CONVEXITY PROPERTIES AND INEQUALITIES CONCERNING THE \((p,k)\)-GAMMA FUNCTION

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Abstract. In this paper, some convexity properties and some inequalities for the \((p,k)\)-analogue of the Gamma function, \(\Gamma_{p,k}(x)\) are given. In particular, a \((p,k)\)-analogue of the celebrated Bohr-Mollerup theorem is given. Furthermore, a \((p,k)\)-analogue of the Riemann zeta function, \(\zeta_{p,k}(x)\) is introduced and some associated inequalities are derived. The established results provide the \((p,k)\)-generalizations of some known results concerning the classical Gamma function.

1. Introduction

In a recent paper [10], the authors introduced a \((p,k)\)-analogue of the Gamma function defined for \(p \in \mathbb{N}, k > 0\) and \(x \in \mathbb{R}^+\) as

\[
\Gamma_{p,k}(x) = \int_0^p t^{x-1} \left(1 - \frac{t^k}{pk}\right)^p \, dt 
\]

\[
= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\ldots(x+pk)} \tag{1.1}
\]

satisfying the basic properties

\[
\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k} \Gamma_{p,k}(x),\tag{1.3}
\]

\[
\Gamma_{p,k}(ak) = \frac{p+1}{p} k^{a-1} \Gamma_p(a), \quad a \in \mathbb{R}^+ 
\]

\[
\Gamma_{p,k}(k) = 1.
\]

The \((p,k)\)-analogue of the Digamma function is defined for \(x > 0\) as

\[
\psi_{p,k}(x) = \frac{d}{dx} \ln \Gamma_{p,k}(x) = \frac{1}{k} \ln(pk) - \sum_{n=0}^p \frac{1}{nk+x} \tag{1.4}
\]

\[
= \frac{1}{k} \ln(pk) - \int_0^\infty \frac{1 - e^{-k(p+1)t}}{1-e^{-kt}} e^{-xt} \, dt.
\]

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Also, the \((p, k)\)-analogue of the Polygamma functions are defined as
\[
\psi_{p,k}^{(m)}(x) = \frac{d^m}{dx^m} \psi_{p,k}(x) = \sum_{n=0}^{P} \frac{(-1)^{m+1}m!}{(nk+x)^{m+1}}
\]
\[
= (-1)^{m+1} \int_{0}^{\infty} \left( \frac{1-e^{-k\alpha+1}}{1-e^{-kt}} \right) t^m e^{-xt} dt
\]
where \(m \in \mathbb{N}\), and \(\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)\).

The functions \(\Gamma_{p,k}(x)\) and \(\psi_{p,k}(x)\) satisfy the following commutative diagrams.

\[
\begin{align*}
\Gamma_{p,k}(x) & \xrightarrow{p \to \infty} \Gamma_k(x) \\
& \xleftarrow{k \to 1} \\
\Gamma_{p}(x) & \xrightarrow{p \to \infty} \Gamma(x)
\end{align*}
\]

\[
\begin{align*}
\psi_{p,k}(x) & \xrightarrow{p \to \infty} \psi_k(x) \\
& \xleftarrow{k \to 1} \\
\psi_{p}(x) & \xrightarrow{p \to \infty} \psi(x)
\end{align*}
\]

The \((p, k)\)-analogue of the classical Beta function is defined as
\[
B_{p,k}(x, y) = \frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}, \quad x > 0, y > 0.
\]

The purpose of this paper is to establish some convexity properties and some inequalities involving the function, \(\Gamma_{p,k}(x)\). In doing so, a \((p, k)\)-analogue of the Bohr-Mollerup theorem is proved. Also, a \((p, k)\)-analogue of the Riemann zeta function, \(\zeta_{p,k}(x)\) is introduced and some associated inequalities relating \(\Gamma_{p,k}(x)\) and \(\zeta_{p,k}(x)\) are derived. We present our findings in the following sections.

