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AN IMPROPER INTEGRAL WITH A SQUARE ROOT

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Abstract. In the paper, the author presents explicit and unified expressions for a sequence of improper integrals in terms of the beta functions and the Wallis ratios. Hereafter, the author derives integral representations for the Catalan numbers originating from combinatorics.

1. Introduction

Let a be a positive number. For $n\ge0$, define

$$I_n=\int_{-a}^{a}x^n\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x. \tag{1}$$

In [1, Section 3], Dana-Picard and Zeitoun computed $I_0=a\pi$ and found a closed form of I_n for $n\in\mathbb{N}$ in three steps:

(1) establishing a formula of recurrence between I_n and I_{n+1} in terms of

$$S_n=\int_{-\pi/2}^{\pi/2}\sin^n\theta\,\mathrm{d}\,\theta; \tag{2}$$

(2) establishing an equation for I_n in terms of S_n;
(3) and establishing different expressions for odd values and even values of n.

The aim of this note is to discuss once again the sequence I_n and correct some errors and typos appeared in [1, Section 3].

2. Explicit and unified expressions for I_n

The sequence I_n can be computed by several methods below.

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Theorem 2.1. *For $n\ge0$, the sequence I_n can be computed by*

$$I_n=a^{n+1}\pi\left[\frac{1+(-1)^n}{n}\frac{1}{B\bigl(\frac12,\frac{n}{2}\bigr)}+\frac{1+(-1)^{n+1}}{n+1}\frac{1}{B\bigl(\frac12,\frac{n+1}{2}\bigr)}\right], \tag{3}$$

where

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}\,t=\int_0^\infty\frac{t^{p-1}}{(1+t)^{p+q}}\,\mathrm{d}\,t=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{4}$$

and

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t$$

for $\Re(p),\Re(q)>0$ and $\Re(z)>0$ denote the Euler integrals of the second kinds (or say, the classical beta and gamma functions) respectively.

Proof. By some properties of definite integral and straightforward computation, we can write

$$\begin{aligned}
I_n&=\int_{-a}^0 x^n\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x+\int_0^a x^n\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x\\
&=\int_a^0(-y)^n\sqrt{\frac{a+(-y)}{a-(-y)}}\,\mathrm{d}(-y)+\int_0^a x^n\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x\\
&=\int_0^a(-1)^ny^n\sqrt{\frac{a-y}{a+y}}\,\mathrm{d}\,y+\int_0^a x^n\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x\\
&=\int_0^a x^n\left[(-1)^n\sqrt{\frac{a-x}{a+x}}+\sqrt{\frac{a+x}{a-x}}\right]\mathrm{d}\,x\\
&=\int_0^a x^n\frac{(a+x)+(-1)^n(a-x)}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x\\
&=\int_0^a x^n\frac{a[1+(-1)^n]+x[1-(-1)^n]}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x\\
&=a[1+(-1)^n]\int_0^a\frac{x^n}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x+[1-(-1)^n]\int_0^a\frac{x^{n+1}}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x.
\end{aligned}$$

In [8, Theorem 3.1], it was obtained that

$$\int_0^a\frac{x^n}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x=\sqrt{\pi}\,a^n\frac{\Gamma\bigl(\frac{n}{2}+\frac12\bigr)}{n\Gamma\bigl(\frac{n}{2}\bigr)}$$

for $a>0$ and $n\ge0$. Accordingly, considering

$$\Gamma\left(\frac12\right)=\sqrt{\pi}\,, \tag{5}$$

we acquire

$$\begin{aligned}
I_n&=a[1+(-1)^n]\sqrt{\pi}\,a^n\frac{\Gamma\bigl(\frac{n}{2}+\frac12\bigr)}{n\Gamma\bigl(\frac{n}{2}\bigr)}+[1-(-1)^n]\sqrt{\pi}\,a^{n+1}\frac{\Gamma\bigl(\frac{n+1}{2}+\frac12\bigr)}{(n+1)\Gamma\bigl(\frac{n+1}{2}\bigr)}\\
&=\sqrt{\pi}\,a^{n+1}\left([1+(-1)^n]\frac{\Gamma\bigl(\frac{n+1}{2}\bigr)}{n\Gamma\bigl(\frac{n}{2}\bigr)}+[1-(-1)^n]\frac{\Gamma\bigl(\frac{n}{2}+1\bigr)}{(n+1)\Gamma\bigl(\frac{n+1}{2}\bigr)}\right)
\end{aligned}$$

