SOME INEQUALITIES BOUNDING CERTAIN RATIOS OF THE
\((p, k)\)-GAMMA FUNCTION

KWARA NANTOMAH

Abstract. In this paper, we establish some inequalities bounding the ratio
\(\Gamma_{p, k}(x)/\Gamma_{p, k}(y)\), where \(\Gamma_{p, k}(\cdot)\) is the \((p, k)\)-analogue of the Gamma function.
Consequently, some previous results are recovered from the obtained results.

1. Introduction

Inequalities that provide bounds for the ratio \(\Gamma(x)/\Gamma(y)\), where \(x\) and \(y\) are numbers of some special form, have been studied intensively by several researchers across the globe. A detailed account on inequalities of this nature can be found in the survey article by Qi [10]. In this study, the focus shall be on the type originating from certain problems of traffic flow.

In 1978, Lew, Frauenthal and Keyfitz [5] by studying certain problems of traffic flow established the double-inequality
\[
2\Gamma\left(n + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right) \Gamma(n + 1) \leq 2^n \Gamma\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}
\]
which can be rearranged as
\[
\frac{2}{\sqrt{\pi}} \leq \frac{\Gamma(n + 1)}{\Gamma(n + \frac{1}{2})} \leq \frac{2^n}{\sqrt{\pi}}, \quad n \in \mathbb{N}.
\]

\[
\left(\frac{x}{x + s}\right)^{1-s} \leq \frac{\Gamma(x + s)}{x^s \Gamma(x)} \leq 1, \quad s \in (0, 1), \quad x > 0
\]
due Wendel [12], extended and refined the inequality (2) by proving the result
\[
\sqrt{x} \leq \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} \leq \sqrt{x + \frac{1}{2}},
\]
for \(x > 0\).

Also, in the paper [6], the authors established the \(q\)-analogue of (4) as
\[
\sqrt{[x]_q} \leq \frac{\Gamma_q(x + 1)}{\Gamma_q(x + \frac{1}{2})} \leq \sqrt{\left[\frac{x + 1}{2}\right]_q},
\]

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for $q \in (0, 1)$ and $x > 0$.

Furthermore, in the paper [7], the authors established the $(q, k)$-analogue of (4) as

$$[x]_q^{1-\frac{1}{k}} \leq \frac{\Gamma_{q,k}(x+k)}{\Gamma_{q,k}(x+\frac{k}{2})} \leq \left[ x + \frac{1}{2} \right]_q^{1-\frac{1}{k}}$$

for $q \in (0, 1)$, $k > 0$ and $x > 0$.

The main objective of this paper is to establish similar inequalities for the $(p, k)$-analogue of the Gamma function.

2. Preliminaries

The classical Euler’s Gamma function, $\Gamma(x)$ is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt = \lim_{n \to \infty} \frac{n!n^x}{x(x+1)(x+2)\ldots(x+n)}$$

Closely related to the Gamma function is the Digamma function, $\psi(x)$ which is defined for $x > 0$ as $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$.

Euler gave another definition of the Gamma function called the $p$-analogue, which is defined for $p \in \mathbb{N}$ and $x > 0$ as (see [1, p. 270])

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\ldots(x+p)}$$

with the $p$-analogue of the Digamma function defined as $\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x)$.

Also, Díaz and Pariguan [2] defined the $k$-analogues of the Gamma and Digamma functions as

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t}{k}} \, dt \quad \text{and} \quad \psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x)$$

for $k > 0$ and $x \in \mathbb{C}\setminus k\mathbb{Z}^-$.

