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Article

Windings of Planar Processes and Applications to the Pricing of Asian Options

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Abstract: Motivated by a common Mathematical Finance topic, this paper surveys several results concerning windings of 2-dimensional processes, including planar Brownian motion, complex-valued Ornstein-Uhlenbeck processes and planar stable processes. In particular, we present Spitzer's asymptotic Theorem for each case. We also relate this study to the pricing of Asian options.

Keywords: planar Brownian motion; complex-valued Ornstein-Uhlenbeck processes; Lévy processes; Stable processes; windings; skew-product representation; Spitzer's Theorem; Bougerol's identity; Asian options; pricing

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1. Introduction

Windings of 2-dimensional processes, and especially of planar Brownian motion have several applications, namely in Finance. More precisely, in Financial Mathematics, the exponential functionals of Brownian motion are of special interest. A fundamental example is the pricing of Asian options (see e.g. [8,11,31,33]), where the payout of an Asian call option is given by:

$$E\left[\left(\frac{1}{t}\int_0^t ds\ \exp(\beta_s+\nu s)-K\right)^+\right],$$

where $(\beta_u, u \geq 0)$ is a real Brownian motion, $\nu \in \mathbb{R}$ and $K \in \mathbb{R}_+$ is the strike price. It is easy to show (for further details, see e.g. [11]) that the computation of this expectation simplifies to the computation of

$$E\left[\left(\int_0^t ds\ \exp(\beta_s+\nu s)-K\right)^+\right],$$

which follows by studying

$$E\left[\int_0^t ds\ \exp(\beta_s+\nu s)\right].$$

In particular, in [31] one can find a more detailed discussion for the distribution of the exponential functional $A_t^{(\nu)} := \int_0^t ds\ \exp(\beta_s+\nu s)$ taken up to a random time T_λ which follows the exponential distribution with parameter $\lambda > 0$. More precisely, Yor [30] obtained that

$$2A_{T_\lambda}^{(\nu)} \overset{(law)}{=} \frac{Q_{1,a}}{2G_b} \overset{(law)}{=} \frac{1-\mathcal{U}^{1/a}}{2G_b},$$

where $Q_{1,a} \sim \text{Beta}(1,a)$, $G_b \sim \text{Gamma}(b)$, $\mathcal{U} \sim U[0,1]$, $a = (\nu/2)+(1/2)\sqrt{2\lambda+\nu^2}$, $b = a-\nu$ and the random variables are independent.

In this paper, we will study this exponential functional in terms of planar Brownian motion, i.e. taken up to a random time different from the case mentioned above, that is (we suppose now that $\nu = 0$ but at the end we will also discuss the case $\nu \neq 0$):

$$\int_0^{T_c^\gamma} ds \ \exp(2\beta_s) \,,$$

where $T_c^\gamma = \inf\{u \geq 0 : \gamma_u = c\}$, $c > 0$, and $(\gamma_u, u \geq 0)$ another real Brownian motion independent from β. For more precise connection with the windings, see Proposition 2 below.

We consider the following processes:

- $(Z_t, t \geq 0)$ a planar Brownian motion (BM),
- $(V_t, t \geq 0)$ a complex-valued Ornstein-Uhlenbeck (OU) process, i.e.: with $\lambda \leq 0$,

$$V_t = V_0 + Z_t - \lambda \int_0^t V_s ds \,, \tag{1}$$

or equivalently, with $(B_t, t \geq 0)$ another planar Brownian motion starting from V_0:

$$V_t = e^{-\lambda t} \left(V_0 + \int_0^t e^{\lambda s} dZ_s \right) = e^{-\lambda t} B_{\alpha_t}, \tag{2}$$

and

- $(U_t, t \geq 0)$ a planar Stable process of index $\alpha \in (0, 2)$,

all of them starting from a point different from 0 (without loss of generality we may consider all of them start from 1).

It is well-known [13] that since $Z_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around it infinitely often. Hence, its continuous winding process $\theta_t^Z = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined. We also recall the skew-product representation of planar BM (see also e.g. [19]):

$$\log |Z_t| + i\theta_t^Z = \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u = H_t^Z = \int_0^t \frac{ds}{|Z_s|^2}} \,, \tag{3}$$

with $(\beta_u + i\gamma_u, u \geq 0)$ denoting another planar Brownian motion starting from $\log 1 + i0 = 0$ (for the Bessel clock H^Z, see also [29]).
We easily deduce that the two σ-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$. Note that the inverse of H^Z is:

$$A^Z(t) = \inf\{u \geq 0, H^Z(u) > t\} = \int_0^t ds \ \exp(2\beta_s) \,. \tag{4}$$

Similarly, for the OU process V we have that its continuous winding process, i.e. $\theta_t^V = \mathrm{Im}(\int_0^t \frac{dV_s}{V_s}), t \geq 0$ is also well defined.

