{(n)} - \delta p_{i}^{(n)}}{\Delta x} = 0,\ \delta v_{i+1}^{(n)} = 0 \tag{35}$$

which can be written as

$$\delta p_{i+1}^{(n)} = \delta p_{i}^{(n)},\ \delta v_{i+1}^{(n)} = 0 \tag{36}$$

For the absorbing boundary condition, the popular first-order absorbing boundary condition (ABC) proposed by Mur is used in the present work. Assuming that the ABC is located at $x = x_i$, sound propagates from the left side to the right side. The ABC can be written as

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$$\left[\frac{\partial f}{\partial x} - \frac{1}{c_0}\frac{\partial f}{\partial t}\right]_{x=x_i} = 0 \tag{37}$$

where the field parameter $f$ can be $\delta p$, $u_x$ or $u_y$ in this equation. This leads to a difference expression for virtual particles that used in the meshfree-FDTD hybrid method.

$$f_{i+1}^{(n)} = f_i^{(n-1)} + \frac{c_0\Delta t - \Delta x}{c_0\Delta t + \Delta x}\left(f_i^{(n)} - f_{i+1}^{(n-1)}\right) \tag{38}$$

The field parameter $f$ in this equation can be $\delta p$ or $\boldsymbol{v}$.

## 4. Sound Propagation Simulation with CSPM

### *4.1. Sound Propagation Model*

Sound propagation along ducts with different boundaries are discussed as shown in Figure 2. In this model, sound propagates from $x < 0$ m to $x \geq 0$ m, and the positive direction of $x$-axis denotes the direction of sound propagation. The CSPM computational region is from -50 m to 150 m, and the propagation time is 1.25 s.

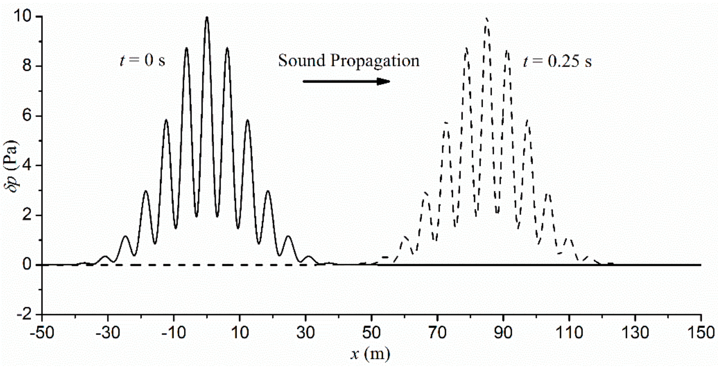


**Figure 2.** The model of sound propagation in a pipe along $x$-axis.

Sound pressure of the acoustic wave [48] in ducts is written as

$$\delta p(t,x) = 2k^2\left[3 + 2\cos(kx - \omega t)\right]\exp\left[\frac{-\ln 2}{200}(kx - \omega t)^2\right] \tag{39}$$

where $t$ denotes time, $x$ is the geometric position in the propagation direction, $\omega$ is the angular frequency of the sound wave, $k = \omega / c_0$ is the wave number. In addition, the sound speed $c_0 = 340$ m/s, and $\omega = 340$ rad/s.

### *4.2. Verification of the Meshfree Algorithm*

Table 1 lists the computational parameters that are used in the CSPM algorithm for sound propagation. The Courant-Friedriches-Lewy number is written as $C_{CFL}$ for short, and $C_{CFL} = u\Delta t / \Delta x$.

**Table 1.** Parameters for the CSPM algorithm for sound propagation modeling.

| Computational Parameters | Values |
|---|---|
| $\Delta x$ | 0.04 m |
| $h$ | 0.058 m |
| Kernel Type | Cubic Spline |
| $C_{CFL}$ | 0.10 |
| $c_0$ | 340 m/s |
| $\rho_0$ | 1.0kg/m$^3$ |

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In order to verify the algorithm, the CSPM algorithms is built to solve acoustic wave equations for sound propagation modeling. Then, the simulation results are compared to theoretical solutions as shown in Figure 3. In this figure, solid lines demonstrate theoretical solutions, and points represent the CSPM simulation results. For clearly identifying different points, they are plotted at intervals of 14 grid points.