Then in a recent paper [8], the authors introduced a $(p, k)$-analogue of the Gamma function defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\Gamma_{p,k}(x) = \int_0^p t^{x-1} \left( 1 - \frac{t^k}{pk} \right)^p \, dt$$

$$= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\ldots(x+pk)}$$
satisfying the properties
\[ \Gamma_{p,k}(x + k) = \frac{p^k x}{x + pk + k} \Gamma_{p,k}(x) \] (5)
\[ \Gamma_{p,k}(ak) = \frac{p + 1}{p} k^{a-1} \Gamma_{p}(a), \quad a \in \mathbb{R}^+ \]
\[ \Gamma_{p,k}(k) = 1. \]

The \((p,k)\)-analogue of the Digamma function is defined for \(x > 0\) as
\[ \psi_{p,k}(x) = \frac{d}{dx} \ln \Gamma_{p,k}(x) = \frac{1}{k} \ln(pk) - \sum_{n=0}^{p} \frac{1}{nk + x} \]
\[ = \frac{1}{k} \ln(pk) - \int_{0}^{\infty} \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} e^{-xt} dt \]

Also, the \((p,k)\)-analogue of the Polygamma functions are defined as
\[ \psi_{p,k}^{(m)}(x) = \frac{d^m}{dx^m} \psi_{p,k}(x) = \sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(nk + x)^{m+1}} \]
\[ = (-1)^{m+1} \int_{0}^{\infty} \left( \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} \right) t^m e^{-xt} dt \]

where \(m \in \mathbb{N}\), and \(\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)\).

The functions \(\Gamma_{p,k}(x)\) and \(\psi_{p,k}(x)\) satisfy the following commutative diagrams.

We now present the main findings of the paper in the following section.

3. MAIN RESULTS

**Lemma 3.1.** Let \(p \in \mathbb{N}\), \(k > 0\) and \(s \in (0, 1)\). Then the inequality
\[ \left( \frac{pkx}{x + pk + k} \right)^{1-s} \leq \frac{\Gamma_{p,k}(x + sk)}{\Gamma_{p,k}(x)} \leq \left( \frac{pk}{x + pk + k} \right)^{s} \]
holds for \(x > 0\).

**Proof.** We employ the Hölder’s inequality for integrals, which is stated for any integrable functions \(f, g : (0, a) \to \mathbb{R}\) as
\[ \int_{0}^{a} |f(t)g(t)| dt \leq \left[ \int_{0}^{a} |f(t)|^\alpha dt \right]^\frac{1}{\alpha} \left[ \int_{0}^{a} |g(t)|^\beta dt \right]^\frac{1}{\beta} \]
where $\alpha > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. We proceed as follows. Let
\[
\alpha = \frac{1}{1-s}, \quad \beta = \frac{1}{s}, \quad f(t) = t^{(1-s)(x-1)} \left(1 - \frac{t^k}{pk}\right)^{p(1-s)}, \quad g(t) = t^{s(x+k-1)} \left(1 - \frac{t^k}{pk}\right)^{ps}.
\]
Then,
\[
\Gamma_{p,k}(x + sk) = \int_0^p t^{x+sk-1} \left(1 - \frac{t^k}{pk}\right)^p dt
= \int_0^p t^{(1-s)(x-1)} \left(1 - \frac{t^k}{pk}\right)^{p(1-s)} t^{s(x+k-1)} \left(1 - \frac{t^k}{pk}\right)^{ps} dt
\leq \left[ \int_0^p \left( t^{(1-s)(x-1)} \left(1 - \frac{t^k}{pk}\right)^{p(1-s)} \right) \frac{1}{t^{1-s}} dt \right]^{1-s} \times
\left[ \int_0^p \left( t^{s(x+k-1)} \left(1 - \frac{t^k}{pk}\right)^{ps} \right) \frac{1}{t^s} dt \right]^s
= \left[ \int_0^p t^{x-1} \left(1 - \frac{t^k}{pk}\right)^p dt \right]^{1-s} \left[ \int_0^p t^{x+k-1} \left(1 - \frac{t^k}{pk}\right)^p dt \right]^s
= \left[ \Gamma_{p,k}(x) \right]^{1-s} \left[ \Gamma_{p,k}(x + k) \right]^s.
\]
That is,
\[
\Gamma_{p,k}(x + sk) \leq \left[ \Gamma_{p,k}(x) \right]^{1-s} \left[ \Gamma_{p,k}(x + k) \right]^s. \tag{7}
\]
Substituting (5) into inequality (7) yields;
\[
\Gamma_{p,k}(x + sk) \leq \left( \frac{pkx}{x + pk + k} \right)^s \Gamma_{p,k}(x). \tag{8}
\]
Replacing $s$ by $1 - s$ in inequality (8) gives
\[
\Gamma_{p,k}(x + k - sk) \leq \left( \frac{pkx}{x + pk + k} \right)^{1-s} \Gamma_{p,k}(x). \tag{9}
\]
Further, upon substituting for $x$ by $x + sk$, we obtain
\[
\Gamma_{p,k}(x + k) \leq \left( \frac{pk(x + sk)}{x + sk + pk + k} \right)^{1-s} \Gamma_{p,k}(x + sk). \tag{10}
\]
Now combining (8) and (10) gives
\[
\frac{\Gamma_{p,k}(x + k)}{\left( \frac{pk(x + sk)}{x + sk + pk + k} \right)}^{1-s} \leq \Gamma_{p,k}(x + sk) \leq \left( \frac{pkx}{x + pk + k} \right)^s \Gamma_{p,k}(x)
\]
which by (5) can be written as
\[
\frac{\left( \frac{pkx}{x + pk + k} \right)}{\left( \frac{pk(x + sk)}{x + sk + pk + k} \right)}^{1-s} \Gamma_{p,k}(x) \leq \Gamma_{p,k}(x + sk) \leq \left( \frac{pkx}{x + pk + k} \right)^s \Gamma_{p,k}(x). \tag{11}
\]
Finally, (11) can be rearranged as
\[
\left( \frac{px}{x+pk+k} \right)^{1-s} \leq \frac{\Gamma_{p,k}(x+sk)}{\Gamma_{p,k}(x)} \leq \left( \frac{px}{x+pk+k} \right)^{s}
\]
concluding the proof.

