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SOME INEQUALITIES BOUNDING CERTAIN RATIOS OF THE (p,k)-GAMMA FUNCTION

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ABSTRACT. In this paper, we establish some inequalities bounding the ratio $\Gamma_{p,k}(x)/\Gamma_{p,k}(y)$, where $\Gamma_{p,k}(.)$ is the (p,k)-analogue of the Gamma function. Consequently, some previous results are recovered from the obtained results.

1. INTRODUCTION

Inequalities that provide bounds for the ratio $\Gamma(x)/\Gamma(y)$, where x and y are numbers of some special form, have been studied intensively by several researchers across the globe. A detailed account on inequalities of this nature can be found in the survey article by Qi [10]. In this study, the focus shall be on the type originating from certain problems of traffic flow.

In 1978, Lew, Frauenthal and Keyfitz [5] by studying certain problems of traffic flow established the double-inequality

$$2\Gamma\left(n+\frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right)\Gamma(n+1) \leq 2^n\Gamma\left(n+\frac{1}{2}\right), \quad n \in \mathbb{N} \tag{1}$$

which can be rearranged as

$$\frac{2}{\sqrt{\pi}} \leq \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \leq \frac{2^n}{\sqrt{\pi}}, \quad n \in \mathbb{N}. \tag{2}$$

Then in 2006, Sándor [11] by using the inequality

$$\left(\frac{x}{x+s}\right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s\Gamma(x)} \leq 1, \quad s \in (0,1),\, x > 0 \tag{3}$$

due Wendel [12], extended and refined the inequality (2) by proving the result

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \leq \sqrt{x+\frac{1}{2}} \tag{4}$$

for $x > 0$.

Also, in the paper [6], the authors established the q-analogue of (4) as

$$\sqrt{[x]_q} \leq \frac{\Gamma_q(x+1)}{\Gamma_q\left(x+\frac{1}{2}\right)} \leq \sqrt{\left[x+\frac{1}{2}\right]_q}$$

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for $q \in (0, 1)$ and $x > 0$.

Furthermore, in the paper [7], the authors established the (q, k)-analogue of (4) as

$$[x]_q^{1-\frac{1}{2k}} \leq \frac{\Gamma_{q,k}(x+k)}{\Gamma_{q,k}\left(x+\frac{1}{2}\right)} \leq \left[x+\frac{1}{2}\right]_q^{1-\frac{1}{2k}}$$

for $q \in (0, 1)$, $k > 0$ and $x > 0$.

The main objective of this paper is to establish similar inequalities for the (p, k)-analogue of the Gamma function.

2. Preliminaries

The classical Euler's Gamma function, $\Gamma(x)$ is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\dots(x+n)}$$

Closely related to the Gamma function is the Digamma function, $\psi(x)$ which is defined for $x > 0$ as $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$.

Euler gave another definition of the Gamma function called the p-analogue, which is defined for $p \in \mathbb{N}$ and $x > 0$ as (see [1, p. 270])

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\dots(x+p)}$$

with the p-analogue of the Digamma function defined as $\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x)$.

Also, Díaz and Pariguan [2] defined the k-analogues of the Gamma and Digamma functions as

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt \quad \text{and} \quad \psi_k(x) = \frac{d}{dx}\ln\Gamma_k(x)$$

for $k > 0$ and $x \in \mathbb{C}\backslash k\mathbb{Z}^-$.

Then in a recent paper [8], the authors introduced a (p, k)-analogue of the Gamma function defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\begin{aligned}\Gamma_{p,k}(x) &= \int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt \\ &= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)}\end{aligned}$$

satisfying the properties

$$\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k}\Gamma_{p,k}(x) \tag{5}$$
$$\Gamma_{p,k}(ak) = \frac{p+1}{p}k^{a-1}\Gamma_p(a), \quad a \in \mathbb{R}^+$$
$$\Gamma_{p,k}(k) = 1.$$

