

SOME QUANTUM ESTIMATES OF HERMITE-HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS

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ABSTRACT. In this study, based on a new quantum integral identity, the authors establish some quantum estimates of Hermite-Hadamard type inequalities for convex functions. These results generalize and improve some known results given in literatures.

1. INTRODUCTION

Convex functions play an important role in mathematical inequalities. One of the most famous inequalities for convex functions is Hermite-Hadamard's inequality, which is stated as follows: Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, where $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$

This famous result can be considered as a necessary and sufficient condition for a function to be convex.

Hermite-Hadamard's inequality has raised many researchers' interest, and a variety of refinements and generalizations have been found (see for example, [1], [2], [3], [4], [6]-[19]).

In [12], Özdemir established the following lemma.

Lemma 1.1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I^0 , $a, b \in I$ with $a < b$ and f'' be integrable on $[a, b]$. Then the following equality holds:*

$$(1.1) \quad \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{(b-a)^2}{2} \int_0^1 s(1-s) f''(sa + (1-s)b) ds.$$

In [13], Özdemir et al. gave some estimates for the result of Lemma 1.1 via m -convexity.

Theorem 1.1. *Let $f : I^0 \rightarrow \mathbb{R}$, where $I^0 \subset [0, \infty)$ be a twice differentiable function on I^0 , $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a, b]$. If $|f''|^r$ is m -convex on $[a, b]$ for some fixed $r > 1$ and $m \in (0, 1]$ then the following inequality holds:*

$$(1.2) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{8} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\frac{|f''(a)|^r + m |f''(\frac{b}{m})|^r}{2} \right)^{\frac{1}{r}},$$

where $p = \frac{r}{r-1}$.

Theorem 1.2. *Let $f : I^0 \rightarrow \mathbb{R}$, where $I^0 \subset [0, \infty)$ be a twice differentiable function on I^0 , $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a, b]$. If $|f''|^r$ is m -convex on $[a, b]$ for some fixed $r > 1$ and $m \in (0, 1]$ then the following inequality holds:*

$$(1.3) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} \left(|f''(a)|^r + m(r+1) \left| f''\left(\frac{b}{m}\right) \right|^r \right)^{\frac{1}{r}}.$$

Theorem 1.3. *Let $f : I^0 \rightarrow \mathbb{R}$, where $I^0 \subset [0, \infty)$ be a twice differentiable function on I^0 , $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a, b]$. If $|f''|^r$ is m -convex on $[a, b]$ for some fixed $r \geq 1$ and $m \in (0, 1]$ then the following inequality holds:*

$$(1.4) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{4} \left(2|f''(a)|^r + m(r+1) \left| f''\left(\frac{b}{m}\right) \right|^r \right)^{\frac{1}{r}}.$$

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In recent years, the topic of quantum calculus has attracted the attention of several scholars. Quantum calculus appeared as a connection between mathematics and physics. It has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities. At present, q -analogues of many identities and inequalities have been established.

In [11], Noor et al. established the following lemma and developed some quantum estimates of it.

Lemma 1.2. *Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a q -differentiable function on I^0 (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$, then*

$$\frac{1}{b-a} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} = \frac{q(b-a)}{1+q} \int_0^1 (1 - (1+q)t) {}_a D_q f((1-t)a + tb) {}_0 d_q t.$$

Theorem 1.4. *Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a q -differentiable function on I^0 (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q f|^r$, $r \geq 1$ is a convex function, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \leq \frac{q(b-a)}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{1-\frac{1}{r}} \left(\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(a)|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(b)|^r \right)^{\frac{1}{r}}.$$

Theorem 1.5. *Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a q -differentiable function on I^0 (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q f|^r$ is a convex function where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) {}_a d_q x - \frac{qf(a) + f(b)}{1+q} \right| \leq \frac{q(b-a)}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{\frac{1}{p}} \left(\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(a)|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3} |{}_aD_q f(b)|^r \right)^{\frac{1}{r}}.$$

The main purpose of this paper is to establish a new quantum integral identity similar to the one given in Lemma 1.1 (Section 4) and develop some quantum estimates of Hermite-Hadamard type inequalities for convex functions (Section 5). Our results in special cases recapture Lemma 1.1 and Theorem 1.1-1.3.

2. PRELIMINARIES

In this section, we first recall some previously known concepts on q -calculus which will be used in this paper.

Let $J = [a, b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant.

Definition 1. [16] *Assume $f : J \rightarrow \mathbb{R}$ is a continuous function and let $x \in J$. Then q -derivative on J of function f at x is defined as*

$$(2.1) \quad {}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, x \neq a, \quad {}_aD_q f(a) = \lim_{x \rightarrow a} {}_aD_q f(x).$$

We say that f is q -differentiable on J provided ${}_aD_q f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then ${}_0D_q f = D_q f$, where D_q is the well-known q -derivative of the function $f(x)$ defined by

$$(2.2) \quad D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Definition 2. [16] *Let $f : J \rightarrow \mathbb{R}$ be a continuous function. We define the second-order q -derivative on interval J , which is denoted as ${}_aD_q^2 f$, provided ${}_aD_q f$ is q -differentiable on J with ${}_aD_q^2 f = {}_aD_q({}_aD_q f) : J \rightarrow \mathbb{R}$. Similarly, we define higher order q -derivative on J , ${}_aD_q^n : J \rightarrow \mathbb{R}$.*

Definition 3. [16] *Let $f : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then q -integral on J is defined by*

$$(2.3) \quad \int_a^x f(t) {}_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a)$$

for $x \in J$. Note that if $a = 0$, then we have the classical q -integral, which is defined by

$$(2.4) \quad \int_0^x f(t) {}_0d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x)$$

for $x \in [0, +\infty)$.