**Theorem 3.2.** Let \( p \in \mathbb{N}, k > 0 \) and \( s \in (0,1) \). Then the inequality
\[
\left( \frac{px}{x+pk+k} \right)^{1-s} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+sk)} \leq \left( \frac{px}{x+sk+pk+k} \right)^{s}
\]
holds for \( x > 0 \).

**Proof.** The inequality (6) implies
\[
\left( \frac{px}{x+pk+k} \right)^{1-s} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+sk)} \leq \left( \frac{px}{x+pk+k} \right)^{s}
\]
which by inversion yields
\[
\left( \frac{px}{x+pk+k} \right)^{-s} \leq \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(x+sk)} \leq \left( \frac{px}{x+sk+pk+k} \right)^{1-s}
\]
Then, substituting the identity (5) into (13) completes the proof.

**Remark 3.3.** Let \( k = 1 \) and \( p \to \infty \) in (12). Then, we obtain
\[
x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+s)^{1-s}
\]
which is an improvement of the Gaußche’s inequality [3, eqn. (7)].

**Corollary 3.4.** Let \( p \in \mathbb{N} \) and \( k > 0 \). Then the inequality
\[
\left( \frac{px}{x+pk+k} \right)^{\frac{1}{2}} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+1/2)} \leq \left( \frac{px}{x+pk+k+1/2} \right)^{1-\frac{1}{2k}}
\]
holds for \( x > 0 \).

**Proof.** This follows from Theorem 3.2 by letting \( s = \frac{1}{2k} \).

**Remark 3.5.** As a consequence of inequality (6), we obtain
\[
\lim_{x \to \infty} \left( \frac{px}{x+pk+k} \right)^{s} \frac{\Gamma_{p,k}(x+sk)}{\Gamma_{p,k}(x)} = 1, \quad s \in (0,1).
\]

**Remark 3.6.** Let \( \alpha, \beta \in (0,1) \). Then by (16), we obtain
\[
\lim_{x \to \infty} \left( \frac{px}{x+pk+k} \right)^{\beta-\alpha} \frac{\Gamma_{p,k}(x+\alpha k)}{\Gamma_{p,k}(x+\beta k)} = 1.
\]
Remark 3.7. We note that the limits (16) and (17) are the \((p,k)\)-analogues of the classical Wendel’s asymptotic relation given by [12]
\[
\lim_{x \to \infty} \frac{\Gamma(x + s)}{x^s \Gamma(x)} = 1.
\]

Remark 3.8. By letting \(p \to \infty\) as \(k \to 1\) in (6), we obtain (3).

