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Article

The Solvency II Standard Formula, Linear Geometry, and Diversification

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Abstract: The core of risk aggregation in the Solvency II Standard Formula is the so-called square root formula. We argue that it should be seen as a means for the aggregation of different risks to an overall risk rather than being associated with variance-covariance based risk analysis. Considering the Solvency II Standard Formula from the viewpoint of linear geometry, we immediately find that it defines a norm and therefore provides a homogeneous and sub-additive tool for risk aggregation. Hence Euler's Principle for the reallocation of risk capital applies and yields explicit formulas for capital allocation in the framework given by the Solvency II Standard Formula. This gives rise to the definition of *diversification functions*, which we define as monotone, subadditive, and homogeneous functions on a convex cone. Diversification functions constitute a class of models for the study of the aggregation of risk, and diversification. The aggregation of risk measures using a diversification function preserves the respective properties of these risk measures. Examples of diversification functions are given by seminorms, which are monotone on the convex cone of non-negative vectors. Each L^p norm has this property, and any scalar product given by a non-negative positive semidefinite matrix does as well. In particular, the Standard Formula is a diversification function and hence a risk measure that preserves homogeneity, subadditivity, and convexity.

Keywords: Solvency II; Standard Formula; Risk Measure; Diversification; Aggregation; Monotony; Homogeneity; Subadditivity; Euler's Principle; Capital Allocation

MSC: 91B30

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1. Introduction

The Solvency II standard formula is a means to assign the so-called solvency capital requirement to an insurance or reinsurance company. The undertaking has to have enough own funds to cover its capital requirement, and the ratio of both is called the solvency ratio, which thereby should be greater or at least be equal to 1.

The solvency capital requirement is the sum of the basic solvency capital requirement – which aggregates the market, life, non-life, health, and counterparty risk – and the adjustments for operational risk, deferred taxes, and others.

Let us denote the solvency capital requirements for the modules market, life, non-life, health, and counterparty risk with $S_1, \ldots, S_5$ respectively, and combine them as a vector $S = (S_1, \ldots, S_5)^{\mathrm{T}}$. Then the basic solvency capital requirement is derived by the formula

$$\text{basic solvency capital requirement} = \sqrt{S^{\mathrm{T}} A S}, \qquad (1)$$

where A is a positive definite matrix of correlation parameters.

The risk modules themselves consist of sub-modules which are aggregated in the same manner. The market risk module, for example, consists of the interest rate, equity, spread, property, currency, and concentration risk sub-modules, which are aggregated by a similar formula, but with a different matrix A of course.

In the following, we will focus on the square-root formula (1) and identify the Solvency II standard formula in the way with which the risk is aggregated, namely the square-root formula as stated above.

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There has been some debate on whether (1) is a reasonable way for the aggregation of risk or not, and how to allocate risk capital in this framework (cf. e.g. [3,5,6]). Yet all these investigations assume, to the best of our knowledge, that there is an additive decomposition of some portfolio (or aggregated outcome) X in terms of sub-portfolios (or contributions) $X_1, \ldots, X_N$ like

$$X = \sum_{k=1}^{N} X_k. \tag{2}$$

This is also assumed in general literature on risk aggregation and allocation (cf. e.g. [4,7,10,12]).

This means that the information on the aggregation of several risks and the diversification effects is encoded in the *joint* distribution of the marginals $X_1, \ldots, X_N$, which then determines the distribution of X. This amounts to knowing the copula of the joint distribution of the marginals.

In this paper we take a different position and do *not* assume that we have a model with an additive structure (2) and do *not* assume that we have a copula, or joint distribution. Instead, we explore the case in which the joint distribution is unknown – and ask whether (1) may be a sound way for the aggregation of risk in case one does not know everything on the joint distribution, i.e. the copula.

Our point of view can be interpreted in the way that we try to find a good model for diversification, which is as easy as possible – however as feasible as needed – and which avoids, in particular, all the model risk and parameter risk which are unavoidable, when one has to build up a joint distribution of risks of different nature.