Theorem 2.1. [16] Assume $f, g : J \rightarrow \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in J$,

$$\begin{aligned} \int_a^x [f(t) + g(t)] {}_a d_q t &= \int_a^x f(t) {}_a d_q t + \int_a^x g(t) {}_a d_q t; \\ \int_a^x (\alpha f)(t) {}_a d_q t &= \alpha \int_a^x f(t) {}_a d_q t. \end{aligned}$$

In addition, we introduce the q -analogues of a and $(x-a)^n$ and the definition of q -Beta function.

Definition 4. [5] For any real number a ,

$$(2.5) \quad [a] = \frac{1-q^a}{1-q}$$

is called the q -analogue of a . In particular, for $n \in \mathbb{Z}^+$, we denote

$$[n] = \frac{1-q^n}{1-q} = q^{n-1} + \dots + q + 1.$$

Definition 5. [5] If n is an integer, the q -analogue of $(x-a)^n$ is the polynomial

$$(2.6) \quad (x-a)_q^n = \begin{cases} 1, & \text{if } n = 0, \\ (x-a)(x-qa) \cdots (x-q^{n-1}a), & \text{if } n \geq 1. \end{cases}$$

Definition 6. [5] For any $t, s > 0$,

$$(2.7) \quad \beta_q(t, s) = \int_0^1 x^{t-1} (1-qx)_q^{s-1} {}_0d_q x$$

is called the q -Beta function. Note that

$$(2.8) \quad \beta_q(t, 1) = \int_0^1 x^{t-1} {}_0d_q x = \frac{1}{[t]},$$

where $[t]$ is the q -analogue of t .

3. SOME SIMPLE CALCULATIONS

In this section, we present some simple calculations that will be used in this paper.

Lemma 3.1. Let $f(x) = 1$, then we have

$$\int_0^1 {}_0d_q x = (1-q) \sum_{n=0}^{\infty} q^n = 1.$$

Lemma 3.2. Let $f(x) = x$ for $x \in [0, 1]$, then we have

$$\int_0^1 x {}_0d_q x = (1-q) \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1+q}.$$

Lemma 3.3. Let $f(x) = x^2$ for $x \in [0, 1]$, then we have

$$\int_0^1 x^2 {}_0d_q x = (1-q) \sum_{n=0}^{\infty} q^{3n} = \frac{1}{1+q+q^2}.$$

Lemma 3.4. Let $f(x) = 1-x$ for $x \in [0, 1]$, then we have

$$\int_0^1 (1-x) {}_0d_q x = \int_0^1 {}_0d_q x - \int_0^1 x {}_0d_q x = 1 - \frac{1}{1+q} = \frac{q}{1+q}.$$

Lemma 3.5. Let $f(x) = 1 - qx$ for $x \in [0, 1]$ where $0 < q < 1$ be a constant, then we have

$$\int_0^1 (1 - qx)_0 d_q x = \int_0^1 1_0 d_q x - q \int_0^1 x_0 d_q x = 1 - q \frac{1}{1+q} = \frac{1}{1+q}.$$

Lemma 3.6. Let $f(x) = x(1 - x)$ for $x \in [0, 1]$, then we have

$$\int_0^1 x(1 - x)_0 d_q x = \int_0^1 (x - x^2)_0 d_q x = \int_0^1 x_0 d_q x - \int_0^1 x^2_0 d_q x = \frac{q^2}{(1+q)(1+q+q^2)}.$$

Lemma 3.7. Let $f(x) = x(1 - qx)$ for $x \in [0, 1]$ where $0 < q < 1$ be a constant, then we have

$$\int_0^1 x(1 - qx)_0 d_q x = \int_0^1 (x - qx^2)_0 d_q x = \int_0^1 x_0 d_q x - q \int_0^1 x^2_0 d_q x = \frac{1}{(1+q)(1+q+q^2)}.$$

Lemma 3.8. Let $f(x) = (1 - x)(1 - qx)$ for $x \in [0, 1]$ where $0 < q < 1$ be a constant, then we have

$$\int_0^1 (1 - x)(1 - qx)_0 d_q x = \int_0^1 (1 - x)_0 d_q x - q \int_0^1 x(1 - x)_0 d_q x = \frac{q}{1+q+q^2}.$$

Lemma 3.9. Let $f(x) = x^2(1 - qx)$ for $x \in [0, 1]$ where $0 < q < 1$ be a constant, then we have

$$\int_0^1 x^2(1 - qx)_0 d_q x = \int_0^1 (x^2 - qx^3)_0 d_q x = \frac{1}{1+q+q^2} - q(1-q) \sum_{n=0}^{\infty} q^{4n} = \frac{1}{(1+q+q^2)(1+q+q^2+q^3)}.$$

Lemma 3.10. Let $f(x) = x(1 - x)(1 - qx)$ for $x \in [0, 1]$ where $0 < q < 1$ be a constant, then we have

$$\begin{aligned} \int_0^1 x(1 - x)(1 - qx)_0 d_q x &= \int_0^1 x_0 d_q x - (1+q) \int_0^1 x^2_0 d_q x + q \int_0^1 x^3_0 d_q x \\ &= \frac{1}{1+q} - (1+q) \frac{1}{1+q+q^2} + q(1-q) \sum_{n=0}^{\infty} q^{4n} = \frac{1}{1+q} - (1+q) \frac{1}{1+q+q^2} + q \frac{1}{1+q+q^2+q^3} \\ &= \frac{q^2}{(1+q+q^2)(1+q+q^2+q^3)}. \end{aligned}$$

Lemma 3.11. Let $f(x) = x^r(1 - x)$ for $x \in [0, 1]$ where $r \geq 1$ be a constant, then we have

$$\int_0^1 x^r(1 - x)_0 d_q x = \frac{q^{r+1}}{[r+1][r+2]}.$$

where $[r+1]$, $[r+2]$ are the q -analogues of $r+1$ and $r+2$.