# 2. Convexity Properties Involving the \((p, k)\)-Gamma Function

Let us begin by recalling the following basic definitions and concepts.

**Definition 1.** A function \(f : (a, b) \to \mathbb{R}\) is said to be convex if
\[
f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)
\]
for all \(x, y \in (a, b)\), where \(\alpha, \beta > 0\) such that \(\alpha + \beta = 1\).

**Lemma 1.** Let \(f : (a, b) \to \mathbb{R}\) be a twice differentiable function. Then \(f\) is said to be convex if and only if \(f''(x) \geq 0\) for every \(x \in (a, b)\).

**Remark 1.** A function \(f\) is said to be concave if \(-f\) is convex, or equivalently, if the inequality (2.1) is reversed.

**Definition 2.** A function \(f : (a, b) \to \mathbb{R}^+\) is said to be logarithmically convex if the inequality
\[
\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y)
\]
or equivalently
\[
f(\alpha x + \beta y) \leq (f(x))^\alpha (f(y))^\beta
\]
holds for all \(x, y \in (a, b)\) and \(\alpha, \beta > 0\) such that \(\alpha + \beta = 1\).

**Theorem 1.** The function, \(\Gamma_{p,k}(x)\) is logarithmically convex.
Proof. Let $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then, by the integral representation (1.1) and by the Hölder’s inequality for integrals, we obtain

$$\Gamma_{p,k}(ax + \beta y) = \int_0^p \left(1 - \frac{tk}{p^k}\right)^p dt$$

$$= \int_0^p t^{\alpha(x-1)t^\beta(y-1)} \left(1 - \frac{tk}{p^k}\right)^p dt$$

$$\leq \left(\int_0^p t^{x-1} \left(1 - \frac{tk}{p^k}\right)^p dt\right)^\alpha \left(\int_0^p t^{y-1} \left(1 - \frac{tk}{p^k}\right)^p dt\right)^\beta$$

$$= (\Gamma_{p,k}(x))^{\alpha} (\Gamma_{p,k}(y))^{\beta}$$

as required. \qed

Remark 2. Since every logarithmically convex function is also convex \cite[pp. 66]{13}, it follows that the function $\Gamma_{p,k}(x)$ is convex.

Remark 3. Theorem 1 was proved in \cite{10} by using a different procedure. In the present work, we provide a much simpler alternative proof by using the Hölder’s inequality for integrals.

The next theorem is the $(p,k)$-analogue of the celebrated Bohr-Mollerup theorem.

**Theorem 2.** Let $f(x)$ be a positive function on $(0, \infty)$. Suppose that

1. $f(k) = 1$,
2. $f(x + k) = \frac{p^kx}{x^p + pk} f(x)$,
3. $\ln f(x)$ is convex.

Then, $f(x) = \Gamma_{p,k}(x)$.

Proof. Define $\phi$ by $e^{\phi(x)} = \frac{f(x)}{\Gamma_{p,k}(x)}$ for $x > 0$, $p \in \mathbb{N}$ and $k > 0$. Then by (a) we obtain

$$e^{\phi(k)} = \frac{f(k)}{\Gamma_{p,k}(k)} = 1$$

implying that $\phi(k) = 0$. Also by (b), we obtain

$$e^{\phi(x+k)} = \frac{f(x + k)}{\Gamma_{p,k}(x + k)} = \frac{f(x)}{\Gamma_{p,k}(x)} = e^{\phi(x)}$$

which implies $\phi(x + k) = \phi(x)$. Thus $\phi(x)$ is periodic with period $k$.

