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Article

The Solvency II Standard Formula, Linear Geometry, and Diversification

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Abstract: We introduce the notions of monotony, subadditivity, and homogeneity for functions defined on a convex cone, call functions with these properties *diversification functions* and obtain the respective properties for the risk aggregation given by such a function. Examples of diversification functions are given by seminorms, which are monotone on the convex cone of non-negative vectors. Any L^p norm has this property, and any scalar product given by a non-negative positive semidefinite matrix as well. In particular, the Standard Formula is a diversification function, hence a risk measure that preserves homogeneity, subadditivity, and convexity.

Keywords: Solvency II; standard formula; risk measure; diversification; aggregation; monotony; homogeneity; subadditivity; Euler's principle; capital allocation

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1. Introduction

The Solvency II standard formula is a means to assign the so-called solvency capital requirement to an insurance or reinsurance company. The undertaking has to have enough own funds so as to cover its capital requirement, and the ratio of both is called solvency ratio, which thereby should be greater or at least be equal to 1.

The solvency capital requirement is the sum of the basic solvency capital requirement – which aggregates the market, life, non-life, health, and counterparty risk – and adjustments for operational risk, deferred taxes, and others.

Let us denote the solvency capital requirements for the modules market, life, non-life, health, and counterparty risk with $S_1, \ldots, S_5$ respectively, and collect them to a vector $S = (S_1, \ldots, S_5)^{\mathrm{T}}$. Then the basic solvency capital requirement is computed by the formula

$$\text{basic solvency capital requirement} = \sqrt{S^{\mathrm{T}} A S}, \tag{1}$$

where A is some positive definite matrix of correlation parameters.

The risk modules themselves consist of sub-modules which are aggregated in the same manner. The market risk module for example consists of the interest rate, equity, spread, property, currency, and concentration risk sub-modules, which are aggregated by a likewise formula, yet with another matrix A of course.

In the following, we will focus on the square-root formula (1) and identify the Solvency II standard formula with the way in which the risk is aggregated, namely the square-root formula as stated above.

In section 1, we present a method suggested once by *Gesamtverband der Deutschen Versicherungswirtschaft* (GDV) for the reallocation of risk within the standard formula and figure out some special case.

In section 2, the method turns out to be the Euler principle applied to the standard formula, which works due to the fact that the standard formula is homogeneous.

Moreover, the standard formula constitutes a norm, which means that we have the triangle inequality at hand. In section 3, we will use this and give some estimates on the sensitivities of the reallocation method.

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In section 4, we consider the standard formula as a risk measure, and as a model for diversification – which applies to the several risks of a given portfolio, or to the risks of sub-portfolios of a portfolio, or to the risks of business lines within a company, or generally speaking to any "portfolio of risks".

So what we do *not* assume is that there is an additive decomposition of some portfolio (or aggregated outcome) X in terms of sub-portfolios (or contributions) $X_1, \ldots, X_N$ like

$$X = \sum_{k=1}^{N} X_k, \tag{2}$$

as it is typically assumed in the literature [3,5,8,10]. Instead, think of different risks within *one* portfolio, or company, as it is the case in the standard formula. Well, one could build up an internal model so as to obtain a stochastic model with the additive structure (2) – but this is another story.

2. The GDV method

The *Gesamtverband der Deutschen Versicherungswirtschaft* (GDV) once suggested an allocation method for re-allocation of risk in the framework of the Solvency II Standard Formula. We therefore call it the GDV method and at first state and prove its properties without relying on geometry. In the next section it will turn out that the GDV method is nothing else than the Euler allocation principle applied to the standard formula.

Lemma 1 (GDV Method). *Let $R \in \mathbb{R}^N$ a vector of risks and A be a symmetric matrix. Let the overall risk be given by*

$$\|R\|_A = \sqrt{R^T A R}. \tag{3}$$

Then the gradient of $R \mapsto \|R\|_A$ is given by

$$\nabla \|R\|_A = \frac{AR}{\|R\|_A} \text{ for all } \|R\|_A > 0. \tag{4}$$

The equation

$$\langle \nabla \|R\|_A, R \rangle = \|R\|_A \tag{5}$$

holds.