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$$
\begin{aligned}
&= a^{n+1}\pi\left[\frac{1+(-1)^n}{n}\frac{\Gamma\bigl(\frac{n+1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)\Gamma\bigl(\frac{n}{2}\bigr)} + \frac{1-(-1)^n}{n+1}\frac{\Gamma\bigl(\frac{n}{2}+1\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)\Gamma\bigl(\frac{n+1}{2}\bigr)}\right]\\
&= a^{n+1}\pi\left[\frac{1+(-1)^n}{n}\frac{1}{B\bigl(\frac{1}{2},\frac{n}{2}\bigr)} + \frac{1+(-1)^{n+1}}{n+1}\frac{1}{B\bigl(\frac{1}{2},\frac{n+1}{2}\bigr)}\right].
\end{aligned}
$$

The proof of Theorem 2.1 is complete. □

Theorem 2.2. *For $n \ge 0$, the sequence I_n can be computed by*

$$
I_n = \frac{1}{2}a^{n+1}\left([1+(-1)^n]B\left(\frac{1}{2},\frac{n+1}{2}\right) + \bigl[1+(-1)^{n+1}\bigr]B\left(\frac{1}{2},\frac{n+2}{2}\right)\right). \tag{6}
$$

Proof. Changing the variable of integration by $x = at$ in (1) and using some other properties of definite integral give

$$
\begin{aligned}
I_n &= \int_{-1}^{1}(at)^n\sqrt{\frac{a+at}{a-at}}\,a\,\mathrm{d}\,t\\
&= a^{n+1}\int_{-1}^{1}t^n\sqrt{\frac{1+t}{1-t}}\,\mathrm{d}\,t\\
&= a^{n+1}\left(\int_{-1}^{0}t^n\sqrt{\frac{1+t}{1-t}}\,\mathrm{d}\,t + \int_{0}^{1}t^n\sqrt{\frac{1+t}{1-t}}\,\mathrm{d}\,t\right)\\
&= a^{n+1}\left[\int_{0}^{1}(-s)^n\sqrt{\frac{1-s}{1+s}}\,\mathrm{d}\,s + \int_{0}^{1}t^n\sqrt{\frac{1+t}{1-t}}\,\mathrm{d}\,t\right]\\
&= a^{n+1}\int_{0}^{1}t^n\left[(-1)^n\sqrt{\frac{1-t}{1+t}} + \sqrt{\frac{1+t}{1-t}}\right]\mathrm{d}\,t\\
&= a^{n+1}\int_{0}^{1}t^n\left[(-1)^n\frac{1-t}{\sqrt{1-t^2}} + \frac{1+t}{\sqrt{1-t^2}}\right]\mathrm{d}\,t\\
&= a^{n+1}\left([1+(-1)^n]\int_{0}^{1}\frac{t^n}{\sqrt{1-t^2}}\,\mathrm{d}\,t + [1-(-1)^n]\int_{0}^{1}\frac{t^{n+1}}{\sqrt{1-t^2}}\,\mathrm{d}\,t\right)\\
&= a^{n+1}\left([1+(-1)^n]\int_{0}^{\pi/2}\frac{\sin^n s}{\sqrt{1-\sin^2 s}}\cos s\,\mathrm{d}\,s\right.\\
&\quad\left. + [1-(-1)^n]\int_{0}^{\pi/2}\frac{\sin^{n+1} s}{\sqrt{1-\sin^2 s}}\cos s\,\mathrm{d}\,s\right)\\
&= a^{n+1}\left([1+(-1)^n]\int_{0}^{\pi/2}\sin^n s\,\mathrm{d}\,s + [1-(-1)^n]\int_{0}^{\pi/2}\sin^{n+1} s\,\mathrm{d}\,s\right).
\end{aligned}
$$

Further making use of the formula

$$
\int_{0}^{\pi/2}\sin^t x\,\mathrm{d}\,x = \frac{1}{2}B\left(\frac{t+1}{2},\frac{1}{2}\right), \quad t > -1
$$

in [8, Remark 6.4] yields

$$
\begin{aligned}
I_n &= a^{n+1}\left([1+(-1)^n]\frac{1}{2}B\left(\frac{1}{2},\frac{n+1}{2}\right) + [1-(-1)^n]\frac{1}{2}B\left(\frac{1}{2},\frac{n+2}{2}\right)\right)\\
&= \frac{1}{2}a^{n+1}\left([1+(-1)^n]B\left(\frac{1}{2},\frac{n+1}{2}\right) + \bigl[1+(-1)^{n+1}\bigr]B\left(\frac{1}{2},\frac{n+2}{2}\right)\right).
\end{aligned}
$$