Following [22,24], we have:

Proposition 1. *The following identity holds:*

$$\theta_t^V = \theta_{\alpha_t}^B, \tag{5}$$

where $\alpha_t = \frac{e^{2\lambda t} - 1}{2\lambda}$.

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Proof. Applying Itô's formula to (2) and dividing by V_s, we obtain:

$$\frac{dV_s}{V_s} = -\lambda \, ds + \frac{dB_{\alpha_s}}{B_{\alpha_s}} \,, \tag{6}$$

hence:

$$\mathrm{Im}\left(\frac{dV_s}{V_s}\right) = \mathrm{Im}\left(\frac{dB_{\alpha_s}}{B_{\alpha_s}}\right),$$

and (5) follows easily. □

Concerning now the Stable process U (see also [3,7,14]), contrary to planar Brownian motion, we cannot define its winding number directly. However, we can consider a path on a finite time interval $[0,t]$ and "fill in" the gaps with line segments in order to obtain the curve of a continuous function $f:[0,1]\to\mathbb{C}$ with $f(0)=1$. 0 is polar and V has no jumps across 0 a.s., thus we have $f(u)\neq 0$ for every $u\in[0,1]$ and the process of the winding number of around 0, $\theta^U=(\theta^U_t, t\geq 0)$ is well-defined, it has cadlag paths of absolute length greater than π and, for all $t\geq 0$,

$$\exp(i\theta^U_t) = \frac{U_t}{|U_t|} \,. \tag{7}$$

We also introduce the clock:

$$H^U(t) = \int_0^t \frac{ds}{|U_s|^\alpha} \,, \tag{8}$$

and its inverse:

$$A^U(u) = \inf\{t\geq 0, H^U(t) > u\} \,. \tag{9}$$

For each process, we will study the exit times from a cone of single and of double border,that is:

$$T^{\theta^W}_c = \inf\{t : \theta^W_t = c\} \,, \tag{10}$$

$$T^{|\theta^W|}_c = \inf\{t : |\theta^W_t| = c\} \,, \tag{11}$$

where $W=Z, V$ or U. Moreover, we will study the asymptotic behavior of each winding process.

The rest of this paper is organized as follows: In Section 2 we discuss windings and the associated version of Spitzer's Asymptotic Theorem (that correspond to the large time asymptotics) (i) for planar Brownian motion, (ii) for complex-valued Ornstein-Uhlenbeck processes, and (iii) for planar Stable processes. In particular, in Subsection 2.1 we characterize the distribution of the exit times from a single and from a double border cone via their Gauss-Laplace transform, that we prove by using Bougerol's identity in law, and we also illustrate the formula for the second random time by giving two examples. We also comment briefly to the small time asymptotics case. Section 3 deals with applications of the previous results to the pricing of Asian options. More precisely, we discuss separately the case of exponential functionals of Brownian motion and of Lévy processes. Finally, in the Appendix A we give a comparative table concerning the clocks associated to each process, i.e. H^Z, H^V and H^U and Spitzer's asymptotic Theorem for planar Brownian motion, for complex-valued Ornstein-Uhlenbeck and for planar Stable processes, followed by some comments and remarks.

2. Windings of planar processes, exponential functionals and Spitzer's Theorem

2.1. The planar Brownian motion case

First, we recall our main tool, that is Bougerol's celebrated identity in law [6], stating that with $(\beta_u, u\geq 0)$ and $(\hat{\beta}_u, u\geq 0)$ denoting two independent linear Brownian motions for $t\geq 0$ fixed,

$$\sinh(\beta_t) \overset{(law)}{=} \hat{\beta}_{A^Z_t(\beta)=\int_0^t ds \exp(2\beta_s)} \,. \tag{12}$$

For the proof and other developments of this identity, see [23] and the references therein. We will study Bougerol's identity in law in terms of planar Brownian motion, which is strongly related to exponential functionals of Brownian motion as one can see below. To that end, we shall also need the following exit times for the BM γ associated to θ^Z:

$$T_c^{\gamma} = \inf\{t : \gamma_t = c\} \, , \quad T_c^{|\gamma|} = \inf\{t : |\gamma_t| = c\} \, .$$

A first result is the following:

Proposition 2. *We have the following relations:*

$$T_c^{\theta^Z} = A^Z_{T_c^{\gamma}}, \text{ and } T_c^{|\theta^Z|} = A^Z_{T_c^{|\gamma|}}.$$

Proof. It follows by the skew-product representation ($\theta_t^Z = \gamma_{H_t^Z}$), using the fact that A^Z is the inverse of H^Z (see also (4)):

$$T_c^{\theta^Z} = \inf\{t : \theta_t^Z = c\} = \inf\{t : \gamma_{H_t^Z} = c\} \overset{s=H_t^Z}{=} \inf\{A_s^Z : \gamma_s = c\} = A^Z_{T_c^{\gamma}} \, . \tag{13}$$

The second relation follows similarly. □

From now on, all the results may be stated either for $A^Z_{T_c^{\gamma}}$ (resp. $A^Z_{T_c^{|\gamma|}}$) or for $T_c^{\theta^Z}$ (resp. $T_c^{|\theta^Z|}$). For the sake of applications in the Mathematical Finance framework, we will mostly use the first notation.