Our main findings are:

- The standard formula (1) allows for the Euler principle of capital allocation, *independent of any assumption on distributions*. It even allows for diversification (i.e. is sub-additive) independently of whether the underlying risk measure is sub-additive or not.
- The standard formula is an example of a general principle of how to aggregate risk, namely an example of a diversification function as defined below. In particular, the standard formula (1) can be interpreted as a risk measure.

We will specify these statements in the following. The article is organized as follows:

In section 2, we present a method once suggested by the *Gesamtverband der Deutschen Versicherungswirtschaft* (GDV) for the reallocation of risk within the standard formula and determine a special case. The GDV method is used in practice because of its economic interpretation.

In section 3, the method turns out to be the Euler principle applied to the standard formula, which works because the standard formula is homogeneous.

Moreover, the standard formula constitutes a norm, which means that we have the triangle inequality at hand. In section 4, we use this and give some estimates on the sensitivities of the reallocation method.

In section 5, we consider the standard formula as a risk measure, and as a model for diversification – which applies to the risks of a given portfolio, or to the risks of sub-portfolios of a portfolio, or to the risks of business lines within a company, or generally speaking to any "portfolio of risks".

2. The GDV method

The *Gesamtverband der Deutschen Versicherungswirtschaft* (GDV) once suggested an allocation method for re-allocation of risk in the framework of the Solvency II Standard Formula (unpublished). We therefore call it the GDV method and at first state and prove its properties without using geometry. In the next section it will turn out that the GDV method is nothing else than the Euler allocation principle applied to the standard formula.

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Lemma 1. *Let $R = (R_1, \dots, R_N)^T \in \mathbb{R}^N$ a vector of risks and A be a symmetric matrix. Let the overall risk be given by*

$$\|R\|_A = \sqrt{R^T A R}. \tag{3}$$

Then the gradient of $R \mapsto \|R\|_A$ is given by

$$\nabla \|R\|_A = \frac{AR}{\|R\|_A} \text{ for all } \|R\|_A > 0. \tag{4}$$

The equation

$$\langle \nabla \|R\|_A, R \rangle = \|R\|_A \tag{5}$$

holds.

Note that A has not to be positive definite or positive semi-definite. It suffices that A be symmetric. In any case, any positive definite or positive semi-definite matrix is symmetric by definition.

Remark 1 (GDV Method)**.** The GDV method is a means to allocate the overall risk $\|R\|_A$ to the sub-risks R_k. It uses the sensitivities (partial derivatives)

$$\omega_k = \frac{\partial \|R\|_A}{\partial R_k} = \frac{1}{\|R\|_A} \sum_{\ell=1}^{N} R_\ell A_{\ell k}, \quad 1 \leq k \leq N, \tag{6}$$

and the risk contribution of risk k to the overall risk is $\omega_k R_k$. Equation (5) shows that the risk contributions add up to the overall risk:

$$\|R\|_A = \omega_1 R_1 + \dots \omega_N R_N. \tag{7}$$

So this method works – because of (7) – and it has the economic interpretation (6), i.e. the sensitivities are nothing else than the marginal contributions of the individual risks to the overall risk.

We make a remark on a special case which is quite often encountered in practice: the concentration risk within the market risk module may vanish. In this case, its marginal contribution vanishes as well, i.e. if there was an increase of concentration risk, the market risk would be affected only slightly.

Remark 2. In case the concentration risk within the Solvency market risk module vanishes, its sensitivity vanishes as well (provided the market risk is positive). This is because the correlation parameters of the concentration risk to the other sub-risks of the market risk are all zero. Therefore market risk can be expressed in terms of concentration risk x as follows:

$$f(x) = \|R\|_A = \sqrt{c + x^2}, \quad c > 0, \tag{8}$$

where $\sqrt{c}$ denotes market risk without concentration risk. The sensitivity of the concentration risk for $x = 0$ is

$$\omega = \frac{df}{dx}(0) = \left. \frac{x}{f(x)} \right|_{x=0} = 0. \tag{9}$$

3. The standard formula as a homogeneous function

There is a reason why equation (5) holds: A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* $\alpha \in \mathbb{R}$, if

$$f(tx) = t^\alpha f(x) \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}. \tag{10}$$