The proof of Theorem 2.2 is complete. □

Corollary 2.1. *For $m \ge 0$, the sequences I_{2m} and I_{2m+1} can be explicitly computed by*

$$I_{2m} = \pi a^{2m+1} \frac{(2m-1)!!}{(2m)!!}$$

and

$$I_{2m+1} = \pi a^{2(m+1)} \frac{(2m+1)!!}{(2m+2)!!}.$$

Proof. From the recurrence relation

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0 \tag{7}$$

and the identity (5), we obtain

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{(2m-1)!!}{2^m}\Gamma\left(\frac{1}{2}\right) = \frac{(2m-1)!!}{2^m}\sqrt{\pi}\,.$$

By this equality and the last expression in (4), we derive

$$B\left(\frac{1}{2},\frac{n}{2}\right) = \frac{\Gamma\bigl(\frac{1}{2}\bigr)\Gamma\bigl(\frac{n}{2}\bigr)}{\Gamma\bigl(\frac{n+1}{2}\bigr)} = \begin{cases} \dfrac{\Gamma\bigl(\frac{1}{2}\bigr)\Gamma(m)}{\Gamma\bigl(m+\frac{1}{2}\bigr)}, & n=2m \\ \dfrac{\Gamma\bigl(\frac{1}{2}\bigr)\Gamma\bigl(m+\frac{1}{2}\bigr)}{\Gamma(m+1)}, & n=2m+1 \end{cases}$$

$$= \begin{cases} \dfrac{\sqrt{\pi}\,(m-1)!}{\frac{(2m-1)!!}{2^m}\sqrt{\pi}}, & n=2m \\ \dfrac{\sqrt{\pi}\,\frac{(2m-1)!!}{2^m}\sqrt{\pi}}{m!}, & n=2m+1 \end{cases} = \begin{cases} 2\dfrac{(2m-2)!!}{(2m-1)!!}, & n=2m; \\ \pi\dfrac{(2m-1)!!}{(2m)!!}, & n=2m+1. \end{cases}$$

Substituting this into (6) reveals

$$I_{2m} = \frac{1}{2}a^{2m+1}\left[2B\left(\frac{1}{2},\frac{2m+1}{2}\right)\right] = a^{2m+1}\pi\frac{(2m-1)!!}{(2m)!!}$$

and

$$I_{2m+1} = \frac{1}{2}a^{2(m+1)}\left[2B\left(\frac{1}{2},\frac{2m+3}{2}\right)\right] = a^{2(m+1)}\pi\frac{(2m+1)!!}{(2m+2)!!}.$$

The proof of Corollary 2.1 is complete. □

3. Integral representations for the Catalan numbers

The Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in various counting problems in combinatorial mathematics. The nth Catalan number can be expressed in terms of the central binomial coefficients $\binom{2n}{n}$ by

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{8}$$

Theorem 3.1. *For $n \ge 0$ and $a > 0$, the Catalan numbers C_n can be represented by*

$$\begin{aligned} C_n &= \frac{1}{\pi}\frac{4^n}{n+1}\frac{1}{a^{2n+1}}\int_{-a}^{a} x^{2n}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x \\ &= \frac{1}{\pi}\frac{2^{2n+1}}{n+1}\frac{1}{a^{2n}}\int_0^a \frac{x^{2n}}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x \\ &= \frac{1}{\pi}\frac{2^{2n+1}}{n+1}\int_0^{\pi/2} \sin^{2n} x\,\mathrm{d}\,x \end{aligned} \tag{9}$$

and

$$
\begin{aligned}
C_n &= \frac{1}{\pi}\frac{2^{2n+1}}{2n+1}\frac{1}{a^{2n+2}}\int_{-a}^{a} x^{2n+1}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x\\
&= \frac{1}{\pi}\frac{2^{2n+2}}{2n+1}\frac{1}{a^{2n+2}}\int_{0}^{a}\frac{x^{2n+2}}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x\\
&= \frac{1}{\pi}\frac{2^{2n+2}}{2n+1}\int_{0}^{\pi/2}\sin^{2n+2}x\,\mathrm{d}\,x.
\end{aligned}
\tag{10}
$$