We state now Spitzer's celebrated asymptotic Theorem for planar BM [20]:

Theorem 3 (Spitzer's Asymptotic Theorem (1958))**.** *The following convergence in law holds:*

$$\frac{2}{\log t}\, \theta_t^Z \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_1 \, , \tag{14}$$

with C_1 denoting a standard Cauchy variable.

There exist several proofs, based on different approaches (see e.g. [2,9,15,17,22,28,32]). Here, following [26] (see also [15]), we propose an easy proof based on Proposition 2.

Proof. First,

$$T_c^{\theta^Z} = A^Z_{T_c^{\gamma}}(\beta) \overset{(law)}{=} A_u^Z(\beta)\Big|_{u=c^2 T_1^{\gamma}}$$

thus:

$$\frac{1}{c^2} T_c^{\theta^Z} \overset{(law)}{=} \int_0^{T_1^{\gamma}} dv \, \exp\left(2c\beta_v\right) \, . \tag{15}$$

For $c \to \infty$, taking logarithms on both sides of (15) and dividing by c, the LHS gives: $\frac{1}{c} \log\left(T_c^{\theta}\right) - \frac{2}{c} \log c$, and the RHS writes:

$$\frac{1}{c} \log\left(\int_0^{T_1^{\gamma}} dv \, \exp\left(2c\beta_v\right)\right) = \log\left(\int_0^{T_1^{\gamma}} dv \, \exp\left(2c\beta_v\right)\right)^{1/c} .$$

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Invoking the classical Laplace argument: $\|f\|_p \overset{p\to\infty}{\longrightarrow} \|f\|_\infty$, the latter converges for $c \to \infty$ to:

$$2 \sup_{v \le T_1^\gamma} (\beta_v) \overset{(law)}{=} 2|\beta|_{T_1^\gamma}.$$

Hence:

$$\frac{1}{c} \log\left(T_c^{\theta^Z}\right) \xrightarrow[c\to\infty]{(law)} 2|\beta|_{T_1^\gamma}. \tag{16}$$

which can be equivalently stated as (recall that $\beta_{T_1^\gamma} \overset{(law)}{=} C_1$)

$$P\left(\log T_c^{\theta^Z} < cx\right) \xrightarrow[c\to\infty]{(law)} P\left(2|C_1| < x\right). \tag{17}$$

The LHS of (17) equals:

$$\begin{aligned} P\left(\log T_c^{\theta^Z} < cx\right) &= P\left(T_c^{\theta^Z} < \exp(cx)\right) = P\left(\sup_{u\le \exp(cx)} \theta_u^Z > c\right) \\ &= P\left(|\theta^Z_{\exp(cx)}| > c\right) = P\left(|\theta_t^Z| > \frac{\log t}{x}\right), \end{aligned} \tag{18}$$

where $t = \exp(cx)$. Recalling now the fact that $|C_1| \overset{(law)}{=} |C_1|^{-1}$, from (17) we get:

$$\textit{for every } x > 0 \textit{ given, } P\left(|\theta_t| > \frac{\log t}{x}\right) \xrightarrow[t\to\infty]{(law)} P\left(|C_1| > \frac{2}{x}\right), \tag{19}$$

hence we obtain precisely Spitzer's Theorem (14). $\square$

Remark 1. Spitzer's Asymptotic Theorem can be equivalently stated in terms of the clock H^Z, i.e.:

$$\frac{4}{(\log t)^2} H^Z(t) \xrightarrow[t\to\infty]{(law)} T_1^\gamma. \tag{20}$$

Following [21,22], we have:

Proposition 4. *The distributions of* $A^Z_{T_c^\gamma}$ *and of* $A^Z_{T_c^{|\gamma|}}$ *are characterized by the following Gauss-Laplace transforms ($x \ge 0$ and $m = \frac{\pi}{2c}$):*

$$c\, E\left[\sqrt{\frac{\pi}{2A^Z_{T_c^\gamma}}} \exp\left(-\frac{x}{2A^Z_{T_c^\gamma}}\right)\right] = \frac{1}{\sqrt{1+x}} \frac{c^2}{\left(c^2 + \log^2(\sqrt{x} + \sqrt{1+x})\right)}, \tag{21}$$

$$c\, E\left[\sqrt{\frac{2}{\pi A^Z_{T_c^{|\gamma|}}}} \exp\left(-\frac{x}{2A^Z_{T_c^{|\gamma|}}}\right)\right] = \frac{1}{\sqrt{1+x}} \frac{2}{(\sqrt{1+x} + \sqrt{x})^m + (\sqrt{1+x} - \sqrt{x})^m}. \tag{22}$$

Remark 2. From e.g. formula (22) with some analytic computations, we can obtain the density function of $A^Z_{T_c^{|\gamma|}}$. For further details, see e.g. [21,22].