The (p,k)-analogue of the Digamma function is defined for $x > 0$ as

$$\psi_{p,k}(x) = \frac{d}{dx}\ln\Gamma_{p,k}(x) = \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+x}$$
$$= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt$$

Also, the (p,k)-analogue of the Polygamma functions are defined as

$$\psi_{p,k}^{(m)}(x) = \frac{d^m}{dx^m}\psi_{p,k}(x) = \sum_{n=0}^{p}\frac{(-1)^{m+1}m!}{(nk+x)^{m+1}}$$
$$= (-1)^{m+1}\int_0^\infty \left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}\right)t^m e^{-xt}\,dt$$

where $m \in \mathbb{N}$, and $\psi_{p,k}^{(0)}(x) \equiv \psi_{p,k}(x)$.

The functions $\Gamma_{p,k}(x)$ and $\psi_{p,k}(x)$ satisfy the following commutative diagrams.

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p\to\infty} & \Gamma_k(x) \\ {\scriptstyle k\to 1}\downarrow & & \downarrow{\scriptstyle k\to 1} \\ \Gamma_p(x) & \xrightarrow[p\to\infty]{} & \Gamma(x) \end{array} \qquad \begin{array}{ccc} \psi_{p,k}(x) & \xrightarrow{p\to\infty} & \psi_k(x) \\ {\scriptstyle k\to 1}\downarrow & & \downarrow{\scriptstyle k\to 1} \\ \psi_p(x) & \xrightarrow[p\to\infty]{} & \psi(x) \end{array}$$

We now present the main findings of the paper in the following section.

3. Main Results

Lemma 3.1. *Let $p \in \mathbb{N}$, $k > 0$ and $s \in (0,1)$. Then the inequality*

$$\frac{\left(\frac{pkx}{x+pk+k}\right)^{1-s}}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \le \frac{\Gamma_{p,k}(x+sk)}{\left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x)} \le 1 \tag{6}$$

holds for $x > 0$.

Proof. We employ the Hölder's inequality for integrals, which is stated for any integrable functions $f, g : (0,a) \to \mathbb{R}$ as

$$\int_0^a |f(t)g(t)|\,dt \le \left[\int_0^a |f(t)|^\alpha\,dt\right]^{\frac{1}{\alpha}}\left[\int_0^a |g(t)|^\beta\,dt\right]^{\frac{1}{\beta}}$$

where $\alpha > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. We proceed as follows. Let
$\alpha = \frac{1}{1-s}$, $\beta = \frac{1}{s}$, $f(t) = t^{(1-s)(x-1)}\left(1 - \frac{t^k}{pk}\right)^{p(1-s)}$, $g(t) = t^{s(x+k-1)}\left(1 - \frac{t^k}{pk}\right)^{ps}$.
Then,

$$\begin{aligned}
\Gamma_{p,k}(x+sk) &= \int_0^p t^{x+sk-1}\left(1-\frac{t^k}{pk}\right)^p dt \\
&= \int_0^p t^{(1-s)(x-1)}\left(1-\frac{t^k}{pk}\right)^{p(1-s)} .t^{s(x+k-1)}\left(1-\frac{t^k}{pk}\right)^{ps} dt \\
&\leq \left[\int_0^p \left(t^{(1-s)(x-1)}\left(1-\frac{t^k}{pk}\right)^{p(1-s)}\right)^{\frac{1}{1-s}} dt\right]^{1-s} \times \\
&\quad \left[\int_0^p \left(t^{s(x+k-1)}\left(1-\frac{t^k}{pk}\right)^{ps}\right)^{\frac{1}{s}} dt\right]^{s} \\
&= \left[\int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt\right]^{1-s}\left[\int_0^p t^{x+k-1}\left(1-\frac{t^k}{pk}\right)^p dt\right]^s \\
&= \left[\Gamma_{p,k}(x)\right]^{1-s}\left[\Gamma_{p,k}(x+k)\right]^s.
\end{aligned}$$

That is,

$$\Gamma_{p,k}(x+sk) \leq \left[\Gamma_{p,k}(x)\right]^{1-s}\left[\Gamma_{p,k}(x+k)\right]^s. \tag{7}$$