Remark 1. *The GDV method is a means to allocate the overall risk $\|R\|_A$ to the sub-risks R_k. It uses the sensitivities (partial derivatives)*

$$\omega_k = \frac{\partial \|R\|_A}{\partial R_k} = \frac{1}{\|R\|_A} \sum_{\ell=1}^{N} R_\ell A_{\ell k}, \quad 1 \le k \le N, \tag{6}$$

and the risk contribution of risk k to the overall risk then is $\omega_k R_k$. Equation (5) shows that the risk contributions add up to the overall risk:

$$\|R\|_A = \omega_1 R_1 + \ldots \omega_N R_N. \tag{7}$$

Remark 2. *In case the concentration risk within the Solvency market risk module vanishes, its sensitivity vanishes as well (provided the market risk is positive). This is because the correlation parameters of the concentration risk to the other sub-risks of the market risk are all zero. Therefore the market risk in terms of the concentration risk x has the form*

$$f(x) = \|R\|_A = \sqrt{c + x^2}, \quad c > 0, \tag{8}$$

where $\sqrt{c}$ denotes the market risk without concentration risk. The sensitivity of the concentration risk for $x = 0$ is

$$\omega = \frac{df}{dx}(0) = \left.\frac{x}{f(x)}\right|_{x=0} = 0. \tag{9}$$

3. The standard formula as a homogeneous function

There is a reason why equation (5) holds: A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* $\alpha \in \mathbb{R}$, if

$$f(tx) = t^\alpha f(x) \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}. \tag{10}$$

A theorem of Euler states that a function is homogeneous of degree $\alpha > 0$ if and only if

$$\langle \nabla f(x), x\rangle = \alpha f(x) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{11}$$

see [9]. A function which is homogeneous of degree 1 is briefly called homogeneous as well. Hence (5) is by Euler's theorem equivalent to the fact that the standard formula is homogeneous, and indeed we have

$$\|tR\|_A = t\,\|R\|_A \text{ for all } t > 0. \tag{12}$$

Capital allocation by (11) is called *Euler's principle* [10] and hence the GDV method (5) amounts to be Euler's principle applied to the standard formula. Under the simplifying assumption that the underlying random variables have a normal distribution, this was observed already by De Angelis and Granito [4].

Lemma 2. *The risk in direction of a risk allocation $R \neq 0$ is proportional to the directional derivative of the risk in this direction, i.e.*

$$\|y\|_A = D_{\nu_R}\,\|R\|_A \cdot |y| \quad \text{for all } y = |y|\,\nu_R, \text{ where } \nu_R = \frac{R}{|R|}. \tag{13}$$

Proof. By Euler's theorem, i.e. (5), the directional derivative of the risk is

$$D_{\nu_R}\,\|R\|_A = \langle \nabla\,\|R\|_A\,, \nu_R\rangle = \left\langle \nabla\,\|R\|_A\,, \frac{R}{|R|}\right\rangle = \frac{\|R\|_A}{|R|} = \|\nu_R\|_A\,. \tag{14}$$

This implies

$$D_{\nu_R}\,\|R\|_A \cdot |y| = \|\,|y|\,\nu_R\|_A = \|y\|_A\,. \tag{15}$$

□

Lemma 3. *Let $R \neq 0$ some risk allocation and $c > 0$. Then the sensitivities of cR and R coincide:*

$$\nabla\,\|cR\|_A = \nabla\,\|R\|_A\,. \tag{16}$$

Proof. By homogeneity, we have for all $v \in \mathbb{R}^n$

$$\begin{aligned}\langle \nabla\,\|cR\|_A\,, v\rangle &= \lim_{h\to 0}\frac{1}{h}\left(\|cR + hv\|_A - \|cR\|_A\right) = \lim_{h\to 0}\frac{c}{h}\left(\left\|R + \frac{h}{c}v\right\|_A - \|R\|_A\right)\\ &= \lim_{h'\to 0}\frac{1}{h'}\left(\|R + h'v\|_A - \|R\|_A\right) = \langle \nabla\,\|R\|_A\,, v\rangle\,.\end{aligned}$$

This implies the assertion. □

More generally, the same proof shows that the gradient of a homogeneous function of degree $\alpha \in \mathbb{R}$ is homogeneous of degree $\alpha - 1$. (The case $\alpha = 0$ is somewhat degenerate, yet true.)

