

CONVEXITY PROPERTIES AND INEQUALITIES CONCERNING THE (p, k) -GAMMA FUNCTION

KWARA NANTOMAH

ABSTRACT. In this paper, some convexity properties and some inequalities for the (p, k) -analogue of the Gamma function, $\Gamma_{p,k}(x)$ are given. In particular, a (p, k) -analogue of the celebrated Bohr-Mollerup theorem is given. Furthermore, a (p, k) -analogue of the Riemann zeta function, $\zeta_{p,k}(x)$ is introduced and some associated inequalities are derived. The established results provide the (p, k) -generalizations of some known results concerning the classical Gamma function.

1. INTRODUCTION

In a recent paper [10], the authors introduced a (p, k) -analogue of the Gamma function defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\Gamma_{p,k}(x) = \int_0^p t^{x-1} \left(1 - \frac{t^k}{pk}\right)^p dt \quad (1.1)$$

$$= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)} \quad (1.2)$$

satisfying the basic properties

$$\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k} \Gamma_{p,k}(x), \quad (1.3)$$

$$\Gamma_{p,k}(ak) = \frac{p+1}{p} k^{a-1} \Gamma_p(a), \quad a \in \mathbb{R}^+$$

$$\Gamma_{p,k}(k) = 1.$$

The (p, k) -analogue of the Digamma function is defined for $x > 0$ as

$$\begin{aligned} \psi_{p,k}(x) &= \frac{d}{dx} \ln \Gamma_{p,k}(x) = \frac{1}{k} \ln(pk) - \sum_{n=0}^p \frac{1}{nk+x} \\ &= \frac{1}{k} \ln(pk) - \int_0^\infty \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} e^{-xt} dt. \end{aligned} \quad (1.4)$$

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Also, the (p, k) -analogue of the Polygamma functions are defined as

$$\begin{aligned}\psi_{p,k}^{(m)}(x) &= \frac{d^m}{dx^m} \psi_{p,k}(x) = \sum_{n=0}^p \frac{(-1)^{m+1} m!}{(nk+x)^{m+1}} \\ &= (-1)^{m+1} \int_0^\infty \left(\frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} \right) t^m e^{-xt} dt\end{aligned}\quad (1.5)$$

where $m \in \mathbb{N}$, and $\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)$.

The functions $\Gamma_{p,k}(x)$ and $\psi_{p,k}(x)$ satisfy the following commutative diagrams.

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p \rightarrow \infty} & \Gamma_k(x) \\ k \rightarrow 1 \downarrow & & \downarrow k \rightarrow 1 \\ \Gamma_p(x) & \xrightarrow{p \rightarrow \infty} & \Gamma(x) \end{array} \qquad \begin{array}{ccc} \psi_{p,k}(x) & \xrightarrow{p \rightarrow \infty} & \psi_k(x) \\ k \rightarrow 1 \downarrow & & \downarrow k \rightarrow 1 \\ \psi_p(x) & \xrightarrow{p \rightarrow \infty} & \psi(x) \end{array}$$

The (p, k) -analogue of the classical Beta function is defined as

$$B_{p,k}(x, y) = \frac{\Gamma_{p,k}(x) \Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)}, \quad x > 0, y > 0. \quad (1.6)$$

The purpose of this paper is to establish some convexity properties and some inequalities involving the function, $\Gamma_{p,k}(x)$. In doing so, a (p, k) -analogue of the Bohr-Mollerup theorem is proved. Also, a (p, k) -analogue of the Riemann zeta function, $\zeta_{p,k}(x)$ is introduced and some associated inequalities relating $\Gamma_{p,k}(x)$ and $\zeta_{p,k}(x)$ are derived. We present our findings in the following sections.

2. CONVEXITY PROPERTIES INVOLVING THE (p, k) -GAMMA FUNCTION

Let us begin by recalling the following basic definitions and concepts.

Definition 1. A function $f : (a, b) \rightarrow \mathbb{R}$ is said to be convex if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \quad (2.1)$$

for all $x, y \in (a, b)$, where $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.

Lemma 1. Let $f : (a, b) \rightarrow \mathbb{R}$ be a twice differentiable function. Then f is said to be convex if and only if $f''(x) \geq 0$ for every $x \in (a, b)$.