Examples

We first illustrate formula (22) by 2 examples that come essentially from [25]:

1. $\boxed{m = 1 \Rightarrow c = \frac{\pi}{2}}$
   Formula (22) states that $(A^Z_{T^{|\gamma|}_{\pi/2}}\ T^{|\theta|}_{\pi/2})$:

$$E\left[\sqrt{\frac{2}{\pi A^Z_{T^{|\gamma|}_{\pi/2}}}}\exp\left(-\frac{x}{2A^Z_{T^{|\gamma|}_{\pi/2}}}\right)\right] \quad = \quad \frac{1}{1+x}\,. \tag{23}$$

   We will verify this formula by simple computations. Indeed, we have that:

$$A^Z_{T^{|\gamma|}_{\pi/2}} = T^{|\theta|}_{\pi/2} = \inf\{t : X_t = 0\} = \inf\{t : X^0_t = 1\},$$

   where $(X^0_t, t \geq 0)$ is another real BM starting from 0. However, with $N \sim \mathcal{N}(0,1)$, $A^Z_{T^{|\gamma|}_{\pi/2}} \overset{(law)}{=} \frac{1}{N^2}$, hence, the LHS of (23) gives:

$$E\left[\sqrt{\frac{2}{\pi}}|N|\exp\left(-\frac{x}{2}N^2\right)\right] = \int_0^\infty dy\; y\; e^{-\frac{x+1}{2}y^2} = \frac{1}{1+x}\,, \tag{24}$$

   thus we get directly (23). Note that on the RHS of (23) we have the Laplace transform of an exponential variable of parameter 1, denoted by $\mathbf{e}_1$.
2. $\boxed{m = 2 \Rightarrow c = \frac{\pi}{4}}$
   Similarly, (22) writes:

$$E\left[\sqrt{\frac{\pi}{4(T_1 \wedge \tilde{T}_1)}}\exp\left(-\frac{x}{T_1 \wedge \tilde{T}_1}\right)\right] \quad = \quad \frac{1}{\sqrt{1+x}}\frac{1}{1+2x}\,. \tag{25}$$

   With $(N, \tilde{N} \sim \mathcal{N}(0,1))$,

$$\begin{aligned} A^Z_{T^{|\gamma|}_{\pi/4}} = T^{|\theta|}_{\pi/4} &= \inf\{t : X_t + Y_t = 0, \text{ or } X_t - Y_t = 0\} \\ &= \inf\{t : \frac{X^0_t + Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \text{ or } \frac{X^0_t - Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}\} \\ &= T_{1/\sqrt{2}} \wedge \tilde{T}_{1/\sqrt{2}} \overset{(law)}{=} \frac{1}{2}\left(T_1 \wedge \tilde{T}_1\right). \end{aligned}$$

   where for every x, $T_x = \inf\{t, \beta_t = x\}$ and $\tilde{T}$ is an independent copy of T. We remark that $T_1 \overset{(law)}{=} \frac{1}{N^2}$, $\tilde{T}_1 \overset{(law)}{=} \frac{1}{\tilde{N}^2}$, which yields ($C$ is a constant):

$$\begin{aligned} &E\left[\left(|N| \vee |\tilde{N}|\right)\exp\left(-x\left(N^2 \vee \tilde{N}^2\right)\right)\right] = 2E\left[|N|\exp\left(-xN^2\right)1_{(|N| \geq |\tilde{N}|)}\right] \\ &= C\int_0^\infty du\; u\; e^{-xu^2}e^{-\frac{u^2}{2}}\int_0^u dy\; e^{-\frac{y^2}{2}}. \end{aligned}$$

   Using Fubini's theorem we get (25). Note that now, on the RHS of (25) we have the Laplace transform of the variable $\frac{N^2}{2} + 2\mathbf{e}_1 \overset{(law)}{=} \gamma_{1/2} + 2\mathbf{e}_1$.