Remark 3.9. By letting \(p \to \infty\) in (15), we obtain
\[
x^{1 - \frac{1}{2k}} \leq \frac{\Gamma(x + k)}{\Gamma_k(x + \frac{1}{2})} \leq \left( x + \frac{1}{2} \right)^{1 - \frac{1}{2k}}
\]
which gives a \(k\)-analogue of (4).

Remark 3.10. By letting \(k \to 1\) in (15), we obtain
\[
\sqrt{\frac{px}{x + p + 1}} \leq \frac{\Gamma_p(x + 1)}{\Gamma_p(x + \frac{1}{2})} \leq \sqrt{\frac{p(x + \frac{1}{2})}{x + p + \frac{3}{2}}}
\]
which gives a \(p\)-analogue of (4).

Remark 3.11. By letting \(p \to \infty\) as \(k \to 1\) in (15), we obtain (4).

Theorem 3.12. Let \(p \in \mathbb{N}\) and \(k > 0\). Then, the inequality
\[
e^{(x-y)\psi_{p,k}(y)} < \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < e^{(x-y)\psi_{p,k}(x)}
\]
holds for \(x > y > 0\).

Proof. Let \(H\) be defined for \(p \in \mathbb{N}, k > 0\) and \(t > 0\) by \(H(t) = \ln \Gamma_{p,k}(t)\). Further, let \((y, x)\) be fixed. Then, by the classical mean value theorem, there exists a \(\lambda \in (y, x)\) such that
\[
H'(\lambda) = \frac{\ln \Gamma_{p,k}(x) - \ln \Gamma_{p,k}(y)}{x - y} = \psi_{p,k}(\lambda).
\]
Thus,
\[
\psi_{p,k}(\lambda) = \frac{1}{x - y} \ln \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)}.
\]
Recall that \(\psi_{p,k}(t)\) is increasing for \(t > 0\) (see [8]). Then for \(\lambda \in (y, x)\) we have
\[
\psi_{p,k}(y) < \frac{1}{x - y} \ln \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < \psi_{p,k}(x).
\]
That is
\[
(x - y)\psi_{p,k}(y) < \ln \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < (x - y)\psi_{p,k}(x).
\]
Then, by taking exponents, we obtain the result (20).
Corollary 3.13. Let \( p \in \mathbb{N} \) and \( k > s > 0 \). Then, the inequality
\[
e^{(k-s)\psi_{p,k}(x+s)} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+s)} < e^{(k-s)\psi_{p,k}(x+k)}
\]
holds for \( x > 0 \).

Proof. This follows from Theorem 3.12 upon replacing \( x \) and \( y \) respectively by \( x+k \) and \( x+s \).

Remark 3.14. In particular, if \( s = \frac{1}{2} \), then inequality (21) becomes
\[
e^{(k-\frac{1}{2})\psi_{p,k}(x+\frac{1}{2})} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+\frac{1}{2})} < e^{(k-\frac{1}{2})\psi_{p,k}(x+k)}
\]

Remark 3.15. The inequality (20) provides a \((p,k)\)-analogue of the result
\[
e^{(x-y)\psi(y)} < \frac{\Gamma(x)}{\Gamma(y)} < e^{(x-y)\psi(x)}
\]
for \( x > y > 0 \), which was established in [9, Corollary 2].

Remark 3.16. Inequality (21) provides a generalization of [4, Theorem 3.1].

4. Conclusion

We have established some inequalities bounding the ratio \( \Gamma_{p,k}(x)/\Gamma_{p,k}(y) \), where \( \Gamma_{p,k}(\cdot) \) is the \((p,k)\)-analogue of the Gamma function. From the established results, we recover some known results in the literature.

REFERENCES

1 Department of Mathematics, University for Development Studies, Navrongo Campus, P. O. Box 24, Navrongo, UE/R, Ghana.

E-mail address: mykwarasoft@yahoo.com, knantomah@uds.edu.gh