Remark 1. A function f is said to be concave if $-f$ is convex, or equivalently, if the inequality (2.1) is reversed.

Definition 2. A function $f : (a, b) \rightarrow \mathbb{R}^+$ is said to be logarithmically convex if the inequality

$$\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y)$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^\alpha (f(y))^\beta$$

holds for all $x, y \in (a, b)$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.

Theorem 1. The function, $\Gamma_{p,k}(x)$ is logarithmically convex.

Proof. Let $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then, by the integral representation (1.1) and by the Hölder's inequality for integrals, we obtain

$$\begin{aligned}\Gamma_{p,k}(\alpha x + \beta y) &= \int_0^p t^{\alpha x + \beta y - 1} \left(1 - \frac{t^k}{pk}\right)^p dt \\ &= \int_0^p t^{\alpha(x-1)} t^{\beta(y-1)} \left(1 - \frac{t^k}{pk}\right)^{p(\alpha+\beta)} dt \\ &= \int_0^p t^{\alpha(x-1)} \left(1 - \frac{t^k}{pk}\right)^{\alpha p} t^{\beta(y-1)} \left(1 - \frac{t^k}{pk}\right)^{\beta p} dt \\ &\leq \left(\int_0^p t^{x-1} \left(1 - \frac{t^k}{pk}\right)^p dt\right)^\alpha \left(\int_0^p t^{y-1} \left(1 - \frac{t^k}{pk}\right)^p dt\right)^\beta \\ &= (\Gamma_{p,k}(x))^\alpha (\Gamma_{p,k}(y))^\beta\end{aligned}$$

as required. \square

Remark 2. Since every logarithmically convex function is also convex [13, p. 66], it follows that the function $\Gamma_{p,k}(x)$ is convex.

Remark 3. Theorem 1 was proved in [10] by using a different procedure. In the present work, we provide a much simpler alternative proof by using the Hölder's inequality for integrals.

The next theorem is the (p, k) -analogue of the celebrated Bohr-Mollerup theorem.

Theorem 2. Let $f(x)$ be a positive function on $(0, \infty)$. Suppose that

- (a) $f(k) = 1$,
- (b) $f(x+k) = \frac{pkx}{x+pk+k} f(x)$,
- (c) $\ln f(x)$ is convex.

Then, $f(x) = \Gamma_{p,k}(x)$.

Proof. Define ϕ by $e^{\phi(x)} = \frac{f(x)}{\Gamma_{p,k}(x)}$ for $x > 0$, $p \in \mathbb{N}$ and $k > 0$. Then by (a) we obtain

$$e^{\phi(k)} = \frac{f(k)}{\Gamma_{p,k}(k)} = 1$$

implying that $\phi(k) = 0$. Also by (b), we obtain

$$e^{\phi(x+k)} = \frac{f(x+k)}{\Gamma_{p,k}(x+k)} = \frac{f(x)}{\Gamma_{p,k}(x)} = e^{\phi(x)}$$

which implies $\phi(x+k) = \phi(x)$. Thus $\phi(x)$ is periodic with period k .

Next we want to show that $\phi(x) = \ln f(x) - \ln \Gamma_{p,k}(x)$ is a constant. That is

$$\phi'(x) = 0 \quad \Leftrightarrow \quad \lim_{h \rightarrow 0} \frac{\phi(x+h) - \phi(x)}{h} = 0.$$

By (c) and Theorem 1, the functions $\ln f(x)$ and $\ln \Gamma_{p,k}(x)$ are convex. This implies $\ln f(x)$ and $\ln \Gamma_{p,k}(x)$ are continuous. Then for $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|\ln f(x+h) - \ln f(x)| < \frac{|h|\varepsilon}{2} \quad \text{whenever} \quad |h| < \delta_1$$

and

$$|\ln \Gamma_{p,k}(x+h) - \ln \Gamma_{p,k}(x)| < \frac{|h|\varepsilon}{2} \quad \text{whenever} \quad |h| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $|h| < \delta$, we have