Substituting (5) into inequality (7) yields;

$$\Gamma_{p,k}(x+sk) \leq \left(\frac{pkx}{x+pk+k}\right)^s \Gamma_{p,k}(x). \tag{8}$$

Replacing s by $1-s$ in inequality (8) gives

$$\Gamma_{p,k}(x+k-sk) \leq \left(\frac{pkx}{x+pk+k}\right)^{1-s} \Gamma_{p,k}(x). \tag{9}$$

Further, upon substituting for x by $x+sk$, we obtain

$$\Gamma_{p,k}(x+k) \leq \left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s} \Gamma_{p,k}(x+sk). \tag{10}$$

Now combining (8) and (10) gives

$$\frac{\Gamma_{p,k}(x+k)}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \leq \Gamma_{p,k}(x+sk) \leq \left(\frac{pkx}{x+pk+k}\right)^s \Gamma_{p,k}(x)$$

which by (5) can be written as

$$\frac{\left(\frac{pkx}{x+pk+k}\right)}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}}\Gamma_{p,k}(x) \leq \Gamma_{p,k}(x+sk) \leq \left(\frac{pkx}{x+pk+k}\right)^s \Gamma_{p,k}(x). \tag{11}$$

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Finally, (11) can be rearranged as

$$\frac{\left(\frac{pkx}{x+pk+k}\right)^{1-s}}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \leq \frac{\Gamma_{p,k}(x+sk)}{\left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x)} \leq 1$$

concluding the proof.

Theorem 3.2. *Let $p \in \mathbb{N}$, $k > 0$ and $s \in (0,1)$. Then the inequality*

$$\left(\frac{pkx}{x+pk+k}\right)^{1-s} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}\left(x+sk\right)} \leq \left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s} \tag{12}$$

holds for $x > 0$.

Proof. The inequality (6) implies

$$\frac{\left(\frac{pkx}{x+pk+k}\right)}{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}} \leq \frac{\Gamma_{p,k}\left(x+sk\right)}{\Gamma_{p,k}(x)} \leq \left(\frac{pkx}{x+pk+k}\right)^{s}$$

which by inversion yields

$$\left(\frac{pkx}{x+pk+k}\right)^{-s} \leq \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}\left(x+sk\right)} \leq \frac{\left(\frac{pk(x+sk)}{x+sk+pk+k}\right)^{1-s}}{\left(\frac{pkx}{x+pk+k}\right)}. \tag{13}$$

Then, substituting the identity (5) into (13) completes the proof.

Remark 3.3. Let $k = 1$ and $p \to \infty$ in (12). Then, we obtain

$$x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma\left(x+s\right)} \leq (x+s)^{1-s} \tag{14}$$

which is an improvement of the Gautschi's inequality [3, eqn. (7)].

Corollary 3.4. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\left(\frac{pkx}{x+pk+k}\right)^{1-\frac{1}{2k}} \leq \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}\left(x+\frac{1}{2}\right)} \leq \left(\frac{pk(x+\frac{1}{2})}{x+pk+k+\frac{1}{2}}\right)^{1-\frac{1}{2k}} \tag{15}$$

holds for $x > 0$.

Proof. This follows from Theorem 3.2 by letting $s = \frac{1}{2k}$.

Remark 3.5. As a consequence of inequality (6), we obtain

$$\lim_{x\to\infty} \frac{\Gamma_{p,k}(x+sk)}{\left(\frac{pkx}{x+pk+k}\right)^{s}\Gamma_{p,k}(x)} = 1, \quad s \in (0,1). \tag{16}$$

Remark 3.6. Let $\alpha, \beta \in (0,1)$. Then by (16), we obtain

$$\lim_{x\to\infty} \left(\frac{pkx}{x+pk+k}\right)^{\beta-\alpha} \frac{\Gamma_{p,k}(x+\alpha k)}{\Gamma_{p,k}(x+\beta k)} = 1. \tag{17}$$

Remark 3.7. We note that the limits (16) and (17) are the (p,k)-analogues of the classical Wendel's asymptotic relation given by [12]

$$\lim_{x\to\infty}\frac{\Gamma(x+s)}{x^s\Gamma(x)}=1.$$

Remark 3.8. By letting $p\to\infty$ as $k\to 1$ in (6), we obtain (3).