Next we want to show that $\phi(x) = \ln f(x) - \ln \Gamma_{p,k}(x)$ is a constant. That is

$$\phi'(x) = 0 \iff \lim_{h \to 0} \frac{\phi(x + h) - \phi(x)}{h} = 0.$$  

By (c) and Theorem 1, the functions $\ln f(x)$ and $\ln \Gamma_{p,k}(x)$ are convex. This implies $\ln f(x)$ and $\ln \Gamma_{p,k}(x)$ are continuous. Then for $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|\ln f(x + h) - \ln f(x)| < \frac{|h| \varepsilon}{2} \text{ whenever } |h| < \delta_1$$
and 
\[ |\ln \Gamma_{p,k}(x+h) - \ln \Gamma_{p,k}(x)| < \frac{|h|\varepsilon}{2} \] 
whenever \( |h| < \delta_2 \).

Let \( \delta = \min\{\delta_1, \delta_2\} \). Then for \( |h| < \delta \), we have
\[
\left| \frac{\phi(x+h) - \phi(x)}{h} \right| = \left| \frac{\ln f(x+h) - \ln \Gamma_{p,k}(x+h) - \ln f(x) + \ln \Gamma_{p,k}(x)}{h} \right|
\leq \left| \frac{\ln f(x+h) - \ln f(x)}{h} \right| + \left| \frac{\ln \Gamma_{p,k}(x+h) - \ln \Gamma_{p,k}(x)}{h} \right|
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
proving that \( \phi'(x) = 0 \). Since \( \phi(x) \) is a constant and \( \phi(k) = 0 \), then \( \phi(x) = 0 \) for every \( x \). Hence \( e^0 = \frac{f(x)}{\Gamma_{p,k}(x)} \). Therefore \( f(x) = \Gamma_{p,k}(x) \).

**Theorem 3.** The function, \( B_{p,k}(x,y) \) as defined by (1.6) is logarithmically convex on \((0, \infty) \times (0, \infty)\).

**Proof.** For \( x, y > 0 \), let \( B_{p,k}(x,y) \) be defined as in (1.6). Then
\[
\ln B_{p,k}(x,y) = \ln \Gamma_{p,k}(x) + \ln \Gamma_{p,k}(y) - \ln \Gamma_{p,k}(x+y).
\]
Without loss of generality, let \( y \) be fixed. Then,
\[
(\ln B_{p,k}(x,y))'' = \psi_{p,k}(x) - \psi_{p,k}(x+y) > 0
\]
since \( \psi_{p,k}(x) \) is decreasing for \( x > 0 \). This completes the proof. \( \square \)

**Remark 4.** Theorem 3 is a \((p,k)\)-analogue of Theorem 6 of [1].

**Corollary 1.** Let \( p \in \mathbb{N} \) and \( k > 0 \). Then the inequality
\[
\psi_{p,k}'(x)\psi_{p,k}'(y) \geq \left[ \psi_{p,k}'(x) + \psi_{p,k}'(y) \right] \psi_{p,k}'(x+y)
\]  
(2.2)
is valid for \( x, y > 0 \).

**Proof.** This follows from the logarithmic convexity of \( B_{p,k}(x,y) \). Let
\[
\phi(x,y) = \ln B_{p,k}(x,y) = \ln \Gamma_{p,k}(x) + \ln \Gamma_{p,k}(y) - \ln \Gamma_{p,k}(x+y).
\]
Since \( \phi(x,y) \) is convex on \((0, \infty) \times (0, \infty)\), then its discriminant, \( \Delta \) is positive semidefinite. That is,
\[
\frac{\partial^2 \phi}{\partial x^2} > 0, \quad \Delta = \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial y^2} - \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \geq 0
\]
implying that
\[
[\psi_{p,k}'(x) - \psi_{p,k}'(x+y)] [\psi_{p,k}'(y) - \psi_{p,k}'(x+y)] - [\psi_{p,k}'(x+y)]^2 \geq 0.
\]
Thus,
\[
\psi_{p,k}'(x)\psi_{p,k}'(y) - [\psi_{p,k}'(x) + \psi_{p,k}'(y)] \psi_{p,k}'(x+y) \geq 0
\]
which completes the proof. \( \square \)