Proof. Applying (2.5) in Definition 4 and (2.8) in Definition 6, we have

$$\begin{aligned} \int_0^1 x^r(1 - x)_0 d_q x &= \int_0^1 x^r_0 d_q x - \int_0^1 x^{r+1}_0 d_q x = \frac{1}{[r+1]} - \frac{1}{[r+2]} = \frac{1-q}{1-q^{r+1}} - \frac{1-q}{1-q^{r+2}} \\ &= (1-q) \left(\frac{1}{1-q^{r+1}} - \frac{1}{1-q^{r+2}} \right) = (1-q) \frac{q^{r+1}(1-q)}{(1-q^{r+1})(1-q^{r+2})} = \frac{q^{r+1}}{\frac{1-q^{r+1}}{1-q} \frac{1-q^{r+2}}{1-q}} = \frac{q^{r+1}}{[r+1][r+2]}. \end{aligned}$$

The proof is completed. \square

Lemma 3.12. Let $f(x) = x^{r+1}(1 - qx)$ for $x \in [0, 1]$ where $0 < q < 1$ and $r \geq 1$ be constants, then

$$\int_0^1 x^{r+1}(1 - qx)_0 d_q x = \frac{1}{[r+2][r+3]}.$$

where $[r+2]$, $[r+3]$ are the q -analogues of $r+2$ and $r+3$.

Proof. Applying (2.5) in Definition 4 and (2.8) in Definition 6, we have

$$\begin{aligned} \int_0^1 x^{r+1}(1 - qx)_0 d_q x &= \int_0^1 x^{r+1}_0 d_q x - q \int_0^1 x^{r+2}_0 d_q x = \frac{1}{[r+2]} - \frac{q}{[r+3]} \\ &= \frac{1-q}{1-q^{r+2}} - \frac{q(1-q)}{1-q^{r+3}} = \frac{(1-q)(1-q)}{(1-q^{r+2})(1-q^{r+3})} = \frac{1}{[r+2][r+3]}. \end{aligned}$$

The proof is completed. \square

Lemma 3.13. Let $f(x) = x^r(1-x)(1-qx)$ for $x \in [0, 1]$ where $0 < q < 1$ and $r \geq 1$ be constants, then we have

$$\int_0^1 x^r(1-x)(1-qx)_0 d_q x = \frac{(1+q)q^{r+1}}{[r+1][r+2][r+3]},$$

where $[r+1]$, $[r+2]$, $[r+3]$ are the q -analogues of $r+1$, $r+2$ and $r+3$.

Proof. Applying (2.5) in Definition 4 and (2.8) in Definition 6, we have

$$\begin{aligned} & \int_0^1 x^r(1-x)(1-qx)_0 d_q x \\ &= \int_0^1 (x^r - (1+q)x^{r+1} + qx^{r+2})_0 d_q x \\ &= \frac{1}{[r+1]} - \frac{1+q}{[r+2]} + \frac{q}{[r+3]} \\ &= \frac{[r+2][r+3] - (1+q)[r+1][r+3] + q[r+1][r+2]}{[r+1][r+2][r+3]} \\ &= \left(\frac{(1-q^{r+2})(1-q^{r+3})}{(1-q)(1-q)} - \frac{(1+q)(1-q^{r+1})(1-q^{r+3})}{(1-q)(1-q)} + \frac{q(1-q^{r+1})(1-q^{r+2})}{(1-q)(1-q)} \right) \\ & \quad \times \left(\frac{1}{[r+1][r+2][r+3]} \right) \\ &= \frac{(1+q)q^{r+1}}{[r+1][r+2][r+3]}. \end{aligned}$$

The proof is completed. \square

4. A NEW QUANTUM INTEGRAL IDENTITY

In this section, we establish a new q -integral identity.

Lemma 4.1. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_a D_q^2 f$ be continuous and integrable on I where $0 < q < 1$. Then the following identity holds:

$$(4.1) \quad \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x = \frac{q^2(b-a)^2}{1+q} \int_0^1 t(1-qt)_a D_q^2 f((1-t)a + tb)_0 d_q t.$$

Proof. From Definition 1 and 2, we have

$$\begin{aligned} & {}_a D_q^2 f((1-t)a + tb) \\ &= {}_a D_q ({}_a D_q f((1-t)a + tb)) \\ &= \frac{{}_a D_q f((1-t)a + tb) - {}_a D_q f((1-qt)a + qtb)}{(1-q)(b-a)t} \\ &= \left[\frac{f((1-t)a + tb) - f((1-qt)a + qtb)}{(1-q)(b-a)t} - \frac{f((1-qt)a + qtb) - f((1-q^2t)a + q^2tb)}{(1-q)q(b-a)t} \right] \\ & \quad \div [(1-q)(b-a)t] \\ &= \frac{qf((1-t)a + tb) - (1+q)f((1-qt)a + qtb) + f((1-q^2t)a + q^2tb)}{(1-q)^2(b-a)^2qt^2}. \end{aligned}$$

