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A DETERMINANTAL EXPRESSION AND A RECURRENCE RELATION FOR THE EULER POLYNOMIALS

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Abstract. In the paper, by a very simple approach, the author establishes an expression in terms of a lower Hessenberg determinant for the Euler polynomials. By the determinantal expression, the author finds a recurrence relation for the Euler polynomials. By the way, the author derives the corresponding expression and recurrence relation for the Euler numbers.

1. Main results

It is known that a matrix $H=(h_{ij})_{n\times n}$ is called a lower (or an upper, respectively) Hessenberg matrix if $h_{ij}=0$ for all pairs (i,j) such that $i+1<j$ (or $j+1<i$, respectively). Correspondingly, we can define a lower (or an upper, respectively) Hessenberg determinant.

It is general knowledge that the Bernoulli numbers and polynomials B_k and $B_k(u)$ can be generated respectively by

$$\frac{z}{e^z-1}=\sum_{k=0}^{\infty}B_k\frac{z^k}{k!}=1-\frac{z}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{z^{2k}}{(2k)!}$$

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and

$$\frac{ze^{uz}}{e^z-1}=\sum_{k=0}^{\infty}B_k(u)\frac{z^k}{k!}$$

for $|z|<2\pi$. Because the function $\frac{x}{e^x-1}-1+\frac{x}{2}$ is odd in $x\in\mathbb{R}$, all of the Bernoulli numbers B_{2k+1} for $k\in\mathbb{N}$ equal 0. The first six Bernoulli numbers B_{2k} for $-\le k\le 5$ are

$$B_0=1,\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66}.$$

In [4, Section 21.5] and [5, p. 1], it was listed that

$$B_k=k!\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1\\
\frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0\\
\frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\frac{1}{(k-1)!} & \frac{1}{(k-2)!} & \frac{1}{(k-3)!} & \cdots & 1 & 0 & 0\\
\frac{1}{k!} & \frac{1}{(k-1)!} & \frac{1}{(k-2)!} & \cdots & \frac{1}{2!} & 1 & 0\\
\frac{1}{(k+1)!} & \frac{1}{k!} & \frac{1}{(k-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & 0
\end{vmatrix}.$$

In [7, Theorem 1.2], the Bernoulli polynomials $B_k(u)$ for $k\in\mathbb{N}$ were expressed by a lower Hessenberg determinant

$$B_k(u)=(-1)^k\left|\frac{1}{\ell+1}\binom{\ell+1}{m}\left[(1-u)^{\ell-m+1}-(-u)^{\ell-m+1}\right]\right|_{1\le\ell\le k,0\le m\le k-1}$$

and, consequently, the Bernoulli numbers B_k for $k\in\mathbb{N}$ were represented by a lower Hessenberg determinant

$$B_k=(-1)^k\left|\frac{1}{\ell+1}\binom{\ell+1}{m}\right|_{1\le\ell\le k,0\le m\le k-1}.$$

It is common knowledge that the Euler numbers E_k and the Euler polynomials $E_k(x)$ can be generated respectively by

$$\frac{2e^{t/2}}{e^t+1}=\sum_{k=0}^{\infty}\frac{E_k}{k!}\left(\frac{t}{2}\right)^k \tag{1}$$

and

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}\frac{E_k(x)}{k!}t^k \tag{2}$$

which converge uniformly with respect to $t\in(-\pi,\pi)$. By these definitions, it is easy to see that

$$E_k=2^kE_k\left(\frac{1}{2}\right). \tag{3}$$

Since the generating function $\frac{2e^{t/2}}{e^t+1}$ of the Euler numbers E_k is even on $(-\pi,\pi)$, then $E_{2k-1}=0$ for all $k\in\mathbb{N}$. The first nine Euler numbers E_{2k} for $0\le k\le 8$ are

$$1,\quad -1,\quad 5,\quad -61,\quad 1385,\quad -50521,\quad 2702765,\quad -199360981$$

and the first six Euler polynomials $E_k(x)$ for $0\le k\le 5$ are

$$1,\quad x-\frac{1}{2},\quad x^2-x,\quad x^3-\frac{3}{2}x^2+\frac{1}{4},\quad x^4-2x^3+x,\quad x^5-\frac{5}{2}x^4+\frac{5}{2}x^2-\frac{1}{2}.$$

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At the website [9], the Euler numbers E_{2k} were represented by a lower Hessenberg determinant

$$E_{2k}=(-1)^k(2k)!\begin{vmatrix}\frac{1}{2!} & 1 & 0 & \cdots & 0 & 0\\ \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \frac{1}{(2k-6)!} & \cdots & \frac{1}{2!} & 1\\ \frac{1}{(2k)!} & \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdots & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix}, \quad k\in\mathbb{N}.$$

In [8, Theorem 1.1], the Euler numbers E_{2k} for $k\in\mathbb{N}$ were represented by a lower Hessenberg and sparse determinant

$$E_{2k}=(-1)^k\left|\binom{i}{j-1}\cos\left((i-j+1)\frac{\pi}{2}\right)\right|_{(2k)\times(2k)}.$$

There have been many explicit expressions for the Euler numbers B_k and the Euler polynomials $E_k(x)$. See, for example, the recently-published papers [2, 3, 8] and plenty of references therein.