4. The standard formula as a norm

A should be positive semidefinite, because it is a correlation matrix. Therefore (3) defines a seminorm. If A is positive definite, (3) defines a norm [6, p. 154].

Note that A may well contain tail-correlations – defined in one or the other manner – rather than linear correlations. Nevertheless, from an economic point of view, A being positive semidefinite is no bad choice in any case. Cf. [2,7] with respect to the interconnection between the standard formula and tail correlations.

Note that A is indeed positive semidefinite in case it contains linear (or Pearson) correlations, as we state in the following well-known Theorem without proof.

Theorem 1. *Let S be the covariation matrix and K the correlation matrix of a random vector Z. Then S and K are positive semidefinite. If there does not exist any linear combination of the entries of Z which is constant, S and K are positive definite.*

Once we know that $\|\cdot\|_A$ defines a seminorm, we can use the triangle inequality.

Lemma 4. *Let A be positive semidefinite and the diagonal elements of A be 1 (as it is the case in the standard formula). Then the absolute value of the sensitivities (6) is not greater than 1.*

Proof. The assumption that the diagonal elements of A are 1 implies

$$\|e_k\|_A = 1 \tag{17}$$

for all unit vectors e_k of the standard basis $(1 \leq k \leq N)$. Hence we obtain by the triangle inequality

$$|\omega_k| = \left|\frac{\partial\, \|R\|_A}{\partial R_k}\right| = \left|\lim_{t\to 0} \frac{\|R + t\, e_k\|_A - \|R\|_A}{t}\right| \leq \lim_{t\to 0} \frac{|t|\, \|e_k\|_A}{|t|} = 1 \tag{18}$$

for all $1 \leq k \leq N$. $\square$

Theorem 2. *Let the matrix A be positive semidefinite, have diagonal elements of 1, and have non-negative entries (as it is the case in the standard formula). Let the risk vector R be non-negative, i.e. every entry of R be non-negative. Let D be a change of the risk vector and*

$$D \geq 0 \quad \text{or} \quad -R \leq D \leq 0. \tag{19}$$

Then the following estimate on the change of the sensitivities holds:

$$\left|\nabla\, \|R + D\|_A - \nabla\, \|R\|_A\right| \leq \frac{\|D\|_A}{\|R + D\|_A}, \tag{20}$$

where the absolute value $|\cdot|$ denotes the maximum norm.

Proof. We derive

$$\begin{aligned} \nabla\, \|R + D\|_A - \nabla\, \|R\|_A &= \frac{A(R+D)}{\|R+D\|_A} - \frac{AR}{\|R\|_A} \\ &= \frac{AR}{\|R\|_A}\left(\frac{\|R\|_A}{\|R+D\|_A} - 1\right) + \frac{AD}{\|D\|_A}\frac{\|D\|_A}{\|R+D\|_A}. \end{aligned} \tag{21}$$

At first we consider the case $D \geq 0$. Then

$$\|R + D\|_A^2 - \|R\|_A^2 = D^{\mathrm{T}} A (R + D) + R^{\mathrm{T}} A D \tag{22}$$

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is non-negative, because there are solely non-negative numbers entering in the products. Hence the bracket in equation (21) is negative or zero. Therefore the two summands on the right hand side of (21) have a different sign in every component. We obtain with Lemma 4

$$\left|\nabla \|R+D\|_A - \nabla \|R\|_A\right| \leq \max\left\{\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right|, \frac{\|D\|_A}{\|R+D\|_A}\right\}. \tag{23}$$

Now the triangle inequality shows

$$\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right| = \frac{\|R+D\|_A - \|R\|_A}{\|R+D\|_A} \leq \frac{\|D\|_A}{\|R+D\|_A}, \tag{24}$$

which implies (20).