Applying this calculation and from Definition 3, we have

$$\begin{aligned} & \int_0^1 t(1-qt)_a D_q^2 f((1-t)a + tb)_0 d_q t \\ &= \int_0^1 t(1-qt) \left(\frac{qf((1-t)a + tb) - (1+q)f((1-qt)a + qtb) + f((1-q^2t)a + q^2tb)}{(1-q)^2(b-a)^2qt^2} \right) d_q t \\ &= \int_0^1 \left[\frac{qf((1-t)a + tb) - (1+q)f((1-qt)a + qtb) + f((1-q^2t)a + q^2tb)}{(1-q)^2(b-a)^2qt} \right] d_q t \end{aligned}$$

$$\begin{aligned}
& -q \left[\frac{qf((1-t)a+tb) - (1+q)f((1-qt)a+qtb) + f((1-q^2t)a+q^2tb)}{(1-q)^2(b-a)^2q} \right] \Big\} {}_0d_q t \\
& = \left\{ \frac{q(1-q) \sum_{n=0}^{\infty} f((1-q^n)a+q^n b)}{(1-q)^2(b-a)^2q} - \frac{(1+q)(1-q) \sum_{n=0}^{\infty} f((1-q^{n+1})a+q^{n+1}b)}{(1-q)^2(b-a)^2q} \right. \\
& \quad \left. + \frac{(1-q) \sum_{n=0}^{\infty} f((1-q^{n+2})a+q^{n+2}b)}{(1-q)^2(b-a)^2q} \right\} - q \left\{ \frac{q(1-q)(b-a) \sum_{n=0}^{\infty} q^n f((1-q^n)a+q^n b)}{(1-q)^2(b-a)^3q} \right. \\
& \quad \left. - \frac{(1+q)(1-q)(b-a) \sum_{n=0}^{\infty} q^{n+1} f((1-q^{n+1})a+q^{n+1}b)}{(1-q)^2(b-a)^3q^2} \right. \\
& \quad \left. + \frac{(1-q)(b-a) \sum_{n=0}^{\infty} q^{n+2} f((1-q^{n+2})a+q^{n+2}b)}{(1-q)^2(b-a)^3q^3} \right\} \\
& = \left\{ \frac{q \left[\sum_{n=0}^{\infty} f((1-q^n)a+q^n b) - \sum_{n=0}^{\infty} f((1-q^{n+1})a+q^{n+1}b) \right]}{(1-q)(b-a)^2q} \right\} \\
& \quad - \left\{ \frac{\sum_{n=0}^{\infty} f((1-q^{n+1})a+q^{n+1}b) - \sum_{n=0}^{\infty} f((1-q^{n+2})a+q^{n+2}b)}{(1-q)(b-a)^2q} \right\} \\
& \quad - q \left\{ \frac{\int_a^b f(x)_a d_q x}{(1-q)^2(b-a)^3} - \frac{(1+q) \int_a^b f(x)_a d_q x - (1+q)(1-q)(b-a)f(b)}{(1-q)^2(b-a)^3q^2} \right. \\
& \quad \left. + \frac{\int_a^b f(x)_a d_q x - (1-q)(b-a)f(b) - (1-q)(b-a)qf((1-q)a+qb)}{(1-q)^2(b-a)^3q^3} \right\} \\
& = \left\{ \frac{q(f(b) - f(a))}{(1-q)(b-a)^2q} - \frac{f((1-q)a+qb) - f(a)}{(1-q)(b-a)^2q} \right\} \\
& \quad - \left\{ \frac{1+q}{(b-a)^3q^2} \int_a^b f(x)_a d_q x + \frac{q^2+q-1}{(1-q)(b-a)^2q^2} f(b) - \frac{qf((1-q)a+qb)}{(1-q)(b-a)^2q^2} \right\} \\
& = \frac{qf(a) + f(b)}{(b-a)^2q^2} - \frac{1+q}{(b-a)^3q^2} \int_a^b f(x)_a d_q x.
\end{aligned}$$

Multiplying both sides by $\frac{q^2(b-a)^2}{1+q}$, we complete the proof. \square

Remark 1. If $q \rightarrow 1$ and substitute $sa + (1-s)b$ for $(1-t)a + tb$, then (4.1) reduces to identity (1.1) in Lemma 1.1.

5. HERMITE-HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS

In this section, we will give some estimates for the left-hand side of the result of (4.1) through convex functions.

Theorem 5.1. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q^2 f|^r$ is convex on $[a, b]$ for $r \geq 1$, then the

following inequality holds:

$$(5.1) \quad \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{(1 + q)^{2 - \frac{1}{r}}} \left(m_1 |{}_a D_q^2 f(a)|^r + m_2 |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}},$$

where

$$m_1 = (1 - q) \sum_{n=0}^{\infty} (q^{2n} - q^{3n})(1 - q^{n+1})^r, \quad m_2 = (1 - q) \sum_{n=0}^{\infty} q^{3n}(1 - q^{n+1})^r.$$

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_a D_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 t(1 - qt) |{}_a D_q^2 f((1 - t)a + tb)|_0 d_q t \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left(\int_0^1 t_0 d_q t \right)^{1 - \frac{1}{r}} \left(\int_0^1 t(1 - qt)^r |{}_a D_q^2 f((1 - t)a + tb)|^r_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.2, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left(\frac{1}{1 + q} \right)^{1 - \frac{1}{r}} \left(|{}_a D_q^2 f(a)|^r \int_0^1 t(1 - t)(1 - qt)^r_0 d_q t + |{}_a D_q^2 f(b)|^r \int_0^1 t^2(1 - qt)^r_0 d_q t \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{(1 + q)^{2 - \frac{1}{r}}} \left(m_1 |{}_a D_q^2 f(a)|^r + m_2 |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} m_1 &= \int_0^1 t(1 - t)(1 - qt)^r_0 d_q t = (1 - q) \sum_{n=0}^{\infty} (q^{2n} - q^{3n})(1 - q^{n+1})^r, \\ m_2 &= \int_0^1 t^2(1 - qt)^r_0 d_q t = (1 - q) \sum_{n=0}^{\infty} q^{3n}(1 - q^{n+1})^r. \end{aligned}$$

The proof is completed. □

Remark 2. If $q \rightarrow 1$, then we have

$$\int_0^1 t(1 - t)^{r+1} dt = \frac{1}{(r + 2)(r + 3)}, \quad \int_0^1 t^2(1 - t)^r dt = \frac{2}{(r + 1)(r + 2)(r + 3)},$$

and (5.1) reduces to the following inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{(b - a)^2}{2^{2 - \frac{1}{r}}} \left(\frac{(r + 1) |f''(a)|^r + 2 |f''(b)|^r}{(r + 1)(r + 2)(r + 3)} \right)^{\frac{1}{r}}.$$