$$\begin{aligned} \left| \frac{\phi(x+h) - \phi(x)}{h} \right| &= \left| \frac{\ln f(x+h) - \ln \Gamma_{p,k}(x+h) - \ln f(x) + \ln \Gamma_{p,k}(x)}{h} \right| \\ &\leq \left| \frac{\ln f(x+h) - \ln f(x)}{h} \right| + \left| \frac{\ln \Gamma_{p,k}(x+h) - \ln \Gamma_{p,k}(x)}{h} \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

proving that $\phi'(x) = 0$. Since $\phi(x)$ is a constant and $\phi(k) = 0$, then $\phi(x) = 0$ for every x . Hence $e^0 = \frac{f(x)}{\Gamma_{p,k}(x)}$. Therefore $f(x) = \Gamma_{p,k}(x)$. \square

Theorem 3. *The function, $B_{p,k}(x, y)$ as defined by (1.6) is logarithmically convex on $(0, \infty) \times (0, \infty)$.*

Proof. For $x, y > 0$, let $B_{p,k}(x, y)$ be defined as in (1.6). Then

$$\ln B_{p,k}(x, y) = \ln \Gamma_{p,k}(x) + \ln \Gamma_{p,k}(y) - \ln \Gamma_{p,k}(x+y).$$

Without loss of generality, let y be fixed. Then,

$$(\ln B_{p,k}(x, y))'' = \psi'_{p,k}(x) - \psi'_{p,k}(x+y) > 0$$

since $\psi'_{p,k}(x)$ is decreasing for $x > 0$. This completes the proof. \square

Remark 4. *Theorem 3 is a (p, k) -analogue of Theorem 6 of [1].*

Corollary 1. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\psi'_{p,k}(x)\psi'_{p,k}(y) \geq [\psi'_{p,k}(x) + \psi'_{p,k}(y)] \psi'_{p,k}(x+y) \quad (2.2)$$

is valid for $x, y > 0$.

Proof. This follows from the logarithmic convexity of $B_{p,k}(x, y)$. Let

$$\phi(x, y) = \ln B_{p,k}(x, y) = \ln \Gamma_{p,k}(x) + \ln \Gamma_{p,k}(y) - \ln \Gamma_{p,k}(x+y).$$

Since $\phi(x, y)$ is convex on $(0, \infty) \times (0, \infty)$, then its discriminant, Δ is positive semidefinite. That is,

$$\frac{\partial^2 \phi}{\partial x^2} > 0, \quad \Delta = \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial y^2} - \left(\frac{\partial^2 \phi}{\partial x \partial y} \right) \left(\frac{\partial^2 \phi}{\partial y \partial x} \right) \geq 0$$

implying that

$$[\psi'_{p,k}(x) - \psi'_{p,k}(x+y)] [\psi'_{p,k}(y) - \psi'_{p,k}(x+y)] - [\psi'_{p,k}(x+y)]^2 \geq 0.$$

Thus,

$$\psi'_{p,k}(x)\psi'_{p,k}(y) - [\psi'_{p,k}(x) + \psi'_{p,k}(y)] \psi'_{p,k}(x+y) \geq 0$$

which completes the proof. \square

Theorem 4. *Let $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then*

$$\psi_{p,k}(\alpha x + \beta y) \geq \alpha \psi_{p,k}(x) + \beta \psi_{p,k}(y). \quad (2.3)$$

Proof. It suffices to show that $-\psi_{p,k}(x)$ is convex on $(0, \infty)$. By (1.5) we obtain

$$-\psi_{p,k}''(x) = \sum_{n=0}^p \frac{2}{(nk+x)^3} > 0.$$

Then (2.3) follows from Definition 1. \square

Theorem 5. Let $p \in \mathbb{N}$, $k > 0$ and $a > 0$. Then the function $Q(x) = a^x \Gamma_{p,k}(x)$ is convex on $(0, \infty)$.

