

ON THE WAVE SOLUTIONS OF CONFORMABLE FRACTIONAL EVOLUTION EQUATIONS

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ABSTRACT. The exact solutions in the wave form are derived for the time fractional KdV and the time fractional Burgers' equations in conformable fractional derivative sense. The fractional variable change using the fundamental properties of the conformable derivative reduces both equations to some nonlinear ODEs. The predicted solution is assumed to be a finite series form of a function satisfying a particular first-order ODE whose solution contains an exponential function in the denominator. The solutions are represented in the explicit forms and illustrated by some choices of the parameters for various fractional orders of the equations.

1. INTRODUCTION

Recent developments in symbolic programming and computer algebra enable to solve more complicated problems in many fields covering engineering, physics, mathematics and the related fields. Moreover, many new techniques have been derived to solve different problems in various forms. The reflections of all stimulate the applied mathematicians to suggest new techniques for solutions of PDEs, particularly the nonlinear ones.

In the last several decades, we all witness that the number of the studies dealing with many problems described by the nonlinear PDEs increases rapidly. Many new methods from the tanh method to different types of expansion methods and the others such as the methods based on ansatzes or first integrals are implemented to the nonlinear PDEs to derive solutions to them. The expansion methods class is a special family of these techniques. There are numerous practical techniques covering the Jacobi elliptic, the exp-function, the hyperbolic tangent expansions and their variations, modifications or generalizations in the literature. The F -expansion method in the generalized form, for example, is used to develop some Jacobi elliptic-type exact solutions, soliton-like and trigonometric type solutions for the Konopelchenko-Dubrovsy equation in two space dimension[1]. The method of (G'/G) -expansion is also a widely used method to derive the solutions to the nonlinear PDEs. In this method, G is chosen as a solution of a second-order ODE. The coupled KdV-mKdV, the KdV-Burgers' and the reaction-diffusion equation have exact solutions represented in the finite series[2].

The variations of the Kudryashov method can also be classified in the expansion methods. The method, briefly, predicts a solution in a finite series form of a function solving a particular first-order ODE. The determination of the coefficients

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used in the series are determined by forcing the solution to satisfy the equation. Kudryashov, himself, describes the method as one of old methods to solve nonlinear differential equations exactly[3]. That study focuses on exact solutions of the Fisher and a higher order nonlinear PDE and proposes exact solutions in a finite series. Kabir's study suggests some solitary wave solutions in traveling form for some higher order nonlinear PDEs [4]. Some exact solutions in series of rational functions with exponential components form are derived by Tandogan et al. to the power non-linear Rosenau-Kawahara equation[5].

The present study aims to determine some explicit wave type exact solutions of the conformable time fractional Burgers' equation (ctfBE) of the form

$$D_t^\alpha(u) + \varepsilon uu_x - \nu u_{xx} = 0, t > 0 \quad (1.1)$$

and the conformable time fractional KdV equation (ctfKdVE)

$$D_t^\alpha(u) + \varepsilon uu_x + \beta u_{xxx} = 0, t > 0 \quad (1.2)$$

where $D_t^\alpha(u)$ stands for the α .th order derivative of the function u with respect to the variable t by implementing the Kudryashov method in modified form. Before starting to describe the method, some significant properties of the conformable derivative are explained in the next section. The following sections involves the implement of the method to the ctfBE and to the ctfKdVE.

2. CONFORMABLE FRACTIONAL DERIVATIVE

Consider a function $f = f(t)$ defined in the positive semi half space $t > 0$. The conformable derivative of order α of f is defined as

$$D_t^\alpha(f(t)) = \lim_{h \rightarrow 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}, t > 0, \alpha \in (0, 1] \quad (2.1)$$

for $f : [0, \infty) \rightarrow \mathbb{R}$ [6]. The conformable derivative defined above satisfies the properties given in the Theorem 1.