Corollary 5.1. In Theorem 5.1, if r is a positive integer, then

$$(1 - qt)^r \leq (1 - qt)_q^r, \quad (1 - t)(1 - qt)^r \leq (1 - qt)_q^{r+1},$$

and (5.1) reduces to

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{(1 + q)^{2 - \frac{1}{r}}} \left(\beta_q(2, r + 2) |{}_a D_q^2 f(a)|^r + \beta_q(3, r + 1) |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}.$$

Theorem 5.2. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_a D_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a D_q^2 f|^r$ is convex on $[a, b]$ where $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$, then

$$(5.2) \quad \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} (k_1)^{\frac{1}{p}} \left(\frac{q^2 |{}_a D_q^2 f(a)|^r + (1 + q) |{}_a D_q^2 f(b)|^r}{(1 + q)(1 + q + q^2)} \right)^{\frac{1}{r}},$$

where

$$k_1 = (1 - q) \sum_{n=0}^{\infty} q^{2n} (1 - q^{n+1})^p.$$

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_aD_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left(\int_0^1 t(1 - qt)^p {}_0d_q t \right)^{\frac{1}{p}} \left(\int_0^1 t |{}_aD_q^2 f((1 - t)a + tb)|^r {}_0d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left(\int_0^1 t(1 - qt)^p {}_0d_q t \right)^{\frac{1}{p}} \left(|{}_aD_q^2 f(a)|^r \int_0^1 t(1 - t) {}_0d_q t + |{}_aD_q^2 f(b)|^r \int_0^1 t^2 {}_0d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.3 and 3.6, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left(\int_0^1 t(1 - qt)^p {}_0d_q t \right)^{\frac{1}{p}} \left(\frac{q^2 |{}_aD_q^2 f(a)|^r + (1 + q) |{}_aD_q^2 f(b)|^r}{(1 + q)(1 + q + q^2)} \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{1 + q} (k_1)^{\frac{1}{p}} \left(\frac{q^2 |{}_aD_q^2 f(a)|^r + (1 + q) |{}_aD_q^2 f(b)|^r}{(1 + q)(1 + q + q^2)} \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$k_1 = \int_0^1 t(1 - qt)^p {}_0d_q t = (1 - q) \sum_{n=0}^{\infty} q^{2n} (1 - q^{n+1})^p.$$

The proof is completed. □

Remark 3. If $q \rightarrow 1$, then we have

$$\int_0^1 t(1 - t)^p dt = \frac{1}{(p + 1)(p + 2)},$$

and (5.2) reduces to the following inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{(b - a)^2}{2} \left(\frac{1}{(p + 1)(p + 2)} \right)^{\frac{1}{p}} \left(\frac{|f''(a)|^r + 2|f''(b)|^r}{6} \right)^{\frac{1}{r}}.$$

Corollary 5.2. In Theorem 5.2, if p is a positive integer, $p > 1$, then

$$(1 - qt)^p \leq (1 - qt)_q^p,$$

and (5.2) reduces to

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} (\beta_q(2, p + 1))^{\frac{1}{p}} \left(\frac{q^2 |{}_aD_q^2 f(a)|^r + (1 + q) |{}_aD_q^2 f(b)|^r}{(1 + q)(1 + q + q^2)} \right)^{\frac{1}{r}}. \end{aligned}$$

Theorem 5.3. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q^2 f|^r$ is convex on $[a, b]$ for $r \geq 1$, then

$$(5.3) \quad \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} \left(h_1 |{}_aD_q^2 f(a)|^r + h_2 |{}_aD_q^2 f(b)|^r \right)^{\frac{1}{r}},$$

where

$$h_1 = (1-q) \sum_{n=0}^{\infty} (q^n)^{r+1} (1-q^n)(1-q^{n+1})^r, \quad h_2 = (1-q) \sum_{n=0}^{\infty} (q^n)^{r+2} (1-q^{n+1})^r.$$

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_a D_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 {}_0 d_q t \right)^{1-\frac{1}{r}} \left(\int_0^1 t^r (1-qt)^r |{}_a D_q^2 f((1-t)a + tb)|^r {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.1, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t^r (1-qt)^r |{}_a D_q^2 f((1-t)a + tb)|^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(|{}_a D_q^2 f(a)|^r \int_0^1 t^r (1-qt)^r (1-t) {}_0 d_q t + |{}_a D_q^2 f(b)|^r \int_0^1 t^{r+1} (1-qt)^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & = \frac{q^2(b-a)^2}{1+q} \left(h_1 |{}_a D_q^2 f(a)|^r + h_2 |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} h_1 &= \int_0^1 t^r (1-qt)^r (1-t) {}_0 d_q t = (1-q) \sum_{n=0}^{\infty} (q^n)^{r+1} (1-q^n)(1-q^{n+1})^r, \\ h_2 &= \int_0^1 t^{r+1} (1-qt)^r {}_0 d_q t = (1-q) \sum_{n=0}^{\infty} (q^n)^{r+2} (1-q^{n+1})^r. \end{aligned}$$

The proof is completed. \square

Remark 4. If $q \rightarrow 1$, then we have

$$\int_0^1 t^r (1-t)^{r+1} dt = \beta(r+1, r+2), \quad \int_0^1 t^{r+1} (1-t)^r dt = \beta(r+2, r+1),$$

and (5.4) reduces to the following inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} (\beta(r+1, r+2) |f''(a)|^r + \beta(r+2, r+1) |f''(b)|^r)^{\frac{1}{r}}.$$