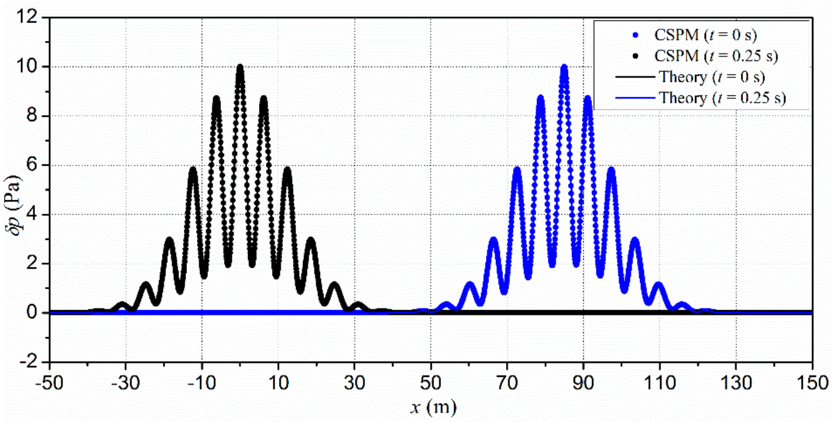


**Figure 3.** Sound pressure computation between CSPM results and theoretical solutions.

From the figure, it can be seen that several peaks and valleys appear in the graph between -50 m and 150 m. The algorithm is able to model the sound propagation process, and the CSPM simulation results agree well with theoretical solutions.

*4.3. Discussion on Computational Parameters*

In this section, effects of the initial particle spacing and the time step on the accuracy of the present CSPM method is discussed. The method is compared with theoretical solutions. The numerical accuracy is evaluated with the relative root mean square errors ($L_{\text{error}}$) and the maximum error ($M_{\text{error}}$), which are given as follows:

$$L_{\text{error}}(\delta p) = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left|\delta p(i) - \delta \overline{p}(i)\right|^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left|\delta \overline{p}(i)\right|^2}} \tag{40}$$

$$M_{\text{error}}(\delta p) = \max_{1\le i\le N}\left|\delta p(i) - \delta \overline{p}(i)\right| \tag{41}$$

where $\delta p(i)$ and $\delta \overline{p}(i)$ are simulation results and theoretical solutions at particle $i$, and $N$ is the total number of particles in the computation domain.

Sound propagation with particle spacing changing from 0.02 to 0.10 m is computed. Then, the CSPM numerical error of sound pressure according to Equations (40) and (41) is shown in Figure 4. In the computation, the ratio of the particle spacing to the smoothing length keeps the same.

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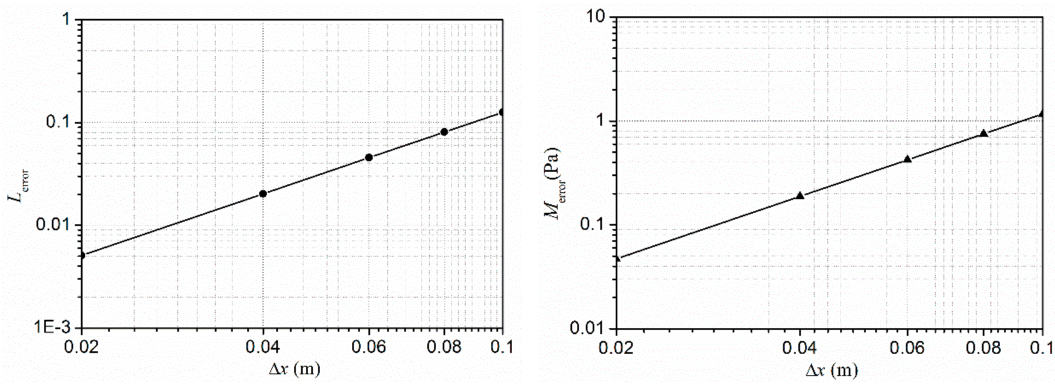
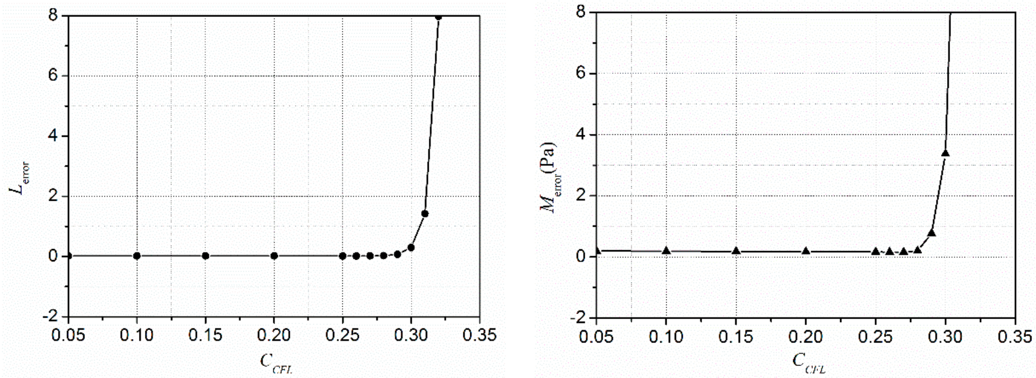