The other case is $-R \leq D \leq 0$. In this case the entries of D in the right hand side of (22) are non-positive, while all other entries are non-negative. Hence (22) is non-positive. This means that the bracket in (21) is non-negative while the entries in the last term are non-positive. Hence the estimate (23) holds in this case as well. We use the triangle inequality in the form

$$\|R\|_A = \|R+D-D\|_A \leq \|R+D\|_A + \|-D\|_A \tag{25}$$

and obtain

$$\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right| = \frac{\|R\|_A - \|R+D\|_A}{\|R+D\|_A} \leq \frac{\|D\|_A}{\|R+D\|_A}. \tag{26}$$

This completes the proof. □

Theorem 3. *Let the matrix A be positive semidefinite and have diagonal elements of 1. Then the following estimate on the change of the sensitivities holds:*

$$\left|\nabla \|R+D\|_A - \nabla \|R\|_A\right| \leq 2\frac{\|D\|_A}{\|R+D\|_A}, \tag{27}$$

where the absolute value $|\cdot|$ denotes the maximum norm.

Proof. We apply the triangle inequality to (21) and use (24) and (26), respectively. □

5. The standard formula as a risk measure, and as a model for diversification

A risk functional is defined on some domain $\mathcal{X} \subseteq L^0$ of random variables (risks) with values in $\mathbb{R} \cup \{\infty\}$. We assume that $\mathcal{X}$ is a convex cone, i.e. $\alpha X + \beta Y \in \mathcal{X}$ for all $X, Y \in \mathcal{X}$ and $\alpha, \beta > 0$. A risk functional R may have one or more of these properties[1]:

- R is *monotone*, if $R(X) \geq R(Y)$ whenever $X \leq Y$ a.s. In this case, the risk functional is called a *risk measure*.
- R is *subadditive*, if $R(X+Y) \leq R(X) + R(Y)$.
- R is *homogeneous*, if $R(tX) = tR(X)$ for all $t > 0$.
- R is *cash invariant*, if $R(X+a) = R(X) - a$ for all $a \in \mathbb{R}$. (For this to make sense we have to assume that $\mathcal{X}$ contains all constants $a \in \mathbb{R}$.)
- R is *convex*, if $R(\alpha X + (1-\alpha)Y) \leq \alpha R(X) + (1-\alpha)R(Y)$ for all $\alpha \in (0,1)$.
- R is *version independent*, if $R(X) = R(Y)$ for all $X \stackrel{d}{=} Y$.
- R is *comonotone additive*, if $R(X+Y) = R(X) + R(Y)$ for X, Y comonotone.

Note that a homogeneous risk functional is subadditive if and only if it is convex.

[1] These properties of risk functionals are taken from [8, p. 142 ff.].

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Definition 1. *Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone and $f : B \to \mathbb{R} \cup \{\infty\}$ a function. Then*

- f *is* monotone (non-decreasing), *if $f(r) \leq f(s)$ for all $r, s \in B$ with $r_k \leq s_k$ for all $1 \leq k \leq n$.*
- f *is* subadditive, *if $f(r+s) \leq f(r) + f(s)$ for all $r, s \in B$.*
- f *is* homogeneous, *if $f(tr) = tf(r)$ for all $r \in B$ and $t > 0$.*
- f *is* convex, *if $f(\alpha r + (1-\alpha)s) \leq \alpha f(r) + (1-\alpha) f(s)$ for all $r, s \in B$ and $\alpha \in (0,1)$.*
- f *is* additive, *if $f(r+s) = f(r) + f(s)$ for all $r, s \in B$.*

Again, a homogeneous function is subadditive if and only if it is convex. – Due to the convention that lower outcomes correspond to a higher risk, the signs in the notion of monotonicity of a risk functional and a function are opposite.

Note that we do not propose a property so as to maintain cash invariance. Cash invariance might not be a helpful concept for the study of diversification effects within a company, or portfolio.