Corollary 5.3. In Theorem 5.3, if r is a positive integer, then

$$(1-qt)^r \leq (1-qt)_q^r, \quad (1-t)(1-qt)^r \leq (1-qt)_q^{r+1},$$

and (5.3) reduces to

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\beta_q(r+1, r+2) |{}_a D_q^2 f(a)|^r + \beta_q(r+2, r+1) |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}. \end{aligned}$$

Theorem 5.4. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_a D_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a D_q^2 f|^r$ is convex on $[a, b]$ where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$,

then

$$(5.4) \quad \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \leq \frac{q^2(b-a)^2}{1+q} (u_1)^{\frac{1}{p}} \left(\frac{q |{}_a D_q^2 f(a)|^r + |{}_a D_q^2 f(b)|^r}{1+q} \right)^{\frac{1}{r}},$$

where

$$u_1 = (1-q) \sum_{n=0}^{\infty} (q^n)^{p+1} (1-q^{n+1})^p.$$

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_a D_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t^p (1-qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left(\int_0^1 |{}_a D_q^2 f((1-t)a + tb)|^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t^p (1-qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left(|{}_a D_q^2 f(a)|^r \int_0^1 (1-t) {}_0 d_q t + |{}_a D_q^2 f(b)|^r \int_0^1 t {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.2 and 3.4, we have

$$\left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \right| \leq \frac{q^2(b-a)^2}{1+q} (u_1)^{\frac{1}{p}} \left(\frac{q |{}_a D_q^2 f(a)|^r + |{}_a D_q^2 f(b)|^r}{1+q} \right)^{\frac{1}{r}}.$$

It is easy to check that

$$u_1 = \int_0^1 t^p (1-qt)^p {}_0 d_q t = (1-q) \sum_{n=0}^{\infty} (q^n)^{p+1} (1-q^{n+1})^p.$$

The proof is completed. □

Remark 5. If $q \rightarrow 1$, then

$$\int_0^1 t^p (1-t)^p dt = \beta(p+1, p+1).$$

Using the properties of Beta function, that is, $\beta(x, x) = 2^{1-2x} \beta(\frac{1}{2}, x)$ and $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(xy)}$, we can obtain that

$$\beta(p+1, p+1) = 2^{1-2(p+1)} \beta\left(\frac{1}{2}, p+1\right) = 2^{-2p-1} \frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)},$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(t)$ is Gamma function:

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx, \quad t > 0.$$

Thus, inequality (5.4) reduces to the $m = 1$ case of inequality (1.2), due to the fact that

$$\begin{aligned} & \frac{(b-a)^2}{2} (\beta(p+1, p+1))^{\frac{1}{p}} \left(\frac{|f''(a)|^r + |f''(b)|^r}{2} \right)^{\frac{1}{r}} \\ & = \frac{(b-a)^2}{2} \left(\frac{2^{-1-2p} \sqrt{\pi} \Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\frac{|f''(a)|^r + |f''(b)|^r}{2} \right)^{\frac{1}{r}} \\ & \leq \frac{(b-a)^2}{8} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\frac{|f''(a)|^r + |f''(b)|^r}{2} \right)^{\frac{1}{r}}. \end{aligned}$$

Corollary 5.4. In Theorem 5.4, if p is a positive integer, $p > 1$, then

$$(1 - qt)^p \leq (1 - qt)_q^p,$$

and (5.4) reduces to

$$\left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} (\beta_q(p + 1, p + 1))^{\frac{1}{p}} \left(\frac{q |{}_a D_q^2 f(a)|^r + |{}_a D_q^2 f(b)|^r}{1 + q} \right)^{\frac{1}{r}}.$$

Theorem 5.5. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_a D_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a D_q^2 f|^r$ is convex on $[a, b]$ where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then

$$(5.5) \quad \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b - a)^2}{1 + q} \left(\frac{1}{[p + 1]} \right)^{\frac{1}{p}} (z_1 |{}_a D_q^2 f(a)|^r + z_2 |{}_a D_q^2 f(b)|^r)^{\frac{1}{r}},$$

where

$$z_1 = (1 - q) \sum_{n=0}^{\infty} (q^n - q^{2n})(1 - q^{n+1})^r, \quad z_2 = (1 - q) \sum_{n=0}^{\infty} q^{2n}(1 - q^{n+1})^r,$$

and $[p + 1]$ is the q -analogue of $p + 1$.

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_a D_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left(\int_0^1 t^p d_q t \right)^{\frac{1}{p}} \left(\int_0^1 (1 - qt)^r |{}_a D_q^2 f((1 - t)a + tb)|^r d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying (2.8) in Definition 6, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left(\frac{1}{[p + 1]} \right)^{\frac{1}{p}} \left(|{}_a D_q^2 f(a)|^r \int_0^1 (1 - qt)^r (1 - t)_0 d_q t + |{}_a D_q^2 f(b)|^r \int_0^1 t(1 - qt)^r d_q t \right)^{\frac{1}{r}} \\ & = \frac{q^2(b - a)^2}{1 + q} \left(\frac{1}{[p + 1]} \right)^{\frac{1}{p}} (z_1 |{}_a D_q^2 f(a)|^r + z_2 |{}_a D_q^2 f(b)|^r)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} z_1 &= \int_0^1 (1 - qt)^r (1 - t)_0 d_q t = (1 - q) \sum_{n=0}^{\infty} (q^n - q^{2n})(1 - q^{n+1})^r, \\ z_2 &= \int_0^1 t(1 - qt)^r d_q t = (1 - q) \sum_{n=0}^{\infty} q^{2n}(1 - q^{n+1})^r. \end{aligned}$$