**Theorem 4.** Let \( x, y > 0 \) and \( \alpha, \beta > 0 \) such that \( \alpha + \beta = 1 \). Then
\[
\psi_{p,k}(\alpha x + \beta y) \geq \alpha \psi_{p,k}(x) + \beta \psi_{p,k}(y).
\]  
(2.3)
Proof. It suffices to show that \( -\psi'_{p,k}(x) \) is convex on \((0, \infty)\). By (1.5) we obtain
\[
-\psi''_{p,k}(x) = \sum_{n=0}^{p} \frac{2}{(nk + x)^3} > 0.
\]
Then (2.3) follows from Definition 1. \(\square\)

**Theorem 5.** Let \( p \in \mathbb{N}, k > 0 \) and \( a > 0 \). Then the function \( Q(x) = a^x \Gamma_{p,k}(x) \) is convex on \((0, \infty)\).

Proof. Recall that \( \Gamma_{p,k}(x) \) is logarithmically convex. Thus,
\[
\Gamma_{p,k}(\alpha x + \beta y) \leq (\Gamma_{p,k}(x))^\alpha (\Gamma_{p,k}(y))^\beta
\]
for \( x, y > 0 \) and \( \alpha, \beta > 0 \) such that \( \alpha + \beta = 1 \). Then,
\[
Q(\alpha x + \beta y) = a^{\alpha x + \beta y} \Gamma_{p,k}(\alpha x + \beta y) \leq a^{\alpha x + \beta y} (\Gamma_{p,k}(x))^\alpha (\Gamma_{p,k}(y))^\beta. \tag{2.4}
\]
Also recall from the Young’s inequality that
\[
u^\alpha v^\beta \leq \alpha u + \beta v \tag{2.5}\]
for \( u, v > 0 \) and \( \alpha, \beta > 0 \) such that \( \alpha + \beta = 1 \). Let \( u = a^x \Gamma_{p,k}(x) \) and \( v = a^y \Gamma_{p,k}(y) \). Then (2.5) becomes
\[
a^{\alpha x + \beta y} (\Gamma_{p,k}(x))^\alpha (\Gamma_{p,k}(y))^\beta \leq \alpha a^x \Gamma_{p,k}(x) + \beta a^y \Gamma_{p,k}(y) = \alpha Q(x) + \beta Q(y). \tag{2.6}
\]
Combining (2.4) and (2.6) yields \( Q(\alpha x + \beta y) \leq \alpha Q(x) + \beta Q(y) \) which concludes the proof. \(\square\)

**Theorem 6.** Let \( p \in \mathbb{N} \) and \( k > 0 \). Then the functions \( A(x) = x \psi_{p,k}(x) \) is strictly convex on \((0, \infty)\).

Proof. Direct computations yield
\[
A''(x) = 2\psi'_{p,k}(x) - x\psi''_{p,k}(x)
\]
which by (1.5) implies
\[
A''(x) = 2 \sum_{n=0}^{p} \frac{1}{(nk + x)^2} - 2 \sum_{n=0}^{p} \frac{x}{(nk + x)^3} = 2 \sum_{n=0}^{p} \frac{nk}{(nk + x)^3} > 0.
\]
Thus, \( A(x) \) is convex. \(\square\)

**Remark 5.** Corollary 1 and Theorems 4, 5 and 6 provide generalizations for some results proved in [14] and [6].

**Definition 3** ([12],[15]). Let \( f : I \subseteq (0, \infty) \to (0, \infty) \) be a continuous function. Then \( f \) is said to be geometrically (or multiplicatively) convex on \( I \) if any of the following conditions is satisfied.
\[
f(\sqrt[n]{x_1 x_2}) \leq \sqrt[n]{f(x_1)f(x_2)}, \tag{2.7}
\]
or more generally
\[
f\left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) \leq \prod_{i=1}^{n} [f(x_i)]^{\lambda_i}, \quad n \geq 2 \tag{2.8}
\]
where \( x_1, x_2, \ldots, x_n \in I \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n > 0 \) with \( \sum_{i=1}^{n} \lambda_i = 1 \). If inequalities (2.7) and (2.8) are reversed, then \( f \) is said to be geometrically (or multiplicatively) concave on \( I \).
Lemma 2 ([12]). Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be a differentiable function. Then $f$ is a geometrically convex function if and only if the function $\frac{xf'(x)}{f(x)}$ is nondecreasing.