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A theorem of Euler states that a function is homogeneous of degree $\alpha > 0$ if and only if

$$\langle \nabla f(x), x \rangle = \alpha f(x) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{11}$$

see [11]. A function which is homogeneous of degree 1 is briefly called homogeneous as well. Hence (5) is by Euler's theorem equivalent to the fact that the standard formula is homogeneous, and indeed we have

$$\|tR\|_A = t\,\|R\|_A \text{ for all } t > 0. \tag{12}$$

Capital allocation by (11) is called *Euler's principle* [12]. Hence, the GDV method (5) amounts to be Euler's principle applied to the standard formula. Under the simplifying assumption that the underlying random variables have a normal distribution, this was already observed by De Angelis and Granito [3].

The following Lemmas work out the economic meaning of homogeneity. They both could be rephrased in saying "Diversification does not hinge on the sheer size of the portfolio, but on its composition only." – Assertion and Proof of Lemma 3 may be known to the reader.

Lemma 2. *The risk in direction of a risk allocation $R \neq 0$ is proportional to the directional derivative of the risk in this direction, i.e.*

$$\|y\|_A = D_{\nu_R}\,\|R\|_A \cdot |y| \quad \textit{for all } y = |y|\,\nu_R, \textit{ where } \nu_R = \frac{R}{|R|}. \tag{13}$$

Proof. By Euler's theorem, i.e. (5), the directional derivative of the risk is

$$D_{\nu_R}\,\|R\|_A = \langle \nabla\,\|R\|_A\,, \nu_R \rangle = \left\langle \nabla\,\|R\|_A\,, \frac{R}{|R|} \right\rangle = \frac{\|R\|_A}{|R|} = \|\nu_R\|_A\,. \tag{14}$$

This implies

$$D_{\nu_R}\,\|R\|_A \cdot |y| = \|\,|y|\,\nu_R\|_A = \|y\|_A\,. \tag{15}$$

□

Lemma 3. *Let $R \neq 0$ be some risk allocation and $c > 0$. Then the sensitivities of cR and R coincide:*

$$\nabla\,\|cR\|_A = \nabla\,\|R\|_A\,. \tag{16}$$

Proof. By homogeneity, we have for all $v \in \mathbb{R}^n$

$$\begin{aligned}\langle \nabla\,\|cR\|_A\,, v\rangle &= \lim_{h\to 0}\frac{1}{h}\left(\|cR + hv\|_A - \|cR\|_A\right) = \lim_{h\to 0}\frac{c}{h}\left(\left\|R + \frac{h}{c}v\right\|_A - \|R\|_A\right)\\ &= \lim_{h'\to 0}\frac{1}{h'}\left(\|R + h'v\|_A - \|R\|_A\right) = \langle \nabla\,\|R\|_A\,, v\rangle\,.\end{aligned}$$

This implies the assertion. □

More generally, the same proof shows that the gradient of a homogeneous function of degree $\alpha \in \mathbb{R}$ is homogeneous of degree $\alpha - 1$. (The case $\alpha = 0$ is somewhat degenerate, yet true.)

4. The standard formula as a norm

The matrix A in (3) should be positive semidefinite, because it is a correlation matrix. Therefore (3) defines a seminorm. If A is positive definite, (3) defines a norm [8, p. 154].

Note that A may well contain tail-correlations – defined in one or the other manner – rather than linear correlations. Nevertheless, from an economic point of view, is not a bad choice to take a positive semidefinite matrix A. Cf. [2,9] with respect to the interconnection between the standard formula and tail correlations.

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Note that A is indeed positive semidefinite in case it contains linear (or Pearson) correlations:

Remark 3. Let S be the covariance matrix and K the correlation matrix of a random vector Z. Then S and K are positive semidefinite. If there is no linear combination of the entries of Z which is constant, S and K are positive definite.