In this paper, by a very simple approach, we will establish a new expression in terms of a lower Hessenberg determinant for the Euler polynomials $E_k(x)$. By the new determinantal expression, we will find a recurrence relation for the Euler polynomials. By the way, we will derive the corresponding expression and recurrence relation for the Euler numbers E_k.

Our main results can be summarized up as two theorems below.

Theorem 1. *For $k\ge 0$, the Euler polynomials $E_k(x)$ can be expressed as*

$$E_k(x)=\frac{(-1)^k}{2^k}\begin{vmatrix}1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0\\ x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0 & 0\\ x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0 & 0\\ x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2 & 0\\ x^{k-1} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2} & 2\\ x^k & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} & \binom{k}{k-1}\end{vmatrix} \tag{4}$$

and, consequently, the Euler numbers E_k can be expressed as

$$E_k=(-1)^k\begin{vmatrix}1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0\\ \frac{1}{2} & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0 & 0\\ \frac{1}{2^2} & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0 & 0\\ \frac{1}{2^3} & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \frac{1}{2^{k-2}} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2 & 0\\ \frac{1}{2^{k-1}} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2} & 2\\ \frac{1}{2^k} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} & \binom{k}{k-1}\end{vmatrix}. \tag{5}$$

As an application of the expression (4), the following recurrence relation for the Euler polynomials and numbers $E_k(x)$ and E_k can be derived.

Theorem 2. *The Euler polynomials and numbers $E_k(x)$ and E_k satisfy the recurrence relations*

$$E_k(x) = -\frac{1}{2}\sum_{\ell=0}^{k-1}\binom{k}{\ell}E_\ell(x) + x^k \tag{6}$$

and, consequently,

$$E_k = -2^{k-1}\sum_{\ell=0}^{k-1}\frac{1}{2^\ell}\binom{k}{\ell}E_\ell + 1. \tag{7}$$

2. Proofs of Theorems 1 and 2

Now we start out to prove our main results.

Proof of Theorem 1. In [6, Section 2.2, p. 849], [7, p. 94], and [8, Lemma 2.1], Exercise 5) in [1, p. 40] was reformulated as the following conclusion. Let $u(x)$ and $v(x) \ne 0$ be two differentiable functions. Let $U_{(n+1)\times1}(x)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(x)$ be an $(n+1)\times n$ matrix whose elements

$$v_{i,j}(x) = \begin{cases} \dbinom{i-1}{j-1}v^{(i-j)}(x), & i-j \ge 0\\ 0, & i-j<0 \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(x)|$ denote the determinant of the $(n+1)\times(n+1)$ matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{pmatrix} U_{(n+1)\times1}(x) & V_{(n+1)\times n}(x)\end{pmatrix}.$$

Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n\frac{\bigl|W_{(n+1)\times(n+1)}(x)\bigr|}{v^{n+1}(x)}. \tag{8}$$

Let $u(t) = e^{xt}$ and $v(t) = e^t + 1$. Then, by virtue of the formula (8),

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{e^{xt}}{e^t+1}\right) = \frac{(-1)^k}{(e^t+1)^{k+1}}$$

$$
\times\begin{vmatrix}
e^{xt} & e^t+1 & 0 & \cdots & 0 & 0 & 0\\
xe^{xt} & \binom{1}{0}e^t & e^t+1 & \cdots & 0 & 0 & 0\\
x^2e^{xt} & \binom{2}{0}e^t & \binom{2}{1}e^t & \cdots & 0 & 0 & 0\\
x^3e^{xt} & \binom{3}{0}e^t & \binom{3}{1}e^t & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
x^{k-2}e^{xt} & \binom{k-2}{0}e^t & \binom{k-2}{1}e^t & \cdots & \binom{k-2}{k-3}e^t & e^t+1 & 0\\
x^{k-1}e^{xt} & \binom{k-1}{0}e^t & \binom{k-1}{1}e^t & \cdots & \binom{k-1}{k-3}e^t & \binom{k-1}{k-2}e^t & e^t+1\\
x^{k}e^{xt} & \binom{k}{0}e^t & \binom{k}{1}e^t & \cdots & \binom{k}{k-3}e^t & \binom{k}{k-2}e^t & \binom{k}{k-1}e^t
\end{vmatrix}
$$

$$
\to\frac{(-1)^k}{2^{k+1}}\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2 & 0\\
x^{k-1} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2} & 2\\
x^{k} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} & \binom{k}{k-1}
\end{vmatrix}
$$

as $t\to 0$ for $k\ge 0$. By the equation (2), the formula (4) is thus proved.