Proof. Recall that $\Gamma_{p,k}(x)$ is logarithmically convex. Thus,

$$\Gamma_{p,k}(\alpha x + \beta y) \leq (\Gamma_{p,k}(x))^\alpha (\Gamma_{p,k}(y))^\beta$$

for $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then,

$$Q(\alpha x + \beta y) = a^{\alpha x + \beta y} \Gamma_{p,k}(\alpha x + \beta y) \leq a^{\alpha x + \beta y} (\Gamma_{p,k}(x))^\alpha (\Gamma_{p,k}(y))^\beta. \quad (2.4)$$

Also recall from the Young's inequality that

$$u^\alpha v^\beta \leq \alpha u + \beta v \quad (2.5)$$

for $u, v > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Let $u = a^x \Gamma_{p,k}(x)$ and $v = a^y \Gamma_{p,k}(y)$. Then (2.5) becomes

$$a^{\alpha x + \beta y} (\Gamma_{p,k}(x))^\alpha (\Gamma_{p,k}(y))^\beta \leq \alpha a^x \Gamma_{p,k}(x) + \beta a^y \Gamma_{p,k}(y) = \alpha Q(x) + \beta Q(y). \quad (2.6)$$

Combining (2.4) and (2.6) yields $Q(\alpha x + \beta y) \leq \alpha Q(x) + \beta Q(y)$ which concludes the proof. \square

Theorem 6. Let $p \in \mathbb{N}$ and $k > 0$. Then the functions $A(x) = x\psi_{p,k}(x)$ is strictly convex on $(0, \infty)$.

Proof. Direct computations yield

$$A''(x) = 2\psi_{p,k}'(x) - x\psi_{p,k}''(x)$$

which by (1.5) implies

$$A''(x) = 2 \sum_{n=0}^p \frac{1}{(nk+x)^2} - 2 \sum_{n=0}^p \frac{x}{(nk+x)^3} = 2 \sum_{n=0}^p \frac{nk}{(nk+x)^3} > 0.$$

Thus, $A(x)$ is convex. \square

Remark 5. Corollary 1 and Theorems 4, 5 and 6 provide generalizations for some results proved in [14] and [6].

Definition 3 ([12],[15]). Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be a continuous function. Then f is said to be geometrically (or multiplicatively) convex on I if any of the following conditions is satisfied.

$$f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1) f(x_2)}, \quad (2.7)$$

or more generally

$$f\left(\prod_{i=1}^n x_i^{\lambda_i}\right) \leq \prod_{i=1}^n [f(x_i)]^{\lambda_i}, \quad n \geq 2 \quad (2.8)$$

where $x_1, x_2, \dots, x_n \in I$ and $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$. If inequalities (2.7) and (2.8) are reversed, then f is said to be geometrically (or multiplicatively) concave on I .

Lemma 2 ([12]). Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be a differentiable function. Then f is a geometrically convex function if and only if the function $\frac{xf'(x)}{f(x)}$ is nondecreasing.

Lemma 3 ([12]). Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be a differentiable function. Then f is a geometrically convex function if and only if the function $\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}}$ holds for any $x, y \in I$.

Theorem 7. Let $f(x) = e^x \Gamma_{p,k}(x)$ for $p \in \mathbb{N}$ and $k \geq 1$. Then f is geometrically convex and the inequality

$$\frac{e^y}{e^x} \left(\frac{x}{y}\right)^{y[1+\psi_{p,k}(y)]} \leq \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} \leq \frac{e^y}{e^x} \left(\frac{x}{y}\right)^{x[1+\psi_{p,k}(x)]} \quad (2.9)$$

is valid for $x > 0$ and $y > 0$.

Proof. We proceed as follows.

$$\ln f(x) = x + \ln \Gamma_{p,k}(x) \quad \text{implying} \quad \frac{f'(x)}{f(x)} = 1 + \psi_{p,k}(x).$$

Then,

$$\begin{aligned} \left(\frac{xf'(x)}{f(x)}\right)' &= 1 + \psi_{p,k}(x) + x\psi'_{p,k}(x) \\ &= 1 + \frac{1}{k} \ln(pk) - \sum_{n=0}^p \frac{1}{nk+x} + \sum_{n=0}^p \frac{x}{(nk+x)^2} \\ &= 1 + \frac{1}{k} \ln(pk) + \sum_{n=1}^p \left[\frac{x}{(nk+x)^2} - \frac{1}{nk+x} \right] \\ &= 1 + \frac{1}{k} \ln(pk) - \sum_{n=1}^p \frac{nk}{(nk+x)^2} \\ &\triangleq h(x). \end{aligned}$$