(**a**) $L_{error}$ (**b**) $M_{error}$

**Figure 4.** Convergence curve for the CSPM method.

From the figure, it can be seen that, with the increasing of particle spacing, $L_{error}$ and $M_{error}$ increase gradually. $L_{error}$ and $M_{error}$ are the smallest at particle spacing $\Delta x$ = 0.02 m, which are 1.5 × $10^{-3}$ and 0.047 Pa respectively. When particle spacing $\Delta x$ = 0.10 m, $L_{error}$ and $M_{error}$ reach 0.126 and 1.17 Pa respectively. The Convergence rate for $L_{error}$ and $M_{error}$ is about 1.997 and 1.998 respectively and 1.998 for average. In conclusion, the CSPM algorithm shows a good convergence in the simulation of sound propagation.

Similarly, numerical error of sound propagation by using the CSPM algorithms with different $C_{CFL}$ is also discussed. When $C_{CFL}$ changes from 0.05 to 0.32, $L_{error}$ and $M_{error}$ are computed as shown in Figure 5.




(**a**) $L_{error}$ (**b**)$M_{error}$

**Figure 5.** Sound pressure error versus $C_{CFL}$ in the CSPM computation.

As can be seen from the figure, in the region of 0.05 ≤ $C_{CFL}$ ≤ 0.28, with the increasing of $C_{CFL}$, $L_{error}$ and $M_{error}$ increase slowly. When $C_{CFL}$ equals to 0.05, $L_{error}$ and $M_{error}$ are 0.02 and 0.19 Pa respectively. When $C_{CFL}$ is 0.28, $L_{error}$ and $M_{error}$ are 0.03 and 0.20 Pa. Moreover, when $C_{CFL}$ is greater than 0.28, $L_{error}$ and $M_{error}$ increase sharply with increased $C_{CFL}$. In general, according to the present case, for maintaining the computational accuracy and efficiency in the numerical simulation, $C_{CFL}$ is preferably set as under 0.28.

## 5. Application of Different Acoustic Boundaries

### *5.1. Soft Boundary*

The sound propagation model with soft boundary is built to validate the boundary treatment technique. The computational domain is -10 m ⩽ $x$ ⩽ 190 m, the computational parameters are shown in Table 1. The soft boundary is located at $x$ = 150 m. Then, Sound pressure at $t$ = 0 s, 0.20 s, 0.40 s, 0.50 s, 0.60 s, 0.80 s are computed, simulation results are shown in Figure 6.