Lemma 3 ([12]). Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be a differentiable function. Then $f$ is a geometrically convex function if and only if the function $\frac{f(x)}{f(y)} \geq \left( \frac{x}{y} \right)^{y[1+\psi_{p,k}(y)]}$ holds for any $x, y \in I$.

Theorem 7. Let $f(x) = e^x \Gamma_{p,k}(x)$ for $p \in \mathbb{N}$ and $k \geq 1$. Then $f$ is geometrically convex and the inequality

$$
\frac{e^y}{e^x} \left( \frac{x}{y} \right)^{y[1+\psi_{p,k}(y)]} \leq \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} \leq \frac{e^y}{e^x} \left( \frac{x}{y} \right)^{x[1+\psi_{p,k}(x)]}
$$

is valid for $x > 0$ and $y > 0$.

Proof. We proceed as follows.

$$
\ln f(x) = x + \ln \Gamma_{p,k}(x) \quad \text{implying} \quad \frac{f'(x)}{f(x)} = 1 + \psi_{p,k}(x).
$$

Then,

$$
\left( \frac{x f'(x)}{f(x)} \right)' = 1 + \psi_{p,k}(x) + x \psi_{p,k}'(x)
$$

$$
= 1 + \frac{1}{k} \ln(pk) - \sum_{n=0}^{p} \frac{1}{nk + x} + \sum_{n=0}^{p} \frac{x}{(nk + x)^2}
$$

$$
= 1 + \frac{1}{k} \ln(pk) + \sum_{n=1}^{p} \left[ \frac{x}{(nk + x)^2} - \frac{1}{nk + x} \right]
$$

$$
= 1 + \frac{1}{k} \ln(pk) - \sum_{n=1}^{p} \frac{nk}{(nk + x)^2}
$$

$$
= \hat{h}(x).
$$

Then $h'(x) = 2 \sum_{n=0}^{p} \frac{nk}{(nk + x)^3} > 0$ implying that $h$ is increasing. Moreover,

$$
h(0) = 1 + \frac{1}{k} \ln(pk) - \sum_{n=1}^{p} \frac{1}{nk}
$$

$$
= 1 + \frac{1}{k} \ln k + \frac{1}{k} \left( \ln p - \sum_{n=1}^{p} \frac{1}{n} \right)
$$

$$
> 1 + \frac{1}{k} \ln k - \frac{1}{k} > 0
$$

since $\ln p - \sum_{n=1}^{p} \frac{1}{n} > -1$ (See eqn. (6) of [2]). Then for $x > 0$, we have $h(x) > h(0) > 0$. Thus $\frac{xf'(x)}{f(x)}$ is nondecreasing. Therefore, by Lemmas 2 and 3, $f$ is geometrically convex and as a result, $\frac{f(x)}{f(y)} \geq \left( \frac{x}{y} \right)^{y[1+\psi_{p,k}(y)]}$. Consequently, we obtain

$$
\frac{e^x \Gamma_{p,k}(x)}{e^y \Gamma_{p,k}(y)} \geq \left( \frac{x}{y} \right)^{y[1+\psi_{p,k}(y)]}
$$

(2.10)
and
\[ e^{y} \Gamma_{p,k}(y) \geq \left( \frac{y}{x} \right)^{x \left[ 1 + \psi_{p,k}(x) \right]} \Gamma_{p,k}(x). \] (2.11)