Proof. From the recurrence relation (7) and the identity (5), it is not difficult to show that the Catalan numbers C_n can be expressed in terms of the gamma function Γ by

$$
C_n = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad n \ge 0.
$$

This implies that

$$
C_n = \frac{1}{\pi}\frac{4^n}{n+1}B\left(\frac{1}{2}, n+\frac{1}{2}\right). \tag{11}
$$

Taking $n = 2p$ in (6) and utilizing (11) lead to

$$
I_{2p} = a^{2p+1}B\left(\frac{1}{2}, \frac{2p+1}{2}\right) = a^{2p+1}\pi\frac{p+1}{4^p}C_n
$$

which is equivalent to

$$
C_n = \frac{4^n}{n+1}\frac{1}{a^{2n+1}\pi}I_{2n} = \frac{1}{\pi}\frac{4^n}{n+1}\frac{1}{a^{2n+1}}\int_{-a}^{a} x^{2n}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x.
$$

The first formula in (9) thus follows.

By similar argument to the deduction of (11), we can discover

$$
C_n = \frac{4^{n+1}}{(2n+1)(2n+2)}\frac{1}{B\left(\frac{1}{2}, n+1\right)}, \quad n \ge 0.
$$

Employing this identity and setting $n = 2p+1$ in (3) figures out

$$
I_{2p+1} = a^{2p+2}\frac{2\pi}{2p+2}\frac{1}{B\left(\frac{1}{2}, p+1\right)} = a^{2p+2}\frac{2\pi}{2p+2}\frac{(2p+1)(2p+2)}{4^{p+1}}C_p
$$

which can be rearranged as

$$
C_p = \frac{1}{a^{2p+2}}\frac{1}{\pi}\frac{2^{2p+1}}{2p+1}I_{2p+1} = \frac{1}{\pi}\frac{1}{a^{2p+2}}\frac{2^{2p+1}}{2p+1}\int_{-a}^{a} x^{2p+1}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x.
$$

The first formula in (10) is thus proved.

The rest integral representations follow from techniques used in the proofs of Theorems (2.1) and (2.2) and from changing variable of integration. $\square$

4. Remarks

Finally, we state several remarks on our main results.

Remark 4.1. The expressions in Corollary 2.1 and the integral representation (9) correct [1, Proposition 3.1 and Corollary 3.2] respectively.

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Remark 4.2. Since

$$B\left(\frac{1}{2},\frac{t+1}{2}\right)B\left(\frac{1}{2},\frac{t}{2}\right)=\frac{2\pi}{t}$$

for $t>0$, the formulas (3) and (6) can be transferred to each other. However, the formula (6) looks simpler.

Remark 4.3. Considering (8), we can rewritten the integral representations in (9) and (10) as

$$\begin{aligned}\binom{2n}{n}&=\frac{1}{\pi}\frac{4^n}{a^{2n+1}}\int_{-a}^{a}x^{2n}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x\\&=\frac{1}{\pi}\frac{2^{2n+1}}{a^{2n}}\int_{0}^{a}\frac{x^{2n}}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x\\&=\frac{1}{\pi}2^{2n+1}\int_{0}^{\pi/2}\sin^{2n}x\,\mathrm{d}\,x\end{aligned}$$

and

$$\begin{aligned}\binom{2n}{n}&=\frac{1}{\pi}\frac{2^{2n+1}(n+1)}{2n+1}\frac{1}{a^{2n+2}}\int_{-a}^{a}x^{2n+1}\sqrt{\frac{a+x}{a-x}}\,\mathrm{d}\,x\\&=\frac{1}{\pi}\frac{2^{2n+2}(n+1)}{2n+1}\frac{1}{a^{2n+2}}\int_{0}^{a}\frac{x^{2n+2}}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x\\&=\frac{1}{\pi}\frac{2^{2n+2}(n+1)}{2n+1}\int_{0}^{\pi/2}\sin^{2n+2}x\,\mathrm{d}\,x.\end{aligned}$$

for $n\ge0$.

Remark 4.4. It is well known that the Wallis ratio is defined by

$$W_n=\frac{(2n-1)!!}{(2n)!!}=\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n+1)},\quad n\in\mathbb{N}.$$

As a result, we have

$$I_{2m}=\pi a^{2m+1}W_m$$

and

$$I_{2m+1}=\pi a^{2m+2}W_{m+1}$$

for $m\ge0$.

The Wallis ratio has been studied and applied by many mathematicians. For more information, please refer to [2, 3, 7, 12, 14], for example, and plenty of literature therein.