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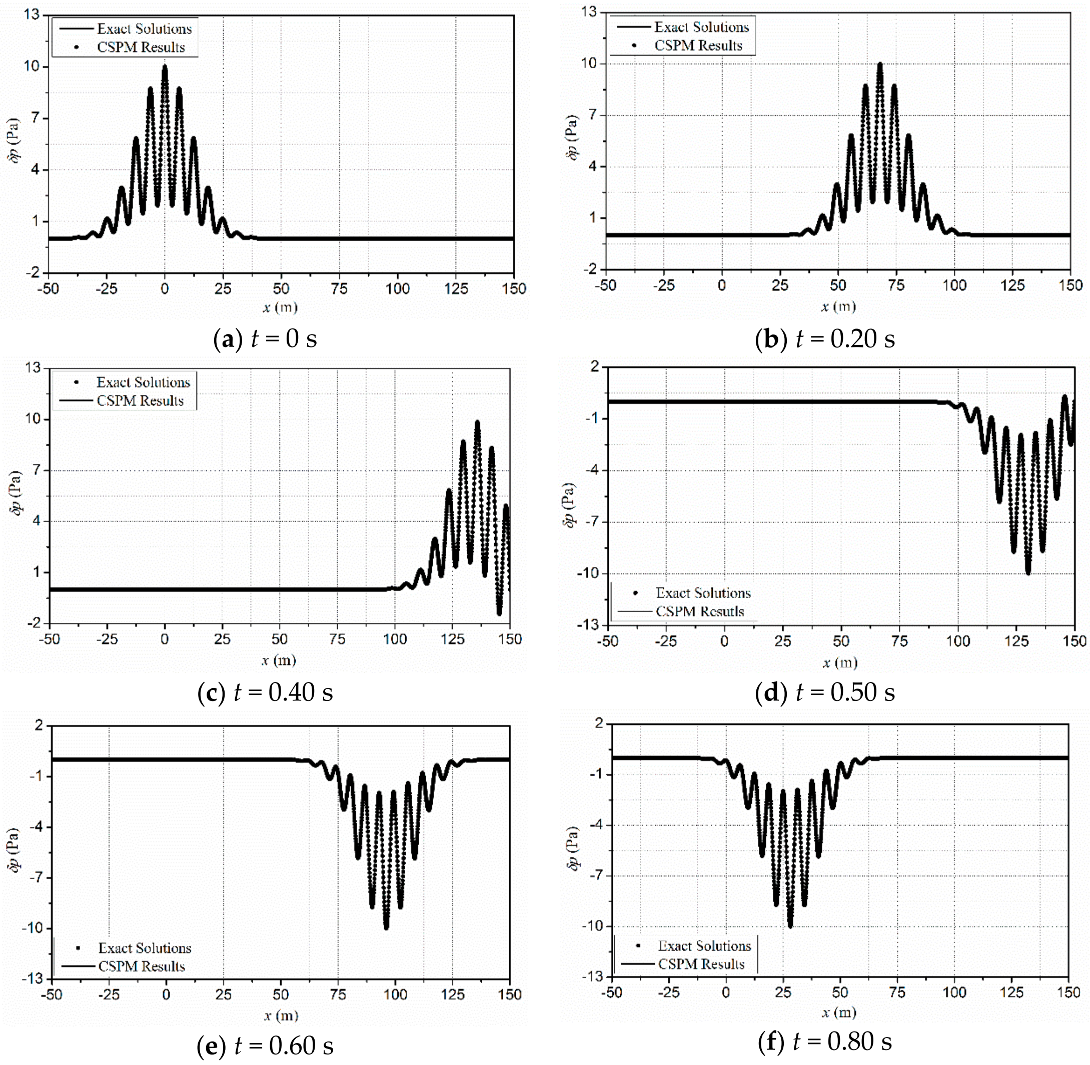


(**a**) $t = 0$ s (**b**) $t = 0.20$ s

(**c**) $t = 0.40$ s (**d**) $t = 0.50$ s

(**e**) $t = 0.60$ s (**f**) $t = 0.80$ s

**Figure 6.** Comparison between CSPM results and exact solutions with sound reflecting from the soft boundary.

Figure 6 (a) shows the sound pressure at initial time. Figure 6 (b) is the sound pressure at $t = 0.20$ s, the sound wave is propagating but has not reached the soft boundary. Figure 6 (c) and (d) is the sound pressure at $t = 0.40$ s and 0.50 s respectively. A reflected sound wave is generated by the soft boundary, and it propagates along the -$x$ direction. There is an overlap between the the incident sound wave and reflected sound. At $t$ = 0.60 s to 0.80 s, the reflected sound wave continues to propagate along the -$x$ direction, as shown in Figure 6 (e) and (f).

It can be seen from the figure that the CSPM results are in good agreement with theoretical solutions, namely, the soft boundary can be accurately represented with the hybrid meshfree-FDTD acoustic boundary treatment technique.

*5.2. Rigid Boundary*

The sound reflection model with rigid boundary at $x$ = 150 m is built. Simulation results are compared to theoretical solutions in Figure 7. Since the sound propagation process before reaching the boundary is the same as in the last section, Figure 7 (a), (b) just give the sound pressure of particles at $t = 0.40$ s and 0.80 s respectively.

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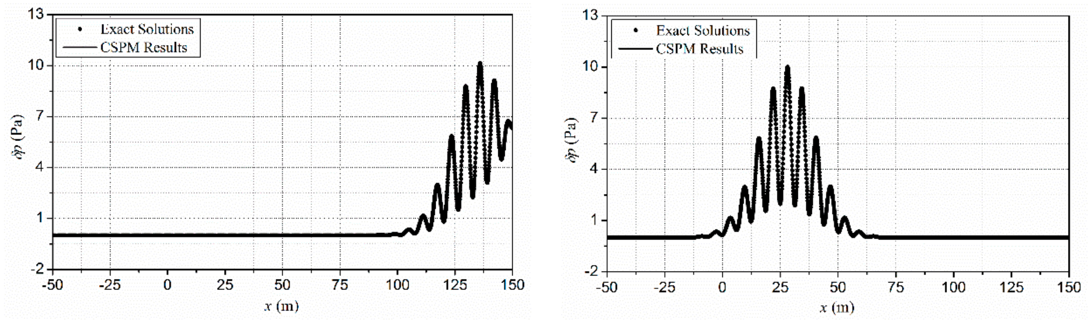