Now combining (2.10) and (2.11) yields the result (2.9) as required. □

**Remark 6.** In particular, by replacing \( x \) and \( y \) respectively by \( x + k \) and \( x + \frac{k}{2} \), inequality (2.9) takes the form:
\[ \frac{1}{\sqrt{e^{x}}} \left( \frac{x + k}{x + \frac{k}{2}} \right)^{(x + \frac{k}{2}) \left[ 1 + \psi_{p,k}(x + \frac{k}{2}) \right]} \leq \frac{\Gamma_{p,k}(x + k)}{\Gamma_{p,k}(x + \frac{k}{2})} \leq \frac{1}{\sqrt{e^{k}}} \left( \frac{x + k}{x + \frac{k}{2}} \right)^{(x + k) \left[ 1 + \psi_{p,k}(x + k) \right]}. \] (2.12)

**Remark 7.** Theorem 7 gives a \((p,k)\)-analogue of the previous results: [3, Theorem 1], [15, Theorem 1.2, Corollary 1.5] and [6, Theorem 3.5]. In particular, by letting \( k = 1 \), we recover the result of [6].

**Remark 8.** Results of type (2.9) and (2.12) can also be found in [9].

3. **Inequalities involving the \((p,k)\)-Riemann zeta function**

**Definition 4.** For \( p \in \mathbb{N}, \ k > 0 \) and \( x > 0 \), let \( \zeta_{p,k}(x) \) be the \((p,k)\)-analogue of the Riemann zeta function, \( \zeta(x) \). Then \( \zeta_{p,k}(x) \) is defined as
\[ \zeta_{p,k}(x) = \frac{1}{\Gamma_{p,k}(x)} \int_{0}^{p} \frac{t^{x-k}}{(1 + \frac{t}{pk})^p - 1} dt, \quad x > k. \] (3.1)

The functions \( \zeta_{p,k}(x) \) satisfies the commutative diagram:

\[ \begin{array}{ccc}
\zeta_{p,k}(x) & \xrightarrow{p \to \infty} & \zeta_k(x) \\
\downarrow k \to 1 & & \downarrow k \to 1 \\
\zeta_p(x) & \xrightarrow{p \to \infty} & \zeta(x)
\end{array} \]

where \( \zeta_p(x) \) and \( \zeta_k(x) \) respectively denote the \( p \) and \( k \) analogues of the Riemann zeta function. See [5] and [4] for instance.

**Lemma 4 ([7]).** Let \( f \) and \( g \) be two nonnegative functions of a real variable, and \( m, n \) be real numbers such that the integrals in (3.2) exist. Then
\[ \int_{a}^{b} g(t) (f(t))^{m} dt \cdot \int_{a}^{b} g(t) (f(t))^{n} dt \geq \left( \int_{a}^{b} g(t) (f(t))^{\frac{m+n}{2}} dt \right)^{2}. \] (3.2)

**Theorem 8.** Let \( p \in \mathbb{N}, \ k > 0 \) and \( x > 0 \). Then the inequality
\[ \frac{x + pk + k}{x + pk + 2k} \zeta_{p,k}(x) \geq \frac{x}{x + k} \zeta_{p,k}(x + k), \quad x > k. \] (3.3)
holds.
Proof. Let \( g(t) = \frac{t}{(1 + \frac{tk}{pk})^{-1}} \), \( f(t) = t \). Then (3.2) implies
\[
\int_0^p \frac{t x - k}{(1 + \frac{tk}{pk})^{p} - 1} dt \cdot \int_0^p \frac{t x + k}{(1 + \frac{tk}{pk})^{p} - 1} dt \geq \left( \int_0^p \frac{t x}{(1 + \frac{tk}{pk})^{p} - 1} dt \right)^2
\]
which by relation (3.1) gives
\[
\zeta_{p,k}(x) \Gamma_{p,k}(x) \cdot \zeta_{p,k}(x + 2k) \Gamma_{p,k}(x + 2k) \geq (\zeta_{p,k}(x + k) \Gamma_{p,k}(x + k))^2. \quad (3.4)
\]
Then by the functional equation (1.3), inequality (3.4) can be rearranged to obtain the desired result (3.3).