The proof is completed. \square

Remark 6. If $q \rightarrow 1$, then

$$\int_0^1 (1 - t)^{r+1} dt = \frac{1}{r + 2}, \quad \int_0^1 t(1 - t)^r dt = \frac{1}{(r + 1)(r + 2)},$$

and (5.5) reduces to

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{(b - a)^2}{2} \left(\frac{1}{p + 1} \right)^{\frac{1}{p}} \left(\frac{(r + 1) |f''(a)|^r + |f''(b)|^r}{(r + 1)(r + 2)} \right)^{\frac{1}{r}}.$$

Corollary 5.5. *In Theorem 5.5, if r is a positive integer, $r > 1$, then*

$$(1-t)^{r+1} \leq (1-qt)_q^{r+1}, \quad (1-t)^r \leq (1-qt)_q^r,$$

and (5.5) reduces to

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\frac{1}{[p+1]} \right)^{\frac{1}{p}} \left(\beta_q(1, r+2) |{}_a D_q^2 f(a)|^r + \beta_q(2, r+1) |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}. \end{aligned}$$

Theorem 5.6. *Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_a D_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a D_q^2 f|^r$ is convex on $[a, b]$ where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then*

$$(5.6) \quad \begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} (\lambda_1)^{\frac{1}{p}} \left(\left(\frac{q^{r+1}}{[r+1][r+2]} \right) |{}_a D_q^2 f(a)|^r + \frac{1}{[r+2]} |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}, \end{aligned}$$

where

$$\lambda_1 = (1-q) \sum_{n=0}^{\infty} q^n (1-q^{n+1})^p,$$

and $[r+1], [r+2]$ are the q -analogues of $r+1$ and $r+2$.

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_a D_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 (1-qt)^p \right)^{\frac{1}{p}} \left(\int_0^1 t^r |{}_a D_q^2 f((1-t)a + tb)|^r {}_0 d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 (1-qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left(|{}_a D_q^2 f(a)|^r \int_0^1 t^r (1-t) {}_0 d_q t + |{}_a D_q^2 f(b)|^r \int_0^1 t^{r+1} {}_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.11 and (2.8) in Definition 6, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 (1-qt)^p {}_0 d_q t \right)^{\frac{1}{p}} \left(\left(\frac{q^{r+1}}{[r+1][r+2]} \right) |{}_a D_q^2 f(a)|^r + \frac{1}{[r+2]} |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}} \\ & = \frac{q^2(b-a)^2}{1+q} (\lambda_1)^{\frac{1}{p}} \left(\left(\frac{q^{r+1}}{[r+1][r+2]} \right) |{}_a D_q^2 f(a)|^r + \frac{1}{[r+2]} |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$\lambda_1 = \int_0^1 (1-qt)^p {}_0 d_q t = (1-q) \sum_{n=0}^{\infty} q^n (1-q^{n+1})^p.$$

The proof is completed. \square

Remark 7. *If $q \rightarrow 1$, then*

$$\int_0^1 (1-t)^p dt = \frac{1}{p+1}.$$

Since

$$\lim_{p \rightarrow \infty} \left(\frac{1}{1+p} \right)^{\frac{1}{p}} = 1 \quad \text{and} \quad \lim_{p \rightarrow 1^+} \left(\frac{1}{1+p} \right)^{\frac{1}{p}} = \frac{1}{2},$$

we have

$$\frac{1}{2} < \left(\frac{1}{1+p} \right)^{\frac{1}{p}} < 1, \quad r \in (1, \infty),$$

hence for $r \in (1, \infty)$,

$$\left(\frac{1}{1+p} \right)^{\frac{1}{p}} < 1 \quad \text{and} \quad \frac{1}{(r+1)(r+2)} < 1.$$

Then inequality (5.6) reduces to the $m = 1$ case of inequality (1.3).

Corollary 5.6. *In Theorem 5.6, if p is a positive integer, $p > 1$, then*

$$(1 - qt)^p \leq (1 - qt)_q^p,$$

and (5.6) reduces to

$$\left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2}{1+q} (\beta_q(1, p+1))^{\frac{1}{p}} \left(\left(\frac{q^{r+1}}{[r+1][r+2]} \right) |{}_a D_q^2 f(a)|^r + \frac{1}{[r+2]} |{}_a D_q^2 f(b)|^r \right)^{\frac{1}{r}}.$$

Theorem 5.7. *Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_a D_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a D_q^2 f|$ is convex on $[a, b]$, then*

$$(5.7) \quad \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2 (q^2 |{}_a D_q^2 f(a)| + |{}_a D_q^2 f(b)|)}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}.$$

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_a D_q^2 f|$ is a convex function, we have

$$\left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2}{1+q} \left(|{}_a D_q^2 f(a)| \int_0^1 t(1-t)(1-qt)_0 d_q t + |{}_a D_q^2 f(b)| \int_0^1 t^2(1-qt)_0 d_q t \right)$$

Applying Lemma 3.9 and 3.10, we have

$$\left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2 (q^2 |{}_a D_q^2 f(a)| + |{}_a D_q^2 f(b)|)}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}.$$

The proof is completed. \square

Remark 8. *If $q \rightarrow 1$, then (5.7) reduces to the following inequality:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2 (f''(a) + f''(b))}{24}.$$

Theorem 5.8. *Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_a D_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a D_q^2 f|^r$ is convex on $[a, b]$ for $r \geq 1$, then*

$$(5.8) \quad \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{r}}} \left(\frac{q^2 |{}_a D_q^2 f(a)|^r + |{}_a D_q^2 f(b)|^r}{1+q+q^2+q^3} \right)^{\frac{1}{r}}.$$