Once we know that $\|\cdot\|_A$ defines a seminorm, we can use the triangle inequality. It immediately follows that the risk contribution of any sub-portfolio cannot be greater than its stand-alone risk:

Lemma 4. *Let A be positive semidefinite and the diagonal elements of A be 1 (as it is the case in the standard formula). Then the absolute value of the sensitivities (6) is not greater than 1.*

Proof. The assumption that the diagonal elements of A are 1 implies

$$\|e_k\|_A = 1 \tag{17}$$

for all unit vectors e_k of the standard basis $(1 \leq k \leq N)$. Hence we obtain by the triangle inequality

$$|\omega_k| = \left|\frac{\partial \|R\|_A}{\partial R_k}\right| = \left|\lim_{t\to 0} \frac{\|R + t\,e_k\|_A - \|R\|_A}{t}\right| \leq \lim_{t\to 0} \frac{|t|\,\|e_k\|_A}{|t|} = 1 \tag{18}$$

for all $1 \leq k \leq N$. □

From the point of view of a manager who is in charge of some sub-portfolio it is a complication that sensitivities can change over time; for this means that the contribution of their sub-portfolio is not at their command alone. Therefore it is important that the sensitivities change only slightly, when the composition of risks change. Of course, if there was turmoil in the portfolio and risks changed considerably, the sensitivities may change considerably as well – but this is reasonable and should be expected.

Theorem 1. *Let the matrix A be positive semidefinite, have diagonal elements of 1, and have non-negative entries (as it is the case in the standard formula). Let the risk vector R be non-negative, i.e. every entry of R be non-negative. Let D be a change of the risk vector and*

$$D \geq 0 \quad \textit{or} \quad -R \leq D \leq 0. \tag{19}$$

Let $\|R\|_A > 0, \|D\|_A > 0, \|R + D\|_A > 0$. Then the following estimate on the change of the sensitivities holds:

$$\left|\nabla \|R + D\|_A - \nabla \|R\|_A\right| \leq \frac{\|D\|_A}{\|R + D\|_A}, \tag{20}$$

where the absolute value $|\cdot|$ denotes the maximum norm.

Proof. We derive

$$\begin{aligned} \nabla \|R + D\|_A - \nabla \|R\|_A &= \frac{A(R + D)}{\|R + D\|_A} - \frac{AR}{\|R\|_A} \\ &= \frac{AR}{\|R\|_A}\left(\frac{\|R\|_A}{\|R + D\|_A} - 1\right) + \frac{AD}{\|D\|_A}\frac{\|D\|_A}{\|R + D\|_A}. \end{aligned} \tag{21}$$

At first we consider the case $D \geq 0$. Then

$$\|R + D\|_A^2 - \|R\|_A^2 = D^{\mathrm{T}} A(R + D) + R^{\mathrm{T}} AD \tag{22}$$

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is non-negative, because there are solely non-negative numbers within the products. Hence the bracket in equation (21) is negative or zero. Therefore the two summands on the right hand side of (21) have a different sign in every component. We obtain with Lemma 4

$$\left|\nabla \|R+D\|_A - \nabla \|R\|_A\right| \leq \max\left\{\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right|, \frac{\|D\|_A}{\|R+D\|_A}\right\}. \tag{23}$$

Now the triangle inequality shows

$$\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right| = \frac{\|R+D\|_A - \|R\|_A}{\|R+D\|_A} \leq \frac{\|D\|_A}{\|R+D\|_A}, \tag{24}$$

which implies (20).

The other case is $-R \leq D \leq 0$. In this case the entries of D in the right hand side of (22) are non-positive, while all other entries are non-negative. Hence (22) is non-positive. This means that the bracket in (21) is non-negative while the entries in the last term are non-positive. Hence, the estimate (23) holds in this case as well. We use the triangle inequality in the form

$$\|R\|_A = \|R+D-D\|_A \leq \|R+D\|_A + \|-D\|_A \tag{25}$$

to obtain

$$\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right| = \frac{\|R\|_A - \|R+D\|_A}{\|R+D\|_A} \leq \frac{\|D\|_A}{\|R+D\|_A}. \tag{26}$$

This completes the proof. □

Note that the assumptions of Theorem 1 are fulfilled in particular if only one risk changes. In general we have the following Theorem 2 with a weaker estimate:

Theorem 2. *Let the matrix A be positive semidefinite and have diagonal elements of 1. Let $R, D \in \mathbb{R}^n$ and $\|R\|_A > 0, \|D\|_A > 0, \|R+D\|_A > 0$. Then the following estimate on the change of the sensitivities holds:*

$$\left|\nabla \|R+D\|_A - \nabla \|R\|_A\right| \leq 2\frac{\|D\|_A}{\|R+D\|_A}, \tag{27}$$

where the absolute value $|\cdot|$ denotes the maximum norm.