Theorem 1. Assume that $\alpha \in (0, 1]$ is the derivative order, and suppose that $v = v(t)$ and $w = w(t)$ are α -differentiable for all positive t . Then,

- $D_t^\alpha(av + bw) = aD_t^\alpha(v) + bD_t^\alpha(w)$
- $D_t^\alpha(t^p) = pt^{p-\alpha}, \forall p \in \mathbb{R}$
- $D_t^\alpha(v(t)) = 0$, for all constant function $v(t) = \lambda$
- $D_t^\alpha(vw) = vD_t^\alpha(w) + wD_t^\alpha(v)$
- $D_t^\alpha\left(\frac{v}{w}\right) = \frac{wD_t^\alpha(v) - vD_t^\alpha(w)}{w^2}$
- $D_t^\alpha(v)(t) = t^{1-\alpha} \frac{dv}{dt}$

for all real a, b [7, 8].

The conformable derivative defined in (2.1) has significant properties like the chain rule and Gronwall's inequality[9]. A useful one is the relation between the conformable derivative and the classical integer ordered derivative in the definition of the composite function.

Theorem 2. Let v be a differentiable and α -conformable differentiable function and w also be defined in the range of v and be differentiable. Then,

$$D_t^\alpha(v \circ w) = t^{1-\alpha} D_t^\alpha(w)(t) D_t^\alpha v(w(t)) \quad (2.2)$$

where $'$ denotes the derivative with respect to t .

3. DESCRIPTION OF THE METHOD

Consider a nonlinear PDE of the form

$$P(u, u_t^\alpha, u_x, u_t^{2\alpha}, u_{xx}, \dots) = 0 \quad (3.1)$$

where $u = u(x, t)$ and the fractional derivative order $\alpha \in (0, 1]$. The classical transformation

$$u(x, t) = u(\xi), \xi = x - \frac{c}{\alpha} t^\alpha \quad (3.2)$$

gives an ODE of the form

$$R(u, u', u'', \dots) = 0 \quad (3.3)$$

where the prime ($'$) stands for the derivative of u with respect to ξ [10].

Consider the equation (3.3) has a solution of the form

$$u(\xi) = a_0 + a_1 Q(\xi) + a_2 Q^2(\xi) + \dots a_n Q^n(\xi) \quad (3.4)$$

for a finite n where $a_n \neq 0$ and all $a_i, 0 \leq i \leq n$ are constants. This polynomial of $Q(\xi)$ is assumed to satisfy the first-order differential equation

$$Q'(\xi) = Q(\xi)(Q(\xi) - 1) \ln A \quad (3.5)$$

Thus, one can determine it as

$$Q(\xi) = \frac{1}{1 + dA\xi}$$

where d and A are nonzero constants with $A > 0$ and $A \neq 1$. The balance between the nonlinear term and the term having the highest order derivative in (3.3) gives the degree n of the power series (3.4). Since (3.4) is a solution, it must satisfy (3.3). Substituting it into (3.3) and rearranging the resultant equation for the powers of $Q(\xi)$ leads a polynomial for $Q(\xi)$. The obtained polynomial equality is solved by equating the coefficients to zero. Thus, the coefficients $a_0, a_1, a_2, \dots, a_n$ are determined algebraically in terms of other parameters originated from the regarding equation, the transformation and the other operations if exist for nonzero a_n .

4. THE SOLUTION OF THE CTFBE

The transformation (3.2) decreases the dimension of the cftBE(1.1) to one as

$$-cu' + \varepsilon uu' - \nu u'' = 0 \quad (4.1)$$

where ($'$) stands for $\frac{d}{d\xi}$. Integrating (4.1) once gives

$$-cu + \varepsilon \frac{1}{2} u^2 - \nu u' = K \quad (4.2)$$

where K is integral constant. The balance of u^2 and u' gives $n = 1$. Thus, the solution should be expressed as

$$u(\xi) = a_0 + a_1 Q(\xi) \quad (4.3)$$

for a nonzero a_1 . Substituting the solution (4.3) and its derivative into (4.2) gives

$$\left(\frac{1}{2} \varepsilon a_1^2 - \nu a_1 \ln(A) \right) Q^2(\xi) + (\varepsilon a_0 a_1 - c a_1 + \nu a_1 \ln(A)) Q(\xi) - c a_0 + \frac{1}{2} \varepsilon a_0^2 - K = 0 \quad (4.4)$$