Then $h'(x) = 2 \sum_{n=0}^p \frac{nk}{(nk+x)^3} > 0$ implying that h is increasing. Moreover,

$$\begin{aligned} h(0) &= 1 + \frac{1}{k} \ln(pk) - \sum_{n=1}^p \frac{1}{nk} \\ &= 1 + \frac{1}{k} \ln k + \frac{1}{k} \left(\ln p - \sum_{n=1}^p \frac{1}{n} \right) \\ &> 1 + \frac{1}{k} \ln k - \frac{1}{k} > 0 \end{aligned}$$

since $\ln p - \sum_{n=1}^p \frac{1}{n} > -1$ (See eqn. (6) of [2]). Then for $x > 0$, we have $h(x) > h(0) > 0$. Thus $\frac{xf'(x)}{f(x)}$ is nondecreasing. Therefore, by Lemmas 2 and 3, f is geometrically convex and as a result, $\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}}$. Consequently, we obtain

$$\frac{e^x \Gamma_{p,k}(x)}{e^y \Gamma_{p,k}(y)} \geq \left(\frac{x}{y}\right)^{y[1+\psi_{p,k}(y)]} \quad (2.10)$$

and

$$\frac{e^y \Gamma_{p,k}(y)}{e^x \Gamma_{p,k}(x)} \geq \left(\frac{y}{x}\right)^{x[1+\psi_{p,k}(x)]}. \tag{2.11}$$

Now combining (2.10) and (2.11) yields the result (2.9) as required. \square

Remark 6. In particular, by replacing x and y respectively by $x+k$ and $x+\frac{k}{2}$, inequality (2.9) takes the form:

$$\frac{1}{\sqrt{e^k}} \left(\frac{x+k}{x+\frac{k}{2}}\right)^{(x+\frac{k}{2})[1+\psi_{p,k}(x+\frac{k}{2})]} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+\frac{k}{2})} \leq \frac{1}{\sqrt{e^k}} \left(\frac{x+k}{x+\frac{k}{2}}\right)^{(x+k)[1+\psi_{p,k}(x+k)]}. \tag{2.12}$$

Remark 7. Theorem 7 gives a (p, k) -analogue of the previous results: [3, Theorem 1], [15, Theorem 1.2, Corollary 1.5] and [6, Theorem 3.5]. In particular, by letting $k=1$, we recover the result of [6].

Remark 8. Results of type (2.9) and (2.12) can also be found in [9].

3. INEQUALITIES INVOLVING THE (p, k) -RIEMANN ZETA FUNCTION

Definition 4. For $p \in \mathbb{N}$, $k > 0$ and $x > 0$, let $\zeta_{p,k}(x)$ be the (p, k) -analogue of the Riemann zeta function, $\zeta(x)$. Then $\zeta_{p,k}(x)$ is defined as

$$\zeta_{p,k}(x) = \frac{1}{\Gamma_{p,k}(x)} \int_0^p \frac{t^{x-k}}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt, \quad x > k. \tag{3.1}$$

The functions $\zeta_{p,k}(x)$ satisfies the commutative diagram:

$$\begin{array}{ccc} \zeta_{p,k}(x) & \xrightarrow{p \rightarrow \infty} & \zeta_k(x) \\ \downarrow k \rightarrow 1 & & \downarrow k \rightarrow 1 \\ \zeta_p(x) & \xrightarrow{p \rightarrow \infty} & \zeta(x) \end{array}$$

where $\zeta_p(x)$ and $\zeta_k(x)$ respectively denote the p and k analogues of the Riemann zeta function. See [5] and [4] for instance.

Lemma 4 ([7]). Let f and g be two nonnegative functions of a real variable, and m, n be real numbers such that the integrals in (3.2) exist. Then

$$\int_a^b g(t) (f(t))^m dt \cdot \int_a^b g(t) (f(t))^n dt \geq \left(\int_a^b g(t) (f(t))^{\frac{m+n}{2}} dt \right)^2. \tag{3.2}$$

Theorem 8. Let $p \in \mathbb{N}$, $k > 0$ and $x > 0$. Then the inequality

$$\frac{x + pk + k}{x + pk + 2k} \cdot \frac{\zeta_{p,k}(x)}{\zeta_{p,k}(x+k)} \geq \frac{x}{x+k} \cdot \frac{\zeta_{p,k}(x+k)}{\zeta_{p,k}(x+2k)}, \quad x > k. \tag{3.3}$$

holds.