Lemma 5. *Let*

$$R_k : \mathcal{X} \to B_k \subseteq \mathbb{R} \cup \{\infty\} \quad (1 \leq k \leq n) \tag{28}$$

be risk functionals and

$$B = B_1 \times \ldots \times B_n \tag{29}$$

be a convex cone. Let $f : B \to \mathbb{R} \cup \{\infty\}$ be a function. Then

$$R = f(R_1, \ldots, R_n) \tag{30}$$

is a risk functional and the following holds:

- *If $R_1, \ldots, R_n$ and f are monotone, then R is monotone, i.e. a risk measure.*
- *If $R_1, \ldots, R_n$ are subadditive, and f is monotone and subadditive, then R is subadditive.*
- *If $R_1, \ldots, R_n$ and f are homogeneous, then R is homogeneous.*
- *If $R_1, \ldots, R_n$ are convex, and f is monotone and convex, then R is convex (albeit not cash invariant in general[2]).*
- *If $R_1, \ldots, R_n$ are version independent, then R is version independent.*
- *If $R_1, \ldots, R_n$ are comonotone additive, and f is additive, then R is comonotone additive.*

The proof is straightforward.

Definition 2. *Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone and $f : B \to \mathbb{R} \cup \{\infty\}$ a function. We call f a* diversification function, *if f is monotone, homogeneous, and subadditive.*

Note that there is a related notion in information theory, namely of an *aggregation function* $f : [0,1]^n \to [0,1]$, which yet aims at a slightly other direction [1].

Lemma 6. *Let R be a risk functional and*

$$S = \max\{R; 0\}. \tag{31}$$

If R is monotone, subadditive, homogeneous, convex, or version independent, then S has the respective property as well.

This shows that one may consider non-negative risk functionals, or risk measures. The proof is straightforward. Lemma 5 implies:

Theorem 4. *Let $R_1, \ldots, R_n : \mathcal{X} \to [0, \infty)$ be non-negative, finite risk measures and $R = (R_1, \ldots, R_n)^T$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a seminorm and the restriction of f to $[0, \infty)^n$ be monotone. Then the following holds:*

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2 For reasons not known to the public a risk measure is called a *convex risk measure* in the literature when it is convex, and cash invariant.

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- *The restriction of f to $[0,\infty)^n$ is a diversification function.*
- *$f(R)$ is a risk measure.*
- *If $R_1,\ldots,R_n$ are homogeneous, then $f(R)$ is a homogeneous risk measure.*
- *If $R_1,\ldots,R_n$ are subadditive, then $f(R)$ is a subadditive risk measure.*
- *If $R_1,\ldots,R_n$ are convex, then $f(R)$ is a risk measure, which is convex (albeit not cash invariant in general).*

It is not unusual for a seminorm that its restriction to non-negative elements is monotone: All L^p-norms have this property. With respect to symmetric bilinear forms we have the following theorem:

Theorem 5. *Let $A \in \mathbb{R}^{n\times n}$ be positive semidefinite. The restriction of the seminorm $\|\cdot\|_A$ to $[0,\infty)^n$ is monotone if and only if A is non-negative, i.e. has non-negative entries only.*

Proof. Let A be non-negative. We show that the restriction of $\|\cdot\|_A$ to $[0,\infty)^n$ is monotone. Let $R,S \in \mathbb{R}^n$ with $R \geq S \geq 0$, i.e. $R_k \geq S_k \geq 0$ for all $1 \leq k \leq n$. Then

$$\|R\|_A^2 - \|S\|_A^2 = (R-S)^{\mathrm{T}}AR + S^{\mathrm{T}}A(R-S) \geq 0, \tag{32}$$

because all contributions in the sum are non-negative.

Now assume that $a_{jk} = a_{kj} < 0$ for some $1 \leq j,k \leq n$. Consider $R = c\,e_j + e_k$ and $S = e_k$, where e_j and e_k are the respective unit vectors of the standard basis and $0 < c < 2|a_{jk}|/|a_{jj}|$ in case $a_{jj} \neq 0$ and 1 otherwise. Then $0 \leq S \leq R$ and

$$\|R\|_A^2 - \|S\|_A^2 = c\,e_j^{\mathrm{T}}A(c\,e_j + e_k) + e_k^{\mathrm{T}}Ac\,e_j = c^2 a_{jj} + 2c\,a_{jk} < 0, \tag{33}$$

which shows that monotonicity is violated. □

Therefore Theorem 4 applies to the standard formula and shows that $\|R\|_A$ is a subadditive, or homogeneous risk measure, whenever the individual risk measures R_k, which are assumed to be non-negative and finite, share the respective property.