Proof of Proposition 4. We apply Bougerol's identity (12) for T^γ_c. Hence ($N \sim \mathcal{N}(0,1)$):

$$\sinh(\beta_{T^\gamma_c}) \overset{(law)}{=} \hat{\beta}_{A^Z_{T^\gamma_c}} \overset{(law)}{=} \sqrt{A^Z_{T^\gamma_c}}\,N,$$

which yields that for fixed $c > 0$:

$$\sinh(C_c) \overset{(law)}{=} \hat{\beta}_{(T_c^\theta)},$$

where $(C_c, c \geq 0)$ is a standard Cauchy process. We identify the densities of the two variables:
LHS: $\frac{1}{\sqrt{1+x^2}}\, h_c(\arg\sinh x) = \frac{1}{\sqrt{1+x^2}}\, h_c(a(x))$;
RHS: $E\left[\frac{1}{\sqrt{2\pi A^Z_{T_c^\gamma}}} \exp\left(-\frac{x^2}{2A^Z_{T_c^\gamma}}\right)\right]$,
and we remark that $a(y) = \arg\sinh(y) = \log(y + \sqrt{1+y^2})$. Changing the variables $x = y^2$ we get (21).
Formula (22) follows by Bougerol's identity for $T_c^{|\gamma|}$ and we apply the same arguments as previously, recalling that the density of $\beta_{T_c^{|\gamma|}}$ is $\frac{1}{2c}\frac{1}{\cosh(my)} = \frac{1}{c}\frac{1}{e^{my}+e^{-my}}$ (see e.g. [4]). $\square$

For other results and variants concerning properties of the random times $A^Z_{T_c^\gamma}$ and $A^Z_{T_c^{|\gamma|}}$, the interested reader is addressed to [25,26] and to the references therein.

2.2. The complex-valued Ornstein-Uhlenbeck case

Concerning the exit times from a cone for complex-valued Ornstein-Uhlenbeck processes, we heve (see also [21,22,24]):

Corollary 5. *The following relations hold:*

$$T_c^{\theta^V} = \frac{1}{2\lambda}\log\left(1 + 2\lambda A^Z_{T_c^B}\right); \tag{26}$$

$$T_c^{|\theta^V|} = \frac{1}{2\lambda}\log\left(1 + 2\lambda A^Z_{T_c^{|B|}}\right). \tag{27}$$

Proof. We prove e.g. (26) ((27) follows similarly). By definition and using (5), we have:

$$T_c^{\theta^V} = \inf\left\{t \geq 0 : \theta_t^V = c\right\} = \inf\left\{t \geq 0 : \theta^B_{\alpha_t} = c\right\},$$

thus:

$$T_c^{\theta^V} = \alpha^{-1}\left(T_c^{\theta^B}\right) = \alpha^{-1}\left(A^Z_{T_c^B}\right), \tag{28}$$

with $\alpha^{-1}(t) = \frac{1}{2\lambda}\log(1 + 2\lambda t)$, which yields (26). $\square$

We can also obtain the analogue of Spitzer's Theorem for OU processes:

Theorem 6. (Spitzer's Theorem for OU processes)
We have that:

$$\frac{\theta_t^V}{t} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_\lambda\,, \tag{29}$$

where, C_σ stands for a Cauchy variable with parameter σ.

Proof. Equation (5) yields:

$$\frac{\theta_t^V}{\lambda t} = \frac{\theta^B_{\alpha_t}}{\lambda t} = \frac{\log\alpha_t}{2\lambda t}\,\frac{2\theta^B_{\alpha_t}}{\log\alpha_t}\,.$$

We remark that:

$$\frac{\log\alpha_t}{2\lambda t} \overset{t\to\infty}{\longrightarrow} 1\,, \tag{30}$$

and applying Spitzer's Theorem (14), we obtain (29). $\square$

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Remark 3. In terms of the clock H^V, Spitzer's Theorem for OU processes may be also stated as:

$$\frac{1}{\lambda^2 t^2} H^V(t) \xrightarrow[t\to\infty]{(law)} T_1^\gamma . \tag{31}$$

2.3. The planar Stable process case

We turn now our interest to planar Stable processes. Bertoin and Werner following [12] obtained the following two Lemmas for $\alpha \in (0,2)$ (for the proofs see [3]):

Lemma 7. *The time-changed process $(\theta^U_{A^U(u)}, u \geq 0)$ is a real-valued symmetric Lévy process, say ρ. It has no Gaussian component and its Lévy measure has support in $[-\pi,\pi]$.*

Let us denote now by dz the Lebesgue measure on $\mathbb{C}$. Then, for every complex number $z \neq 0$, $\phi(z)$ stands for the determination of its argument valued in $(-\pi,\pi]$.