Proof. Let $g(t) = \frac{1}{\left(1 + \frac{t^k}{pk}\right)^p - 1}$, $f(t) = t$, $m = x - k$, $n = x + k$, $a = 0$ and $b = p$.

Then (3.2) implies

$$\int_0^p \frac{t^{x-k}}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \cdot \int_0^p \frac{t^{x+k}}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \geq \left(\int_0^p \frac{t^x}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \right)^2$$

which by relation (3.1) gives

$$\zeta_{p,k}(x)\Gamma_{p,k}(x) \cdot \zeta_{p,k}(x+2k)\Gamma_{p,k}(x+2k) \geq (\zeta_{p,k}(x+k)\Gamma_{p,k}(x+k))^2. \quad (3.4)$$

Then by the functional equation (1.3), inequality (3.4) can be rearranged to obtain the desired result (3.3). \square

Remark 9. (i) By letting $p \rightarrow \infty$ in (3.3), we obtain the result of Theorem 3.1 of [4].

(ii) By setting $k = 1$ in (3.3), we obtain the result of Theorem 6 of [5].

(iii) By letting $p \rightarrow \infty$ and $k = 1$ in (3.3), we obtain the result of Theorem 2.2 of [7].

Theorem 9. Let $p \in \mathbb{N}$ and $k > 0$. Then for $x > k$, $y > k$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ such that $\frac{x}{\alpha} + \frac{y}{\beta} > k$, the inequality

$$\frac{\Gamma_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)}{(\Gamma_{p,k}(x))^{\frac{1}{\alpha}} (\Gamma_{p,k}(y))^{\frac{1}{\beta}}} \leq \frac{(\zeta_{p,k}(x))^{\frac{1}{\alpha}} (\zeta_{p,k}(y))^{\frac{1}{\beta}}}{\zeta_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)} \quad (3.5)$$

holds.

Proof. We employ the Hölder's inequality:

$$\int_a^b f(t)g(t) dt \leq \left(\int_a^b (f(t))^\alpha dt \right)^{\frac{1}{\alpha}} \left(\int_a^b (g(t))^\beta dt \right)^{\frac{1}{\beta}} \quad (3.6)$$

where $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let $f(t) = \frac{t^{\frac{x-k}{\alpha}}}{\left(\left(1 + \frac{t^k}{pk}\right)^p - 1\right)^{\frac{1}{\alpha}}}$, $g(t) = \frac{t^{\frac{y-k}{\beta}}}{\left(\left(1 + \frac{t^k}{pk}\right)^p - 1\right)^{\frac{1}{\beta}}}$, $a = 0$

and $b = p$. Then (3.6) implies

$$\int_0^p \frac{t^{\frac{x}{\alpha} + \frac{y}{\beta} - k}}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \leq \left(\int_0^p \frac{t^{x-k}}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \right)^{\frac{1}{\alpha}} \left(\int_0^p \frac{t^{y-k}}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \right)^{\frac{1}{\beta}}.$$

By relation (3.1) we obtain

$$\Gamma_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \zeta_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \leq (\Gamma_{p,k}(x)\zeta_{p,k}(x))^{\frac{1}{\alpha}} (\Gamma_{p,k}(y)\zeta_{p,k}(y))^{\frac{1}{\beta}}$$

which when rearranged gives (3.5) as required. \square

Remark 10. (i) By letting $p \rightarrow \infty$ in (3.5), we obtain the result of Theorem 3.3 of [4].

(ii) By letting $p \rightarrow \infty$ and $k = 1$ in (3.5), we obtain the result of Theorem 7 of [5].

(iii) In particular, let $k = 1$ in (3.5). Then by replacing x and y respectively by $x - 1$ and $y + 1$, we obtain

$$\frac{\Gamma_p \left(\frac{x-1}{\alpha} + \frac{y+1}{\beta} \right)}{(\Gamma_p(x-1))^{\frac{1}{\alpha}} (\Gamma_p(y+1))^{\frac{1}{\beta}}} \leq \frac{(\zeta_p(x-1))^{\frac{1}{\alpha}} (\zeta_p(y+1))^{\frac{1}{\beta}}}{\zeta_p \left(\frac{x-1}{\alpha} + \frac{y+1}{\beta} \right)}$$

which corresponds to Theorem 2.7 of [8].