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_aD_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t(1-qt)_0 d_q t \right)^{1-\frac{1}{r}} \left(\int_0^1 t(1-qt) |{}_aD_q^2 f((1-t)a+tb)|^r {}_0d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t(1-qt)_0 d_q t \right)^{1-\frac{1}{r}} \\ & \quad \times \left(|{}_aD_q^2 f(a)|^r \int_0^1 t(1-t)(1-qt)_0 d_q t + |{}_aD_q^2 f(b)|^r \int_0^1 t^2(1-qt)_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.7, 3.9 and 3.10, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\frac{1}{(1+q)(1+q+q^2)} \right)^{1-\frac{1}{r}} \left(\frac{q^2 |{}_aD_q^2 f(a)|^r + |{}_aD_q^2 f(b)|^r}{(1+q+q^2)(1+q+q^2+q^3)} \right)^{\frac{1}{r}} \\ & = \frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{r}}} \left(\frac{q^2 |{}_aD_q^2 f(a)|^r + |{}_aD_q^2 f(b)|^r}{1+q+q^2+q^3} \right)^{\frac{1}{r}}. \end{aligned}$$

The proof is completed. \square

Remark 9. If $q \rightarrow 1$, then (5.8) reduces to the following inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2^{2+\frac{1}{r}}} (|f''(a)|^r + |f''(b)|^r)^{\frac{1}{r}}.$$

Theorem 5.9. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q^2 f|^r$ is convex on $[a, b]$ for $r \geq 1$, then

$$(5.9) \quad \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2}{1+q} \left(\frac{1}{1+q} \right)^{1-\frac{1}{r}} \left(\frac{(1+q)q^{r+1}}{[r+1][r+2][r+3]} |{}_aD_q^2 f(a)|^r + \frac{1}{[r+2][r+3]} |{}_aD_q^2 f(b)|^r \right)^{\frac{1}{r}},$$

where $[r+1]$, $[r+2]$, $[r+3]$ are the q -analogues of $r+1$, $r+2$ and $r+3$.

Proof. Using Lemma 4.1, Hölder's inequality and the fact that $|{}_aD_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 (1-qt)_0 d_q t \right)^{1-\frac{1}{r}} \left(\int_0^1 t^r(1-qt) |{}_aD_q^2 f((1-t)a+tb)|^r {}_0d_q t \right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 (1-qt)_0 d_q t \right)^{1-\frac{1}{r}} \\ & \quad \times \left(|{}_aD_q^2 f(a)|^r \int_0^1 t^r(1-t)(1-qt)_0 d_q t + |{}_aD_q^2 f(b)|^r \int_0^1 t^{r+1}(1-qt)_0 d_q t \right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.5, 3.12 and 3.13, we have

$$\left| \frac{qf(a) + f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right|$$

$$\leq \frac{q^2(b-a)^2}{1+q} \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}} \left(\frac{(1+q)q^{r+1}}{[r+1][r+2][r+3]} |{}_aD_q^2 f(a)|^r + \frac{1}{[r+2][r+3]} |{}_aD_q^2 f(b)|^r\right)^{\frac{1}{r}}.$$

The proof is completed. □

Remark 10. If $q \rightarrow 1$, then (5.9) reduces to

$$(5.10) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{4} \left(\frac{2}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}} (2|f''(a)|^r + (r+1)|f''(b)|^r)^{\frac{1}{r}}.$$

Since $\left(\frac{2}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}} \leq 1, r \in [1, \infty)$, then inequality (5.10) reduces to the $m = 1$ case of inequality (1.4).

Theorem 5.10. Let $f : I = [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice q -differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q^2 f|^r$ is convex on $[a, b]$ where $p, r > 1, \frac{1}{p} + \frac{1}{r} = 1$, then

$$(5.11) \quad \left| \frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \leq \frac{q^2(b-a)^2}{1+q} (\beta_q(p+1, 2))^{\frac{1}{p}} \left(\frac{(q+q^2)|{}_aD_q^2 f(a)|^r + |{}_aD_q^2 f(b)|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 4.1, Hölder’s inequality and the fact that $|{}_aD_q^2 f|^r$ is a convex function, we have

$$\begin{aligned} & \left| \frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t^p(1-qt)_0 d_q t\right)^{\frac{1}{p}} \left(\int_0^1 (1-qt) |{}_aD_q^2 f((1-t)a+tb)|^r {}_0 d_q t\right)^{\frac{1}{r}} \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t^p(1-qt)_0 d_q t\right)^{\frac{1}{p}} \\ & \quad \times \left(|{}_aD_q^2 f(a)|^r \int_0^1 (1-t)(1-qt)_0 d_q t + |{}_aD_q^2 f(b)|^r \int_0^1 t(1-qt)_0 d_q t\right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 3.7, 3.8 and the fact that $(1-qt) = (1-qt)_q^1$, we have

$$\begin{aligned} & \left| \frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_a d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left(\int_0^1 t^p(1-qt)_q^1 d_q t\right)^{\frac{1}{p}} \left(\frac{q|{}_aD_q^2 f(a)|^r}{1+q+q^2} + \frac{|{}_aD_q^2 f(b)|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}} \\ & = \frac{q^2(b-a)^2}{1+q} (\beta_q(p+1, 2))^{\frac{1}{p}} \left(\frac{(q+q^2)|{}_aD_q^2 f(a)|^r + |{}_aD_q^2 f(b)|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}. \end{aligned}$$

The proof is completed. □

Remark 11. If $q \rightarrow 1$, then

$$\beta(p+1, 2) = \int_0^1 t^p(1-t)dt = \frac{1}{(p+1)(p+2)},$$

and (5.11) reduces to

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{(p+1)(p+2)} \right)^{\frac{1}{p}} \left(\frac{2|f''(a)|^r + |f''(b)|^r}{6} \right)^{\frac{1}{r}}.$$

6. CONCLUSIONS

Quantum calculus has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities. In this paper, we first establish a new quantum integral identity and then develop some quantum estimates of Hermite-Hadamard type inequalities for convex functions. These results in some special cases recapture the known results. We hope that our results may be helpful for further study.

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