Proof. We apply the triangle inequality to (21) and use (24) and (26), respectively. □

5. The standard formula as a risk measure, and as a model for diversification

A risk functional is defined on a domain $\mathcal{X} \subseteq L^0$ of random variables (risks) with values in $\mathbb{R} \cup \{\infty\}$. We assume that $\mathcal{X}$ is a convex cone, i.e. $\alpha X + \beta Y \in \mathcal{X}$ for all $X, Y \in \mathcal{X}$ and $\alpha, \beta > 0$. A risk functional R may have one or more of these properties[1]:

- R is *monotone*, if $R(X) \geq R(Y)$ whenever $X \leq Y$ a.s. In this case, the risk functional is called a *risk measure*.
- R is *subadditive*, if $R(X+Y) \leq R(X) + R(Y)$.
- R is *homogeneous*, if $R(tX) = tR(X)$ for all $t > 0$.
- R is *cash invariant*, if $R(X+a) = R(X) - a$ for all $a \in \mathbb{R}$. (For this to make sense we have to assume that $\mathcal{X}$ contains all constants $a \in \mathbb{R}$.)
- R is *convex*, if $R(\alpha X + (1-\alpha)Y) \leq \alpha R(X) + (1-\alpha)R(Y)$ for all $\alpha \in (0,1)$.
- R is *version independent*, if $R(X) = R(Y)$ for all $X \overset{d}{=} Y$.

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[1] There is a lot of literature on risk measures. For our purposes it is more than enough to rely on the textbook [10, p. 142 ff.].

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- R is *comonotone additive*, if $R(X+Y) = R(X) + R(Y)$ for X, Y comonotone.

Note that a homogeneous risk functional is subadditive if and only if it is convex.

Definition 1. Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone and $f : B \to \mathbb{R} \cup \{\infty\}$ a function. Then

- f is *monotone (non-decreasing)*, if $f(r) \leq f(s)$ for all $r, s \in B$ with $r_k \leq s_k$ for all $1 \leq k \leq n$.
- f is *subadditive*, if $f(r+s) \leq f(r) + f(s)$ for all $r, s \in B$.
- f is *homogeneous*, if $f(tr) = tf(r)$ for all $r \in B$ and $t > 0$.
- f is *convex*, if $f(\alpha r + (1-\alpha)s) \leq \alpha f(r) + (1-\alpha) f(s)$ for all $r, s \in B$ and $\alpha \in (0,1)$.
- f is *additive*, if $f(r+s) = f(r) + f(s)$ for all $r, s \in B$.

As is the case for risk functionals, a homogeneous function is subadditive if and only if it is convex. – Due to the convention that lower outcomes correspond to a higher risk, the signs in the notion of monotony of a risk functional and a function are opposite. If the unit vectors $e_1, \ldots, e_n$ of the standard basis in $\mathbb{R}^n$ are elements of B, monotony is equivalent to monotony in every argument, i.e.

$$f(x) \leq f(x + ce_k) \text{ for all } x \in B, c > 0, 1 \leq k \leq n. \tag{28}$$

Note that we do not propose a property so as to maintain cash invariance. Cash invariance might not be a helpful concept for the study of diversification effects within a company or a portfolio.