Considering the relation (3) and taking $x=\frac{1}{2}$ in (4) lead to the formula (5) readily. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let

$$
D_{k+1}(x)=\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2 & 0\\
x^{k-1} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2} & 2\\
x^{k} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} & \binom{k}{k-1}
\end{vmatrix}
$$

for $k\ge 0$. Expanding $D_{k+1}(x)$ according to the last column consecutively and inductively reveals

$$
D_{k+1}(x)=\binom{k}{k-1}\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2\\
x^{k-1} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2}
\end{vmatrix}
$$

$$
- 2\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2\\
x^{k} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2}
\end{vmatrix}
$$

$$
= \binom{k}{k-1}\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-3} & 2\\
x^{k-1} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2}
\end{vmatrix}
$$

$$
- 2\binom{k}{k-2}\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
x^{k-3} & \binom{k-3}{0} & \binom{k-3}{1} & \binom{k-3}{2} & \cdots & \binom{k-3}{k-4} & 2\\
x^{k-2} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{k-4} & \binom{k-2}{k-3}
\end{vmatrix}
$$

$$
+ 2^2\binom{k}{k-3}\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
x^{k-4} & \binom{k-4}{0} & \binom{k-4}{1} & \binom{k-4}{2} & \cdots & \binom{k-4}{k-5} & 2\\
x^{k-3} & \binom{k-3}{0} & \binom{k-3}{1} & \binom{k-3}{2} & \cdots & \binom{k-3}{k-5} & \binom{k-3}{k-4}
\end{vmatrix}
$$

$$
- 2^3\begin{vmatrix}
1 & 2 & 0 & 0 & \cdots & 0 & 0\\
x & \binom{1}{0} & 2 & 0 & \cdots & 0 & 0\\
x^2 & \binom{2}{0} & \binom{2}{1} & 2 & \cdots & 0 & 0\\
x^3 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
x^{k-4} & \binom{k-4}{0} & \binom{k-4}{1} & \binom{k-4}{2} & \cdots & \binom{k-4}{k-5} & 2\\
x^{k} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-5} & \binom{k}{k-4}
\end{vmatrix}
$$

$$
= \sum_{\ell=0}^{m}(-1)^{\ell}2^{\ell}\binom{k}{k-\ell-1}D_{k-\ell}(x) + (-1)^{m+1}2^{m+1}
$$

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$$\times\begin{vmatrix} 1 & 2 & 0 & \cdots & 0 & 0 \\ x & \binom{1}{0} & 2 & \cdots & 0 & 0 \\ x^2 & \binom{2}{0} & \binom{2}{1} & \cdots & 0 & 0 \\ x^3 & \binom{3}{0} & \binom{3}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots\vdots & \vdots \\ x^{k-m-2} & \binom{k-m-2}{0} & \binom{k-m-2}{1} & \cdots & \binom{k-m-2}{k-m-3} & 2 \\ x^k & \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k-m-3} & \binom{k}{k-m-2} \end{vmatrix}$$

$$= \sum_{\ell=0}^{k-2}(-1)^\ell 2^\ell \binom{k}{k-\ell-1} D_{k-\ell}(x) + (-1)^{k-1}2^{k-1}\begin{vmatrix} 1 & 2 \\ x^k & \binom{k}{0} \end{vmatrix}$$

$$= \sum_{\ell=0}^{k-2}(-1)^\ell 2^\ell \binom{k}{k-\ell-1} D_{k-\ell}(x) + (-1)^{k-1}2^{k-1}\left[\binom{k}{0} - 2x^k\right]$$

$$= \sum_{\ell=0}^{k-1}(-1)^\ell 2^\ell \binom{k}{k-\ell-1} D_{k-\ell}(x) + (-1)^k 2^k x^k$$

$$= \sum_{\ell=0}^{k-1}(-1)^\ell 2^\ell \binom{k}{k-\ell-1} \frac{2^{k-\ell-1}}{(-1)^{k-\ell-1}}\left[\frac{(-1)^{k-\ell-1}}{2^{k-\ell-1}} D_{k-\ell}(x)\right] + (-1)^k 2^k x^k$$

$$= (-1)^{k-1}2^{k-1}\sum_{\ell=0}^{k-1}\binom{k}{k-\ell-1} E_{k-\ell-1}(x) + (-1)^k 2^k x^k.$$

Accordingly, it follows that

$$E_k(x) = \frac{(-1)^k}{2^k} D_{k+1}(x) = -\frac{1}{2}\sum_{\ell=0}^{k-1}\binom{k}{k-\ell-1} E_{k-\ell-1}(x) + x^k$$
$$= -\frac{1}{2}\sum_{\ell=0}^{k-1}\binom{k}{\ell} E_\ell(x) + x^k.$$

The relation (6) is thus proved.

Taking $x = \frac{1}{2}$ in (6) and making use of the relation (3) result in the relation (7). The proof of Theorem 2 is complete. □

Remark 1. The expression (5) can also be obtained by applying the formula (8) to the functions $u(t) = e^{t/2}$ and $v(t) = e^t + 1$ and considering the generating function $\frac{2e^{t/2}}{e^t+1}$ in the equation (1).

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