Remark 4.5. In [3], the formula

$$\int_{0}^{\pi/2}\sin^t x\,\mathrm{d}\,x=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t+2}{2}\right)},\quad t>-1 \tag{12}$$

was stated. See also [7, p. 16, Eq. (2.18)]. In [6, p. 142, Eq. 5.12.2], the formula

$$\int_{0}^{\pi/2}\sin^{2a-1}\theta\cos^{2b-1}\theta\,\mathrm{d}\,\theta=\frac{1}{2}B(a,b),\quad \Re(a),\Re(b)>0 \tag{13}$$

was listed. By (12) or (13), we can find that the quantity S_n defined in (2) is

$$S_n=\int_{-\pi/2}^{0}\sin^n x\,\mathrm{d}\,x+\int_{0}^{\pi/2}\sin^n x\,\mathrm{d}\,x$$

$$
\begin{aligned}
&= \int_0^{\pi/2} (-1)^n \sin^n x \,\mathrm{d}\, x + \int_0^{\pi/2} \sin^n x \,\mathrm{d}\, x \\
&= \frac{1+(-1)^n}{2} B\biggl(\frac{n+1}{2}, \frac{1}{2}\biggr).
\end{aligned}
$$

Remark 4.6. In [11, Theorem 2.3], among other things, the integral formulas

$$
\begin{gathered}
\int_a^b \biggl(\frac{b-t}{t-a}\biggr)^{\lambda} \,\mathrm{d}\, t = (b-a)\frac{\lambda\pi}{\sin(\lambda\pi)}, \\
\int_a^b \biggl(\frac{b-t}{t-a}\biggr)^{\lambda} \frac{1}{t} \,\mathrm{d}\, t = \frac{\pi}{\sin(\lambda\pi)}\biggl[\biggl(\frac{b}{a}\biggr)^{\lambda} - 1\biggr], \\
\int_a^b \biggl(\frac{b-t}{t-a}\biggr)^{\lambda} \frac{1}{t^{k+1}} \,\mathrm{d}\, t = \frac{\pi}{\sin(\lambda\pi)}\biggl(\frac{b}{a}\biggr)^{\lambda} \frac{1}{a^k} \sum_{\ell=0}^{k} \frac{\langle\lambda\rangle_\ell}{\ell!} \binom{k-1}{\ell-1} \biggl(1-\frac{a}{b}\biggr)^{\ell}
\end{gathered}
$$

for $b > a > 0$, $k \in \mathbb{N}$, and $\lambda \in (-1,1) \setminus \{0\}$ were derived, where

$$
\langle x\rangle_n = \begin{cases} \prod_{k=0}^{n-1} (x-k), & n \ge 1 \\ 1, & n = 0 \end{cases}
$$

is called the falling factorial. In [11, Remark 4.4], the integral formula

$$
\begin{aligned}
\int_a^b \biggl(\frac{b-t}{t-a}\biggr)^{\lambda} \frac{1}{t} \ln\frac{b-t}{t-a} \,\mathrm{d}\, t& \\
&= \begin{cases} \dfrac{\pi}{\sin(\lambda\pi)}\biggl\{\biggl(\dfrac{b}{a}\biggr)^{\lambda} \ln\dfrac{b}{a} - \pi\cot(\lambda\pi)\biggl[\biggl(\dfrac{b}{a}\biggr)^{\lambda} - 1\biggr]\biggr\}, & \lambda \ne 0 \\ \dfrac{1}{2}\biggl(\ln\dfrac{b}{a}\biggr)^2, & \lambda = 0 \end{cases}
\end{aligned}
$$

was concluded from [11, Theorem 2.3]. By comparing the forms of these integrals and I_n, we naturally propose a problem: can one explicitly compute the integrals

$$
\int_a^b \biggl(\frac{b-t}{t-a}\biggr)^{\lambda} t^{\alpha} \,\mathrm{d}\, t \quad\text{and}\quad \int_a^b t^{\alpha} \biggl(\frac{b-t}{t-a}\biggr)^{\lambda} \ln\frac{b-t}{t-a} \,\mathrm{d}\, t
$$

for

$$
\alpha \in \begin{cases} \mathbb{R}, & b > a > 0 \\ \mathbb{N}, & b > 0 > a \end{cases}
$$

and $\lambda \in (-1,1)$?

Remark 4.7. In recent years, the Catalan numbers C_n has been analytically generalized and studied in [4, 5, 9, 10, 13, 15, 16, 17] and closely-related references therein.

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