Remark 3.9. By letting $p\to\infty$ in (15), we obtain

$$x^{1-\frac{1}{2k}} \le \frac{\Gamma_k(x+k)}{\Gamma_k\left(x+\frac{1}{2}\right)} \le \left(x+\frac{1}{2}\right)^{1-\frac{1}{2k}} \tag{18}$$

which gives a k-analogue of (4).

Remark 3.10. By letting $k\to 1$ in (15), we obtain

$$\sqrt{\left(\frac{px}{x+p+1}\right)} \le \frac{\Gamma_p(x+1)}{\Gamma_p\left(x+\frac{1}{2}\right)} \le \sqrt{\left(\frac{p(x+\frac{1}{2})}{x+p+\frac{3}{2}}\right)} \tag{19}$$

which gives a p-analogue of (4).

Remark 3.11. By letting $p\to\infty$ as $k\to 1$ in (15), we obtain (4).

Theorem 3.12. *Let $p\in\mathbb{N}$ and $k>0$. Then, the inequality*

$$e^{(x-y)\psi_{p,k}(y)} < \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < e^{(x-y)\psi_{p,k}(x)} \tag{20}$$

holds for $x>y>0$.

Proof. Let H be defined for $p\in\mathbb{N}$, $k>0$ and $t>0$ by $H(t)=\ln\Gamma_{p,k}(t)$. Further, let (y,x) be fixed. Then, by the classical mean value theorem, there exists a $\lambda\in(y,x)$ such that

$$H'(\lambda)=\frac{\ln\Gamma_{p,k}(x)-\ln\Gamma_{p,k}(y)}{x-y}=\psi_{p,k}(\lambda).$$

Thus,

$$\psi_{p,k}(\lambda)=\frac{1}{x-y}\ln\frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)}.$$

Recall that $\psi_{p,k}(t)$ is increasing for $t>0$ (see [8]). Then for $\lambda\in(y,x)$ we have

$$\psi_{p,k}(y) < \frac{1}{x-y}\ln\frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < \psi_{p,k}(x).$$

That is

$$(x-y)\psi_{p,k}(y) < \ln\frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} < (x-y)\psi_{p,k}(x).$$

Then, by taking exponents, we obtain the result (20).

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Corollary 3.13. *Let $p \in \mathbb{N}$ and $k > s > 0$. Then, the inequality*

$$e^{(k-s)\psi_{p,k}(x+s)} < \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+s)} < e^{(k-s)\psi_{p,k}(x+k)} \tag{21}$$

holds for $x > 0$.

Proof. This follows from Theorem 3.12 upon replacing x and y respectively by $x + k$ and $x + s$.

Remark 3.14. In particular, if $s = \frac{1}{2}$, then inequality (21) becomes

$$e^{(k-\frac{1}{2})\psi_{p,k}(x+\frac{1}{2})} < \frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+\frac{1}{2})} < e^{(k-\frac{1}{2})\psi_{p,k}(x+k)} \tag{22}$$

Remark 3.15. The inequality (20) provides a (p, k)-analogue of the result

$$e^{(x-y)\psi(y)} < \frac{\Gamma(x)}{\Gamma(y)} < e^{(x-y)\psi(x)} \tag{23}$$

for $x > y > 0$, which was established in [9, Corollary 2] .

Remark 3.16. Inequality (21) provides a generalization of [4, Theorem 3.1].

4. Conclusion

We have established some inequalities bounding the ratio $\Gamma_{p,k}(x)/\Gamma_{p,k}(y)$, where $\Gamma_{p,k}(.)$ is the (p, k)-analogue of the Gamma function. From the established results, we recover some known results in the literature.

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