Equating the coefficients of each power of $Q(\xi)$ and the constant term to zero yields the algebraic system of equations

$$\begin{aligned} -K - ca_0 + \frac{1}{2} \varepsilon a_0^2 &= 0 \\ \varepsilon a_0 a_1 - ca_1 + \nu a_1 \ln(A) &= 0 \\ \frac{1}{2} \varepsilon a_1^2 - \nu a_1 \ln(A) &= 0 \end{aligned} \quad (4.5)$$

This system has various solutions for $a_1 \neq 0$:

Solution 1: When the solution of the system (4.5) is chosen as

$$\begin{aligned} a_0 &= -\frac{\nu \ln(A) + \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K}}{\varepsilon} \\ a_1 &= 2 \frac{\nu \ln(A)}{\varepsilon} \\ c &= -\sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K} \end{aligned} \quad (4.6)$$

the solution of (4.2) is constructed as

$$u(\xi) = -\frac{\nu \ln(A) + \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K}}{\varepsilon} + 2 \frac{\nu \ln(A)}{\varepsilon} \frac{1}{1 + dA\xi}, \quad (4.7)$$

where $\sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K} \geq 0$ and $\varepsilon \neq 0$. Thus, the solution of the ctfBE (1.1) is expressed as

$$u_1(x, t) = -\frac{\nu \ln(A) + \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K}}{\varepsilon} + 2 \frac{\nu \ln(A)}{\varepsilon} \frac{1}{1 + dA^{x + \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K} \frac{t^\alpha}{\alpha}}}, \quad (4.8)$$

Solution 2: When the solution of the system (4.5) is chosen as

$$\begin{aligned} a_0 &= -\frac{\nu \ln(A) - \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K}}{\varepsilon} \\ a_1 &= 2 \frac{\nu \ln(A)}{\varepsilon} \\ c &= \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K} \end{aligned} \quad (4.9)$$

the solution of the ODE (4.2) can be written as

$$u(\xi) = -\frac{\nu \ln(A) - \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K}}{\varepsilon} + 2 \frac{\nu \ln(A)}{\varepsilon} \frac{1}{1 + dA\xi}, \quad (4.10)$$

with the conditions $\sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K} \geq 0$ and $\varepsilon \neq 0$. Thus, the exact solution of the ctfBE (1.1) is written in an explicit form as

$$u_2(x, t) = -\frac{\nu \ln(A) - \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K}}{\varepsilon} + 2 \frac{\nu \ln(A)}{\varepsilon} \frac{1}{1 + dA^{x - \sqrt{\nu^2 (\ln(A))^2 - 2\varepsilon K} \frac{t^\alpha}{\alpha}}}, \quad (4.11)$$

5. THE SOLUTION OF THE CTFKdVE

The transformation (3.2) converts the ctfKdVE to

$$-cu' + \varepsilon uu' + \beta u''' = 0 \quad (5.1)$$

Integrating once changes the ODE above to

$$-cu + \varepsilon \frac{1}{2}u^2 + \beta u'' = K \quad (5.2)$$

where K is the integration constant. The balance of u^2 and u'' gives $n = 2$. Substituting the predicted solution $u(\xi) = a_0 + a_1 Q(\xi) + a_2 Q^2(\xi)$, $a_2 \neq 0$ into (5.2) yields

$$\begin{aligned} & \left(\frac{1}{2} \varepsilon a_2^2 + 6 \beta a_2 (\ln(A))^2 \right) Q^4(\xi) + \left(2 \beta a_1 (\ln(A))^2 - 10 \beta a_2 (\ln(A))^2 + \varepsilon a_1 a_2 \right) Q^3(\xi) \\ & + \left(\frac{1}{2} \varepsilon a_1^2 + \varepsilon a_0 a_2 - 3 \beta a_1 (\ln(A))^2 + 4 \beta a_2 (\ln(A))^2 - c a_2 \right) Q^2(\xi) \\ & + \left(-c a_1 + \varepsilon a_0 a_1 + \beta a_1 (\ln(A))^2 \right) Q(\xi) + \frac{1}{2} \varepsilon a_0^2 - K - c a_0 = 0 \end{aligned} \quad (5.3)$$