Lemma 8. *The Lévy measure of $\theta_{A^U(\cdot)}$ is the image of the Lévy measure ν of U by the mapping $z \to \phi(1+z)$. Consequently, $E[(\theta_{A^U(u)})^2] = uk(\alpha)$, where*

$$k(\alpha) = \frac{\alpha\, 2^{-1+\alpha/2}\Gamma(1+\alpha/2)}{\pi\Gamma(1-\alpha/2)} \int_{\mathbb{C}} |z|^{-2-\alpha} |\phi(1+z)|^2 dz . \tag{32}$$

Concerning now U, we use the analogue of the skew product representation for planar BM which is the Lamperti correspondence for stable processes. Hence, there exist two real-valued Lévy processes $(\xi_u, u \geq 0)$ and $(\rho_u, u \geq 0)$, the first one non-symmetric whereas the second one symmetric, both starting from 0, such that:

$$\log|U_t| + i\theta^U_t = (\xi_u + i\rho_u)\Big|_{u=H^U_t} .$$

Remark 4. $|Z|$ and $Z_{A^U(\cdot)}/|Z_{A^U(\cdot)}|$ are NOT independent. Indeed, the processes $|Z_{A^U(\cdot)}|$ and $Z_{A^U(\cdot)}/|Z_{A^U(\cdot)}|$ jump at the same times hence they cannot be independent. Moreover, $A^U(\cdot)$ depends only upon $|Z|$, hence $|Z|$ and $Z_{A^U(\cdot)}/|Z_{A^U(\cdot)}|$ are not independent. For further discussion on the independence, see e.g. [16], where is shown that an isotropic α-self-similar Markov process has a skew-product structure if and only if its radial and its angular part do not jump at the same time.

Bertoin and Werner in [3] obtained the analogue of Spitzer's asymptotic Theorem 3 for isotropic stable Lévy processes of index $\alpha \in (0,2)$:

Theorem 9. *The family of processes*

$$\left(c^{-1/2}\theta^U_{\exp(ct)}, t \geq 0\right)$$

converges in distribution on $D([0,\infty),\mathbb{R})$ endowed with the Skorohod topology, as $c \to \infty$, to $\left(\sqrt{r(\alpha)}\beta_t, t \geq 0\right)$, where $(\beta_s, s \geq 0)$ is a real valued Brownian motion and

$$r(\alpha) = \frac{\alpha\, 2^{-1-\alpha/2}}{\pi} \int_{\mathbb{C}} |z|^{-2-\alpha} |\phi(1+z)|^2 dz . \tag{33}$$

Proof. We refer to two different proofs:

1. Bertoin and Werner (1996) [3], using an "Ornstein-Uhlenbeck type" process and ergodicity arguments, and
2. Doney and Vakeroudis (2012) [7], using the continuity of the composition function $\rho_{H^U(\cdot)}$ (see [27]).

□

Remark 5. Bertoin and Werner in [3] showed also that the clock $H^U(t)$ satisfies the following a.s. convergence:

$$\frac{1}{\log t} H^U(t) \xrightarrow[t\to\infty]{a.s.} 2^{-\alpha} \frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} . \tag{34}$$

For discussions concerning the small time asymptotic behavior ($t \to 0$) of the windings of a Stable process, we address the interested reader to [7] for the case where our process is issued from a point different from the origin and to [14] when it is issued from the origin. For the latter, one can either apply classical scaling arguments, or use a novel method appealing to tools from the theory of self-similar Markov processes involving path transformations and time changes. This approach may give access to windings of some conditioned versions of stable processes and to the so-called one-dimensional windings, that is the upcrossings of stable processes over the origin.

3. Applications to the pricing of Asian options

3.1. Asian options and exponential functionals of Brownian motion

In this Subection, we return to the initial Financial Mathematics problem, that is the characterisation of the distribution of

$$A_t = \int_0^t \exp(2\beta_u) du,$$

in order to compute $E\left[\left(\frac{1}{t}A_t - K\right)^+\right]$. To that end, one may use the previously stated results to access the distribution of A_t via Williams' so called 'pinching method' [17,28]. Loosely speaking, when Williams studied windings of Brownian motion, instead of working out directly the asymptotics of the winding process θ, he studied the asymptotic behaviour of this process taken at a random time, depending on θ (for similar results but with the use of a random time independent of θ, see e.g. [22]). Next, one simply remarks that the difference between the initial winding process and the subordinated process is finite, and renormalising appropriately, this difference converges to 0. Hence, the asymptotic study of the renormalised subordinated process yields similar results for the renormalised initial one. We propose here to mimic this method for our benefit, by invoking the time changes discussed in the previous Sections.

Proposition 10. *The following convergence in law holds*

$$\frac{1}{t} \log A_t^Z \xrightarrow[t\to\infty]{(law)} 2|\beta|_{T_1^\gamma} \overset{(law)}{=} 2|C_1|, \tag{35}$$

where C_1 is a standard Cauchy random variable.