**Remark 9.**

(i) By letting \( p \to \infty \) in (3.3), we obtain the result of Theorem 3.1 of [4].

(ii) By letting \( k = 1 \) in (3.3), we obtain the result of Theorem 6 of [5].

(iii) By letting \( p \to \infty \) and \( k = 1 \) in (3.3), we obtain the result of Theorem 2.2 of [7].

**Theorem 9.** Let \( p \in \mathbb{N} \) and \( k > 0 \). Then for \( x > k \), \( y > k \), \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \) such that \( \frac{x}{\alpha} + \frac{y}{\beta} > k \), the inequality
\[
\frac{\Gamma_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)}{(\Gamma_{p,k}(x))^{\frac{1}{\alpha}} (\Gamma_{p,k}(y))^{\frac{1}{\beta}}} \leq \frac{(\zeta_{p,k}(x))^{\frac{1}{\alpha}} (\zeta_{p,k}(y))^{\frac{1}{\beta}}}{\zeta_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)} \quad (3.5)
\]
holds.

Proof. We employ the Hölder’s inequality:
\[
\int_a^b f(t)g(t) \, dt \leq \left( \int_a^b (f(t))^\alpha \, dt \right)^{\frac{1}{\alpha}} \left( \int_a^b (g(t))^\beta \, dt \right)^{\frac{1}{\beta}} \quad (3.6)
\]
where \( \alpha > 1 \), \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \). Let \( f(t) = \frac{t^{\frac{\alpha}{\alpha} + \frac{\beta}{\beta} - \frac{k}{pk}}}{(1 + \frac{tk}{pk})^{\alpha} - 1} \), \( g(t) = \frac{t^{\frac{\alpha}{\alpha} + \frac{\beta}{\beta} + \frac{k}{pk}}}{(1 + \frac{tk}{pk})^{\beta} - 1} \), \( a = 0 \) and \( b = p \). Then (3.6) implies
\[
\int_0^p \frac{t^{\frac{\alpha}{\alpha} + \frac{\beta}{\beta} - \frac{k}{pk}}}{(1 + \frac{tk}{pk})^{\alpha} - 1} dt \leq \left( \int_0^p \frac{t^{\frac{\alpha}{\alpha} + \frac{\beta}{\beta} + \frac{k}{pk}}}{(1 + \frac{tk}{pk})^{\beta} - 1} dt \right)^{\frac{1}{\alpha}} \left( \int_0^p \frac{t^{\frac{\alpha}{\alpha} + \frac{\beta}{\beta} - \frac{k}{pk}}}{(1 + \frac{tk}{pk})^{\alpha} - 1} dt \right)^{\frac{1}{\beta}}.
\]
By relation (3.1) we obtain
\[
\Gamma_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \zeta_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \leq (\Gamma_{p,k}(x) \zeta_{p,k}(x))^{\frac{1}{\alpha}} (\Gamma_{p,k}(y) \zeta_{p,k}(y))^{\frac{1}{\beta}}
\]
which when rearranged gives (3.5) as required.

**Remark 10.**

(i) By letting \( p \to \infty \) in (3.5), we obtain the result of Theorem 3.3 of [4].

(ii) By letting \( p \to \infty \) and \( k = 1 \) in (3.5), we obtain the result of Theorem 7 of [5].
(iii) In particular, let \( k = 1 \) in (3.5). Then by replacing \( x \) and \( y \) respectively by \( x - 1 \) and \( y + 1 \), we obtain
\[
\frac{\Gamma_p \left( \frac{x-1}{\alpha} + \frac{y+1}{p} \right)}{(\Gamma_p (x-1))^k (\Gamma_p (y+1))^k} \leq \frac{(\zeta_p (x-1))^{\frac{1}{k}} (\zeta_p (y+1))^{\frac{1}{k}}}{\zeta_p \left( \frac{x-1}{\alpha} + \frac{y+1}{p} \right)}
\]
which corresponds to Theorem 2.7 of [8].