Lemma 5. *Let*

$$R_k : \mathcal{X} \to B_k \subseteq \mathbb{R} \cup \{\infty\} \quad (1 \leq k \leq n) \tag{29}$$

be risk functionals and

$$B = B_1 \times \ldots \times B_n \tag{30}$$

be a convex cone. Let $f : B \to \mathbb{R} \cup \{\infty\}$ be a function. Then

$$R = f(R_1, \ldots, R_n) \tag{31}$$

is a risk functional and the following holds:

- *If $R_1, \ldots, R_n$ and f are monotone, then R is monotone, i.e. a risk measure.*
- *If $R_1, \ldots, R_n$ are subadditive, and f is monotone and subadditive, then R is subadditive.*
- *If $R_1, \ldots, R_n$ and f are homogeneous, then R is homogeneous.*
- *If $R_1, \ldots, R_n$ are convex, and f is monotone and convex, then R is convex (albeit not cash invariant in general[2]).*
- *If $R_1, \ldots, R_n$ are version independent, then R is version independent.*
- *If $R_1, \ldots, R_n$ are comonotone additive, and f is additive, then R is comonotone additive.*

The proof is straightforward, and the several facts of the lemma may be known to the experts. The point here is that the lemma paves the way for the following definition that establishes a class of models, which are useful for the study of the aggregation of risk, and diversification.

Definition 2. Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone and $f : B \to \mathbb{R} \cup \{\infty\}$ a function. We call f a *diversification function*, if f is monotone, homogeneous, and subadditive.

Note that there is a related notion in information theory, namely of an *aggregation function* $f : [0,1]^n \to [0,1]$, which has a slightly different aim [1].

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[2] For reasons not known to the public a risk measure is called a *convex risk measure* in literature when it is convex, and cash invariant.

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Lemma 6. *Let R be a risk functional and*

$$S = \max\{R; 0\}. \tag{32}$$

If R is monotone, subadditive, homogeneous, convex, or version independent, then S has the respective property as well.

This shows that one may consider non-negative risk functionals, or risk measures. The proof is straightforward. Lemma 5 implies:

Theorem 3. *Let $R_1, \ldots, R_n : \mathcal{X} \to [0, \infty)$ be non-negative, finite risk measures and $R = (R_1, \ldots, R_n)^T$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a seminorm and the restriction of f to $[0, \infty)^n$ be monotone. Then the following holds:*

- *The restriction of f to $[0, \infty)^n$ is a diversification function.*
- *$f(R)$ is a risk measure.*
- *If $R_1, \ldots, R_n$ are homogeneous, then $f(R)$ is a homogeneous risk measure.*
- *If $R_1, \ldots, R_n$ are subadditive, then $f(R)$ is a subadditive risk measure.*
- *If $R_1, \ldots, R_n$ are convex, then $f(R)$ is a risk measure, which is convex (albeit not cash invariant in general).*

It is not unusual for a seminorm that its restriction to non-negative elements is monotone: All L^p-norms have this property. With respect to symmetric bilinear forms we have the following theorem:

Theorem 4. *Let $A \in \mathbb{R}^{n\times n}$ be positive semidefinite. The restriction of the seminorm $\|\cdot\|_A$ to $[0, \infty)^n$ is monotone if and only if A is non-negative, i.e. has non-negative entries only.*

Proof. Let A be non-negative. We show that the restriction of $\|\cdot\|_A$ to $[0, \infty)^n$ is monotone. Let $R, S \in \mathbb{R}^n$ with $R \geq S \geq 0$, i.e. $R_k \geq S_k \geq 0$ for all $1 \leq k \leq n$. Then

$$\|R\|_A^2 - \|S\|_A^2 = (R - S)^{\mathrm{T}} A R + S^{\mathrm{T}} A (R - S) \geq 0, \tag{33}$$

because all contributions in the sum are non-negative.

Now assume that $a_{jk} = a_{kj} < 0$ for some $1 \leq j, k \leq n$. Consider $R = c\, e_j + e_k$ and $S = e_k$, where e_j and e_k are the respective unit vectors of the standard basis and $0 < c < 2|a_{jk}|/|a_{jj}|$ in case $a_{jj} \neq 0$ and 1 otherwise. Then $0 \leq S \leq R$ and

$$\|R\|_A^2 - \|S\|_A^2 = c\, e_j^{\mathrm{T}} A (c\, e_j + e_k) + e_k^{\mathrm{T}} A c\, e_j = c^2 a_{jj} + 2c\, a_{jk} < 0, \tag{34}$$

which shows that monotony is violated. □

Therefore Theorem 3 applies to the standard formula and shows that $\|R\|_A$ is a subadditive risk measure, and homogeneous risk measure respectively, whenever the individual risk measures R_k, which are assumed to be non-negative and finite, share the respective property.