Proof. First, remark that

$$\log\left(\frac{A^Z_{T_t^\gamma}}{A_t^Z}\right) = \log\left(\frac{\int_0^{T_t^\gamma} \exp(2\beta_u)du}{\int_0^t \exp(2\beta_u)du}\right),$$

which is a random variable that exists (and which seems to be of no other interest here). Renormalising by t, we get

$$\frac{1}{t}\left(\log A^Z_{T_t^\gamma} - \log A_t^Z\right) = \frac{1}{t}\log\left(\frac{A^Z_{T_t^\gamma}}{A_t^Z}\right) \xrightarrow[t\to\infty]{(P)} 0.$$

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Hence, studying asymptotically $t^{-1}\log A^Z_{T^\gamma_t}$ as $t\to\infty$ would yield similar results for $t^{-1}\log A^Z_t$. Following [26], by applying the scaling property of Brownian motion and changing variables we have that (recall that $T^\gamma_t \overset{(law)}{=} t^2 T^\gamma_1$)

$$A^Z_{T^\gamma_t} = \int_0^{T^\gamma_t} e^{2\beta_v} dv \overset{(law)}{=} t^2 \int_0^{T^\gamma_1} e^{2t\beta_u} du.$$

Taking now logarithms and dividing both parts by t we get

$$\begin{aligned}\frac{1}{t}\log A^Z_{T^\gamma_t} &\overset{(law)}{=} \frac{1}{t}\log\left(t^2\int_0^{T^\gamma_1} e^{2t\beta_u}du\right)\\ &= \frac{2\log t}{t} + \log\left(\int_0^{T^\gamma_1} e^{2t\beta_u}du\right)^{1/t},\end{aligned}$$

where, using the fact that the p-norm converges to the ∞-norm when $p\to\infty$, the latter converges for $t\to\infty$ towards $2\sup_{0\le u\le T^\gamma_1}\beta_u$. We recall that invoking the reflexion principle (see e.g. [19])

$$2\sup_{0\le u\le T^\gamma_1}\beta_u \overset{(law)}{=} |\beta|_{T^\gamma_1} \overset{(law)}{=} |C_1|.$$

Hence,

$$\frac{1}{t}\log T^\gamma_t \xrightarrow[t\to\infty]{(law)} 2|C_1|$$

and the result for A^Z_t follows immediately. □

The distribution of A_t may also be characterized by the following result due to Dufresne [8].

Proposition 11. *For every $x\ge 0$ and with $a(u)=\arg\sinh(u)$,*

$$E\left[\frac{1}{\sqrt{2\pi A^Z_t}}\exp\left(-\frac{x}{2A^Z_t}\right)\right] = \frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{1+x}}\exp\left(-\frac{(a(\sqrt{x}))^2}{2t}\right).$$

Proof. We appeal again to Bougerol's identity in law: for every $t>0$ fixed,

$$\sinh(\beta_t) \overset{(law)}{=} \hat{\beta}_{A_t(\beta)},$$

and we identify the densities of the two parts, i.e.
on the LHS: $\frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{1+y^2}}\exp\left(-\frac{(a(y))^2}{2t}\right)$,
and on the RHS: $E\left[\frac{1}{\sqrt{2\pi A^Z_t}}\exp\left(-\frac{y^2}{2A^Z_t}\right)\right]$.
The proof finishes by changing variables: $x=y^2$. □

Remark 6. These results may easily be generalized for

$$A^{Z,(\nu)}_t = \int_0^t \exp(\beta_s+\nu s)ds.$$

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Indeed, following [1] or [23], we have access to its distribution by the following relation: with ν, μ two real numbers, for every t fixed (β, B and δ are three independent Brownian motions),

$$\sinh(Y_t^{(\nu,\mu)}) \overset{(law)}{=} \int_0^t \exp(\beta_s + \nu s) d(B_s + \mu s) = \delta_{\int_0^t \exp(2(\beta_s+\nu s))ds}, \tag{36}$$

where $(Y_t^{(\nu,\mu)}, t \geq 0)$ is a diffusion with infinitesimal generator:

$$\frac{1}{2}\frac{d^2}{dy^2} + \left(\nu \tanh(y) + \frac{\mu}{\cosh(y)}\right)\frac{d}{dy},$$

starting from $y = \arg\sinh(x)$. Here, without loss of generality we may consider $\mu = 0$. Then, one can mimic the approach where also $\nu = 0$ which was presented above.

3.2. *Asian options and exponential functionals of Lévy processes*

We turn now our interest to the case of Asian options in relation with Lévy processes, that is the case where the exponential functional of interest is $A_t^U \equiv \int_0^t \exp(\alpha\xi_s)ds$. Recall from Subsection 2.3 that U is an isotropic planar Stable process, and ξ, ρ are two real-valued Lévy processes, the first one non-symmetric and the second one symmetric. Hence, we have:

$$T_c^{\theta^U} = \inf\{t : \theta_t^U = c\}, = (H^U)_u^{-1}\Big|_{u=T_c^\rho} = \int_0^{T_c^\rho} ds \exp(\alpha\xi_s) \equiv A_{T_c^\rho}^U, \tag{37}$$

and similarly

$$T_c^{|\theta^U|} = A_{T_c^{|\rho|}}^U. \tag{38}$$

We state the following Proposition only for $A_{T_c^\rho}^U$ but a similar results holds also for $A_{T_c^{|\rho|}}^U$. One can now mimic the approach of the previous Subsection in order to extend the result to A_t^U.