in the arranged form. Forcing the coefficients of the powers of $Q(\xi)$ and the constant term to be zero gives an algebraic system

$$\begin{aligned} \frac{1}{2} \varepsilon a_0^2 - K - c a_0 &= 0 \\ -c a_1 + \varepsilon a_0 a_1 + \beta a_1 (\ln(A))^2 &= 0 \\ \frac{1}{2} \varepsilon a_1^2 + \varepsilon a_0 a_2 - 3 \beta a_1 (\ln(A))^2 + 4 \beta a_2 (\ln(A))^2 - c a_2 &= 0 \\ 2 \beta a_1 (\ln(A))^2 - 10 \beta a_2 (\ln(A))^2 + \varepsilon a_1 a_2 &= 0 \\ \frac{1}{2} \varepsilon a_2^2 + 6 \beta a_2 (\ln(A))^2 &= 0 \end{aligned} \quad (5.4)$$

Solution 1: The solution

$$\begin{aligned} a_0 &= \frac{-\beta (\ln(A))^2 + \sqrt{\beta^2 (\ln(A))^4 - 2 \varepsilon K}}{\varepsilon} \\ a_1 &= 12 \frac{\beta (\ln(A))^2}{\varepsilon} \\ a_2 &= -12 \frac{\beta (\ln(A))^2}{\varepsilon} \\ c &= \sqrt{\beta^2 (\ln(A))^4 - 2 \varepsilon K} \end{aligned} \quad (5.5)$$

of the system (5.4) gives the solution of the ODE (5.2) as

$$\begin{aligned} u(\xi) &= \frac{-\beta (\ln(A))^2 + \sqrt{\beta^2 (\ln(A))^4 - 2 \varepsilon K}}{\varepsilon} + 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{1 + dA\xi}, \\ & - 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{(1 + dA\xi)^2} \end{aligned} \quad (5.6)$$

where $\sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K} \geq 0$ and $\varepsilon \neq 0$. Thus, the exact solution of the ct-fKdVE (1.2) is written in an explicit form as

$$u_3(x, t) = \frac{-\beta (\ln(A))^2 + \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K}}{\varepsilon} + 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{1 + dA^{x - \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K} \frac{t^\alpha}{\alpha}}},$$

$$- 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{\left(1 + dA^{x - \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K} \frac{t^\alpha}{\alpha}}\right)^2} \quad (5.7)$$

Solution 2: Similarly, the solution

$$a_0 = -\frac{\beta (\ln(A))^2 + \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K}}{\varepsilon}$$

$$a_1 = 12 \frac{\beta (\ln(A))^2}{\varepsilon}$$

$$a_2 = -12 \frac{\beta (\ln(A))^2}{\varepsilon}$$

$$c = -\sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K} \quad (5.8)$$

of the system (5.4) gives the solution of the ODE (5.2) as

$$u(\xi) = -\frac{\beta (\ln(A))^2 + \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K}}{\varepsilon} + 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{1 + dA^\xi}, \quad (5.9)$$

$$- 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{(1 + dA^\xi)^2}$$

where $\sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K} \geq 0$ and $\varepsilon \neq 0$. Thus, the exact solution of the ct-fKdVE (1.2) is written in an explicit form as

$$u_4(x, t) = -\frac{\beta (\ln(A))^2 + \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K}}{\varepsilon} + 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{1 + dA^{x + \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K} \frac{t^\alpha}{\alpha}}},$$

$$- 12 \frac{\beta (\ln(A))^2}{\varepsilon} \frac{1}{\left(1 + dA^{x + \sqrt{\beta^2 (\ln(A))^4 - 2\varepsilon K} \frac{t^\alpha}{\alpha}}\right)^2} \quad (5.10)$$

6. CONCLUSION

Some conformable time fractional partial differential equations are solved by using the modified Kudryashov method. Both equations are reduced to some nonlinear ODEs of integer order. The balance between the nonlinear term and the term with

the highest order derivative gives the highest power of the series forming the solution. Substituting the solution into the resultant ODEs and some computer algebra give the relations between the parameters of the equations and the coefficients of the finite series solution.

Some explicit solutions are given for the conformable time fractional Burgers' and the conformable time fractional KdV equations.

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