Lemma 5 ([11]). Let $f : (0, \infty) \rightarrow (0, \infty)$ be a differentiable, logarithmically convex function. Then the function

$$g(x) = \frac{(f(x))^\alpha}{f(\alpha x)}, \quad \alpha \geq 1$$

is decreasing on its domain.

Lemma 6. Let $p \in \mathbb{N}$, $k > 0$ and $\alpha \geq 1$. Then the inequality

$$\frac{[\Gamma_{p,k}(y+k)]^\alpha}{\Gamma_{p,k}(\alpha y+k)} \leq \frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)} \leq 1 \quad (3.7)$$

holds for $0 \leq x \leq y$.

Proof. Note that the function $f(x) = \Gamma_{p,k}(x+k)$ is differentiable and logarithmically convex. Then by Lemma 5, $G(x) = \frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)}$ is decreasing and for $0 \leq x \leq y$, we have $G(y) \leq G(x) \leq G(0)$ yielding the result. \square

Theorem 10. Let $p \in \mathbb{N}$, $k > 0$ and $\alpha \geq 1$. Then the inequality

$$\frac{[\Gamma_{p,k}(y+k)\zeta_{p,k}(y+k)]^\alpha}{\Gamma_{p,k}(\alpha y+k)\zeta_{p,k}(\alpha y+k)} \leq \frac{[\zeta_{p,k}(x+k)]^\alpha}{\zeta_{p,k}(\alpha x+k)} \quad (3.8)$$

is satisfied for $0 < x \leq y$.

Proof. Let H be defined $x > 0$ by

$$H(x) = \Gamma_{p,k}(x+k)\zeta_{p,k}(x+k) = \int_0^p \frac{t^x}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt. \quad (3.9)$$

Then for $x, y > 0$ and $a, b > 0$ such that $a + b = 1$, we have

$$\begin{aligned} H(ax + by) &= \int_0^p \frac{t^{ax+by}}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \\ &= \int_0^p \frac{t^{ax}}{\left(\left(1 + \frac{t^k}{pk}\right)^p - 1\right)^a} \cdot \frac{t^{by}}{\left(\left(1 + \frac{t^k}{pk}\right)^p - 1\right)^b} dt \\ &\leq \left(\int_0^p \frac{t^x}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \right)^a \left(\int_0^p \frac{t^y}{\left(1 + \frac{t^k}{pk}\right)^p - 1} dt \right)^b \\ &= (H(x))^a (H(y))^b. \end{aligned}$$

Therefore, $H(x)$ is logarithmically convex. Then by Lemma 5, the function

$$T(x) = \frac{[\Gamma_{p,k}(x+k)\zeta_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)\zeta_{p,k}(\alpha x+k)}$$

is decreasing. Hence for $0 < x \leq y$, we have

$$\frac{[\Gamma_{p,k}(y+k)\zeta_{p,k}(y+k)]^\alpha}{\Gamma_{p,k}(\alpha y+k)\zeta_{p,k}(\alpha y+k)} \leq \frac{[\Gamma_{p,k}(x+k)\zeta_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)\zeta_{p,k}(\alpha x+k)}.$$

Then by the right hand side of (3.7), we obtain

$$\frac{[\Gamma_{p,k}(y+k)\zeta_{p,k}(y+k)]^\alpha}{\Gamma_{p,k}(\alpha y+k)\zeta_{p,k}(\alpha y+k)} \leq \frac{[\zeta_{p,k}(x+k)]^\alpha}{\zeta_{p,k}(\alpha x+k)}$$

concluding the proof. \square

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KWARA NANTOMAH: DEPARTMENT OF MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY FOR DEVELOPMENT STUDIES, NAVRONGO CAMPUS, P. O. BOX 24, NAVRONGO, UE/R, GHANA.

E-mail address: mykwarasoft@yahoo.com, knantomah@uds.edu.gh



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