This is another hint that it could be an unfortunate choice to define A by the correlation matrix of the underlying random variables: In general one would encounter negative correlation coefficients, which would lead to a risk functional that is not a risk measure in general because of the lack of monotony. Unless, for example, the underlying risks are elliptically distributed, have a mean of zero, and the risk measures are a value at risk at the same level of confidence.

In any case, the risk measure of the standard formula *is* a value at risk. Value at risk is a homogeneous, cash invariant, version independent, comonotone additive risk measure, yet not subadditive nor convex in general [10, p. 147]. Hence the standard formula with value at risk is a homogeneous, version independent risk measure. (To be precise, value at risk has to be maximized with zero to fit.) Of course, if the joint distribution of the overall risk was known, then it is by no means guaranteed that the standard formula resembles the value at risk of the joint distribution;

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however, given the marginal risk measures only the standard formula is indeed a risk measure of the overall risk in spite of the fact of there being no information on its distribution.

We have two concluding remarks on diversification functions and the standard formula. The first one is on the composition of diversification functions, due to the fact that the standard formula is actually nested.

Lemma 7. *Let $B_1, \dots, B_m$ be convex cones. Then $B = B_1 \times \dots \times B_m$ is a convex cone, and vice versa.*

Lemma 8. *Let $C \subseteq (\mathbb{R} \cup \{\infty\})^m$, $B_k \subseteq \mathbb{R} \cup \{\infty\}$ be convex cones, and $g_k : C \to B_k$ be diversification functions $(1 \le k \le n, m \in \mathbb{N})$, and $g = (g_1, \dots, g_n)^T$. Let*

$$f : B_1 \times \dots \times B_n \longrightarrow \mathbb{R} \cup \{\infty\} \tag{35}$$

be a diversification function. Then

$$f \circ g : C \longrightarrow \mathbb{R} \cup \{\infty\} \tag{36}$$

is a diversification function. The sensitivity ω_ℓ $(1 \le \ell \le m)$ of $f \circ g$ with respect to an allocation $S \in C$ is given by

$$\omega_\ell = \frac{\partial (f \circ g)}{\partial S_\ell}(S) = \langle \nabla f(g(S)), \partial_\ell g(S) \rangle. \tag{37}$$

Remark 4. One often encounters the special case, in which there is only one function g_k, which depends on a coordinate S_ℓ. In this case, equation (37) means

$$\omega_\ell = \partial_k f(g(S))\, \partial_\ell g_k(S). \tag{38}$$

Secondly, Lemma 4 can be restated using monotony and subadditivity instead of the triangle inequality.

Definition 3. Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone, and $f : B \to \mathbb{R} \cup \{\infty\}$ a diversification function. We call f a *normalized diversification function*, if $e_k \in B$ for all unit vectors $e_1, \dots, e_n$ of the standard basis in $\mathbb{R}^n$, and

$$f(e_k) = 1 \text{ for all } 1 \le k \le n. \tag{39}$$

Theorem 5. *The sensitivities of a differentiable, normalized diversification function are not negative and not greater than 1, i.e.*

$$\frac{\partial f}{\partial x_k}(x) \in [0,1] \text{ for all } x \in B \text{ and } 1 \le k \le n. \tag{40}$$

Proof. We have by monotony

$$f(x + h\, e_k) \ge f(x) \text{ for all } x \in B, h > 0 \text{ and } 1 \le k \le n, \tag{41}$$

and we obtain by subadditivity

$$f(x + h\, e_k) \le f(x) + f(h\, e_k) \text{ for all } x \in B, h > 0 \text{ and } 1 \le k \le n. \tag{42}$$

So

$$0 \le \lim_{h \searrow 0} \frac{f(x + h\, e_k) - f(x)}{h} = \frac{\partial f}{\partial x_k}(x) \le \lim_{h \searrow 0} \frac{f(h\, e_k)}{h} = 1, \tag{43}$$

as was to be shown. □

This applies to the sensitivities of the standard formula as well, for the standard formula is a normalized diversification function.

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