This is another hint that it could be an unskilled choice to define A by the correlation matrix of the underlying random variables: In general one would encounter negative correlation coefficients which would lead to a risk functional which would not enjoy to be a risk measure in general because of the lack of monotony. Unless, for example, the underlying risks are elliptically distributed, have mean zero, and the risk measures are value at risk of same level of confidence.

In any case, the risk measure of the standard formula *is* value at risk. Value at risk is a homogeneous, cash invariant, version independent, comonotone additive risk measure, yet not subadditive nor convex in general [8, p. 147]. Hence the standard formula with value at risk is a homogeneous, version independent risk measure. (To be precise, value at risk has to be maximized with zero to fit.)

We have two concluding remarks on diversification functions, and the standard formula. Due to the fact that the standard formula is actually nested, the first one is on the composition of diversification functions.

Lemma 7. *Let $B_1,\ldots,B_m$ be convex cones. Then $B = B_1 \times \ldots \times B_m$ is a convex cone, and vice versa.*

Lemma 8. *Let $C \subseteq (\mathbb{R}\cup\{\infty\})^m$, $B_k \subseteq \mathbb{R}\cup\{\infty\}$ be convex cones, and $g_k : C \to B_k$ be diversification functions $(1 \leq k \leq n, m \in \mathbb{N})$. Let*

$$f : B_1 \times \ldots \times B_n \longrightarrow \mathbb{R}\cup\{\infty\} \tag{34}$$

be a diversification function. Then

$$f \circ g : C \longrightarrow \mathbb{R}\cup\{\infty\} \tag{35}$$

is a diversification function. The sensitivity ω_ℓ $(1 \le \ell \le m)$ of $f \circ g$ with respect to an allocation $S \in C$ is given by

$$\omega_\ell = \frac{\partial (f \circ g)}{\partial S_\ell}(S) = \left\langle \nabla f(g(S)), \begin{pmatrix} \partial_\ell g_1(S) \\ \vdots \\ \partial_\ell g_n(S) \end{pmatrix} \right\rangle. \tag{36}$$

Remark 3. *One often encounters the special case, in which there is only one function g_k, which depends on a coordinate S_ℓ. In this case, equation (36) means*

$$\omega_\ell = \partial_k f(g(S)) \partial_\ell g_k(S). \tag{37}$$

Finally, Lemma 4 can be restated using monotony and subadditivity instead of the triangle inequality.

Definition 3. *Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone, and $f : B \to \mathbb{R} \cup \{\infty\}$ a diversification function. We call f a* normalized diversification function, *if $e_k \in B$ for all unit vectors $e_1, \ldots, e_n$ of the standard basis in $\mathbb{R}^n$, and*

$$f(e_k) = 1 \textit{ for all } 1 \le k \le n. \tag{38}$$

Theorem 6. *The sensitivities of a differentiable, normalized diversification function are not negative and not greater than 1, i.e.*

$$\frac{\partial f}{\partial x_k}(x) \in [0, 1] \textit{ for all } x \in B \textit{ and } 1 \le k \le n. \tag{39}$$

Proof. We have by monotony

$$f(x + h\, e_k) \ge f(x) \text{ for all } x \in B, h > 0 \text{ and } 1 \le k \le n, \tag{40}$$

and we obtain by subadditivity

$$f(x + h\, e_k) \le f(x) + f(h\, e_k) \text{ for all } x \in B, h > 0 \text{ and } 1 \le k \le n. \tag{41}$$

So

$$0 \le \lim_{h \searrow 0} \frac{f(x + h\, e_k) - f(x)}{h} = \frac{\partial f}{\partial x_k}(x) \le \lim_{h \searrow 0} \frac{f(h\, e_k)}{h} = 1, \tag{42}$$

as was to be shown. □

This applies to the sensitivities of the standard formula as well, for the standard formula is a normalized diversification function.

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