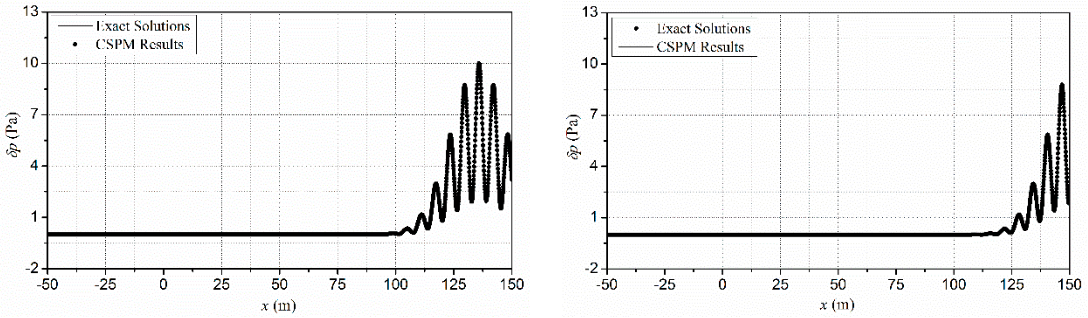

(**a**) $t$ = 0.40 s (**b**) $t$ = 0.80 s

**Figure 7.** Comparison between CSPM results and exact solutions with sound reflecting from the rigid boundary.

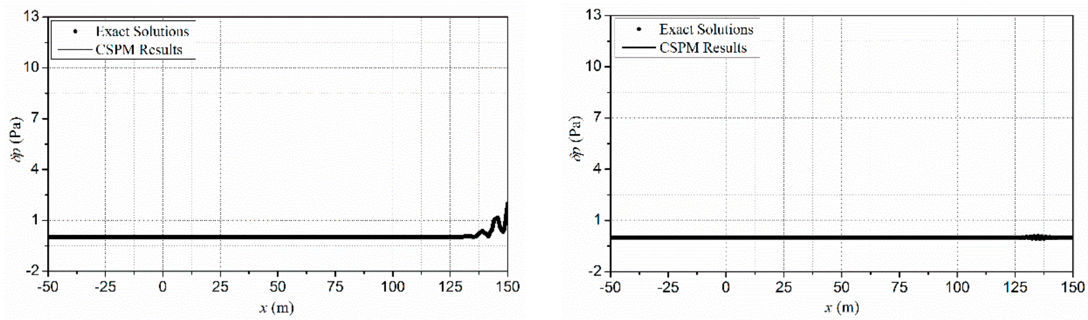
Figure show that the CSPM method predicts each peak of the sound waves almost the same as theoretical solutions. The numerical simulation can handle the process of sound propagation and reflection correctly.

### *5.3. Absorbing Boundary*

To model sound propagation in unbounded domain, the artificial boundary condition have to be used to eliminate the reflection from the edges of the computation domain. The sound propagation model with the implementation of absorbing boundary is built, and the absorbing boundary is located at $x$ = 150 m. Simulation results are shown in Figure 8.




(**a**) $t$ = 0.40 s (**b**) $t$ = 0.45 s




(**c**) $t$ = 0.50 s (**d**) $t$ = 0.60 s

**Figure 8.** Comparison between CSPM results and exact solutions with sound propagation through absorbing boundary.

Unlike soft and rigid boundary, under the effect of absorbing boundary, the incident wave is absorbed with no reflected waves. It can be seen from the figure, CSPM simulation results are in good agreement with theoretical solutions at each time. There is almost no reflected sound pressure in the last figure. The absorbing boundary works well in the computation.

## 6. Conclusions

The Lagrangian meshfree CSPM method is proposed to improve the accuracy in solving acoustic wave equations, and different acoustic boundary conditions are implemented with a novel boundary treatment technique based on the hybrid meshfree-FDTD method. The findings lead to the following conclusions:

1. The CSPM method is proposed to simulate sound propagation in the time domain by solving acoustic wave equations. Numerical results agree well with theoretical solutions in modeling sound propagation in pipes.

2. The CSPM method has good convergence while maintaining a constant ratio of the particle spacing to the smoothing length. According to the present work, the convergence rate is about 1.998 and CCFL is suggested to be under 0.28.

3. A hybrid meshfree-FDTD method is developed and used as an acoustic boundary treatment technique for the meshfree method, and different boundaries are built for virtual particles by using this technique.

4. Sound propagation and reflection computed with soft, rigid, and absorbing boundaries agree well with theoretical solutions in modeling sound propagation.

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**Conflicts of Interest:** The authors declare that there is no conflict of interests regarding the publication of this paper.

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