**Lemma 5** ([11]). Let \( f : (0, \infty) \to (0, \infty) \) be a differentiable, logarithmically convex function. Then the function
\[
g(x) = \frac{(f(x))^\alpha}{f(\alpha x)}, \quad \alpha \geq 1
\]
is decreasing on its domain.

**Lemma 6.** Let \( p \in \mathbb{N}, k > 0 \) and \( \alpha \geq 1 \). Then the inequality
\[
\frac{[\Gamma_{p,k}(y+k)]^\alpha}{\Gamma_{p,k}(\alpha y + k)} \leq \frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x + k)} \leq 1 \tag{3.7}
\]
holds for \( 0 \leq x \leq y \).

**Proof.** Note that the function \( f(x) = \Gamma_{p,k}(x+k) \) is differentiable and logarithmically convex. Then by Lemma 5, \( G(x) = \frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x + k)} \) is decreasing and for \( 0 \leq x \leq y \), we have \( G(y) \leq G(x) \leq G(0) \) yielding the result. \( \square \)

**Theorem 10.** Let \( p \in \mathbb{N}, k > 0 \) and \( \alpha \geq 1 \). Then the inequality
\[
\frac{[\Gamma_{p,k}(y+k) \zeta_{p,k}(y+k)]^\alpha}{\Gamma_{p,k}(\alpha y + k) \zeta_{p,k}(\alpha y + k)} \leq \frac{[\zeta_{p,k}(x+k)]^\alpha}{\zeta_{p,k}(\alpha x + k)} \tag{3.8}
\]
is satisfied for \( 0 < x \leq y \).

**Proof.** Let \( H \) be defined \( x > 0 \) by
\[
H(x) = \Gamma_{p,k}(x+k) \zeta_{p,k}(x+k) = \int_0^p \frac{t^x}{(1 + \frac{t}{pk})^p - 1} \, dt. \tag{3.9}
\]
Then for \( x, y > 0 \) and \( a, b > 0 \) such that \( a + b = 1 \), we have
\[
H(ax + by) = \int_0^p \frac{t^{ax+by}}{(1 + \frac{t}{pk})^p - 1} \, dt
= \int_0^p \left( \int_0^\alpha \left( 1 + \frac{t^b}{pk} \right)^p - 1 \right)^a \left( \int_0^\beta \left( 1 + \frac{t^a}{pk} \right)^p - 1 \right)^b \, dt
\leq \left( \int_0^p \frac{t^x}{(1 + \frac{t}{pk})^p - 1} \, dt \right)^a \left( \int_0^p \frac{t^y}{(1 + \frac{t}{pk})^p - 1} \, dt \right)^b
= (H(x))^a (H(y))^b.
\]
Therefore, \( H(x) \) is logarithmically convex. Then by Lemma 5, the function
\[
T(x) = \frac{[\Gamma_{p,k}(x+k) \zeta_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x + k) \zeta_{p,k}(\alpha x + k)}
\]
is decreasing. Hence for $0 < x \leq y$, we have

$$\frac{[\Gamma_{p,k}(y + k)\zeta_{p,k}(y + k)]^\alpha}{\Gamma_{p,k}(\alpha y + k)\zeta_{p,k}(\alpha y + k)} \leq \frac{[\Gamma_{p,k}(x + k)\zeta_{p,k}(x + k)]^\alpha}{\Gamma_{p,k}(\alpha x + k)\zeta_{p,k}(\alpha x + k)}.$$ 

Then by the right hand side of (3.7), we obtain

$$\frac{[\Gamma_{p,k}(y + k)\zeta_{p,k}(y + k)]^\alpha}{\Gamma_{p,k}(\alpha y + k)\zeta_{p,k}(\alpha y + k)} \leq \frac{[\zeta_{p,k}(x + k)]^\alpha}{\zeta_{p,k}(\alpha x + k)},$$

concluding the proof. □

References


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