Proposition 12. *For $a > 0$, the following convergence in law holds*

$$\frac{1}{t}\log A_{T_{\sqrt{at}}^\rho}^U \xrightarrow[t\to\infty]{(law)} \tau_{\sqrt{a/r(\alpha)}}^{(1/2)}, \tag{39}$$

where $r(\alpha)$ is given by (33) and, with β denoting again a real Brownian motion, for any $x > 0$, $\tau_x^{(1/2)} \equiv \inf\{t : \beta_t = x\}$ which is a $\frac{1}{2}-$stable subordinator.

Proof. Following [7], for every $a > 0$ we use (39) and Theorem 9 and we have

$$\begin{aligned}
\frac{1}{t}\log A_{T_{\sqrt{at}}^\rho}^U &= \frac{1}{t}\log\left(T_{\sqrt{at}}^{\theta^U}\right) = \frac{1}{t}\log\left(\inf\left\{u : \theta_u^U > \sqrt{at}\right\}\right) \\
&\overset{u=\exp(ts)}{=} \frac{1}{t}\log\left(\inf\left\{e^{ts} : \frac{1}{\sqrt{t}}\,\theta_{\exp(ts)}^U > \sqrt{a}\right\}\right) \\
&= \inf\left\{s : \frac{1}{\sqrt{t}}\,\theta_{\exp(ts)}^U > \sqrt{a}\right\} \\
&\xrightarrow[t\to\infty]{(law)} \inf\left\{s : \beta_{r(\alpha)s} > \sqrt{a}\right\} \\
&= \inf\left\{s : \sqrt{r(\alpha)}\beta_s > \sqrt{a}\right\} \equiv \tau_{\sqrt{a/r(\alpha)}}^{(1/2)},
\end{aligned}$$

which finishes the proof. □

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Appendix. Comparative table between 2-dimensional Brownian motion, Ornstein-Uhlenbeck and Stable processes

We note that $\overset{(d)}{\Longrightarrow}$ denotes weak convergence in distribution in the sense of Skorohod whereas $\overset{(law)}{\Longrightarrow}$ is used for convergence in distribution in the common sense (see e.g. [5] for further details).

Table 1: Spitzer's Asymptotic Theorem and Clock H^W, $W = Z, V, U$

	Planar BM (Z) ($\alpha = 2$)	**Complex-valued OU (V)**	**Planar Stable Process (U)** ($\alpha \in (0,2)$)
Clock H^W	$H^Z(t) = \int_0^t \frac{ds}{\lvert Z_s\rvert^2}$	$H^V(t) = \int_0^{\alpha(t)} \frac{ds}{\lvert B_s\rvert^2}$, $\alpha(t) = (2\lambda)^{-1}(\exp(2\lambda t) - 1)$ B: planar BM	$H^U(t) = \int_0^t \frac{ds}{\lvert U_s\rvert^\alpha}$
Spitzer's law	$\frac{2}{\log t}\,\theta_t^Z \xrightarrow[t\to\infty]{(law)} C_1$	$\frac{1}{t}\,\theta_t^V \xrightarrow[t\to\infty]{(law)} C_\lambda$	$\frac{1}{\sqrt{\log t}}\,\theta_t^U \xrightarrow[t\to\infty]{(d)} \sqrt{r(\alpha)}N,$ $N \sim \mathcal{N}(0,1)$
$t \to \infty$	$\frac{4}{(\log t)^2}\,H^Z(t) \xrightarrow[t\to\infty]{(law)} T_1$, $T_1 = \inf\{t : \beta_t = 1\}$	$\frac{1}{\lambda^2 t^2}\,H^V(t) \xrightarrow[t\to\infty]{(law)} T_1$, $T_1 = \inf\{t : \beta_t = 1\}$	$\frac{1}{\log t}\,H^U(t) \xrightarrow[t\to\infty]{a.s.} 2^{-\alpha}\frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)}$

Comments

1. We recall that Spitzer's theorem for BM can be extended in order to get convergence only in the sense of finite-dimensional distributions but not in the sense of Skorohod [10], contrary to what happens in the Stable case.
2. Following Bertoin and Werner [3], U is transient hence the difference between θ^U and the winding number around an arbitrary fixed point different from 1 is bounded and converges as time goes to infinity. Thus, we can obtain easily a multidimensional version of Spitzer's analogue (that is the so-called windings around several points), contrary to the BM case. For the latter, one needs to decompose θ^Z in small and big windings (see e.g. Pitman and Yor [18]).
3. For windings of 3-dimensional BM around certain curves, see e.g. Le-Gall and Yor [15].

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