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SOME INTEGRAL TRANSFORMS OF THE GENERALIZED k -MITTAG-LEFFLER FUNCTION

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ABSTRACT. In the paper, the authors generalize the notion “ k -Mittag-Leffler function”, establish some integral transforms of the generalized k -Mittag-Leffler function, and derive several special and known conclusions in terms of the generalized Wright function and the generalized k -Wright function.

1. PRELIMINARIES

Throughout this paper, let \mathbb{C} , \mathbb{R} , \mathbb{R}_0^+ , \mathbb{R}^+ , \mathbb{Z}_0^- , and \mathbb{N} denote respectively the sets of complex numbers, real numbers, non-negative numbers, positive numbers, non-positive integers, and positive integers.

The Pochhammer symbol $(\lambda)_\nu$ can be defined for $\lambda, \nu \in \mathbb{C}$ by $(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$, where

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

is called the classical gamma function and its reciprocal $\frac{1}{\Gamma}$ is analytic on the whole complex plane \mathbb{C} . See [14, Chapter 5], [16, Section 1], and [24, Section 1.1]. In particular, when $\nu \in \{0\} \cup \mathbb{N}$, the quantity

$$(\lambda)_n = \begin{cases} 1, & \nu = 0 \\ \lambda(\lambda+1) \cdots (\lambda+n-1), & n \in \mathbb{N} \end{cases}$$

is called the rising factorial. See [17] and closely-related references therein.

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The k -Pochhammer symbol $(\lambda)_{n,k}$ was defined in [2] for $\lambda, \nu \in \mathbb{C}$ and $k \in \mathbb{R}$ by

$$(\lambda)_{\nu,k} = \frac{\Gamma_k(\lambda + \nu k)}{\Gamma_k(\lambda)}, \quad (1.1)$$

where

$$\Gamma_k(z) = k^{z/k-1} \Gamma\left(\frac{z}{k}\right) \quad (1.2)$$

is called the k -gamma function. In particular,

$$(\lambda)_{n,k} = \begin{cases} 1, & n = 0; \\ \lambda(\lambda + k) \cdots (\lambda + (n-1)k), & n \in \mathbb{N}. \end{cases}$$

In 1903, Mittag-Leffler, a Swedish mathematician, introduced and investigated in [12, 13] the so-called Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

for $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}_0^+$. In 1905, Wiman [25] generalized $E_\alpha(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

where $z \in \mathbb{C}$, $\alpha, \beta \in \mathbb{C}$, and $\Re(\alpha), \Re(\beta) > 0$. In 1971, Prabhakar [15] introduced the function

$$E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}$$

for $z \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{C}$, and $\Re(\alpha), \Re(\beta), \Re(\gamma) > 0$. In 2012, Dorrego and Cerutti [3] introduced the k -Mittag-Leffler function

$$E_{k,\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!},$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $k \in \mathbb{R}$, $\Re(\alpha), \Re(\beta), \Re(\gamma) > 0$, and $(\gamma)_{n,k}$ is the k -Pochhammer symbol. In 2012, a generalization of the k -Mittag-Leffler function

$$GE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}$$

for $\alpha, \beta, \gamma \in \mathbb{C}$, $k \in \mathbb{R}$, $\Re(\alpha), \Re(\beta), \Re(\gamma) > 0$, and $q \in (0, 1) \cup \mathbb{N}$ was introduced and studied in [6]. For more information on generalizations of the Mittag-Leffler function, please refer to the papers [1, 9, 19, 20, 21] and closely-related references therein.

In this paper, we consider a more general generalization

$$E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{(\delta)_n}, \quad (1.3)$$

where $\alpha, \beta, \gamma, \delta, \tau \in \mathbb{C}$, $k \in \mathbb{R}$, $\Re(\alpha), \Re(\beta) > 0$, and $\delta \neq 0, -1, -2, \dots$. It is clear that $E_{k,\alpha,\beta,1}^{\gamma,\tau}(z) = GE_{k,\alpha,\beta}^{\gamma,\tau}(z)$ and $E_{k,\alpha,\beta,1}^{\gamma,1}(z) = E_{k,\alpha,\beta}^\gamma(z)$.

It is well known [14, 18] that the generalized hypergeometric function can be defined by

$${}_pF_q[(\alpha_p); (\beta_q); z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (\alpha_j)_n}{\prod_{j=1}^q (\beta_j)_n} \frac{z^n}{n!}$$

for $|z| < 1$ and $p \leq q$ with $p = q + 1$ and that the generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is given by the series

$${}_p\Psi_q(z) = {}_p\Psi_q[(a_i, \alpha_i)_{1,p}; (b_j, \beta_j)_{1,q}; z] = \frac{\prod_{j=1}^q \Gamma(\beta_j)}{\prod_{i=1}^p \Gamma(\alpha_i)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!} \quad (1.4)$$

for $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}$ with $1 \leq i \leq p$ and $1 \leq j \leq q$. Asymptotic behavior of the function ${}_p\Psi_q(z)$ for large values of argument of $z \in \mathbb{C}$ were studied in [5, 26, 27] under the condition $\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1$. If putting $\alpha_1 = \dots = \alpha_p = \beta_1 = \dots = \beta_q = 1$ in (1.4), then

$$\begin{aligned} {}_p\Psi_q(z) &= {}_p\Psi_q[(a_1, 1), \dots, (a_p, 1); (b_1, 1), \dots, (b_q, 1); z] \\ &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{i=1}^p \Gamma(a_i)} {}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z]. \end{aligned}$$

The generalized k -Wright function was introduced in [7] as

$${}_p\Psi_q^k(z) = {}_p\Psi_q^k[(a_i, \alpha_i)_{1,p}; (b_j, \beta_j)_{1,q}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma_k(a_i + \alpha_i n)}{\prod_{j=1}^q \Gamma_k(b_j + \beta_j n)} \frac{z^n}{n!}$$

for $k \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R} \setminus \{0\}$, and $a_i + \alpha_i n, b_j + \beta_j n \in \mathbb{C} \setminus k\mathbb{Z}^-$ with $1 \leq p$ and $1 \leq j \leq q$. The Euler transform of a function $f(z)$ is defined by

$$B\{f(z); \alpha, \beta\} = \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} f(z) dz \quad (1.5)$$

for $\alpha, \beta \in \mathbb{C}$ and $\Re(\alpha), \Re(\beta) > 0$. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L\{f(t); s\} = \int_0^{\infty} e^{-st} f(t) dt, \quad \Re(s) > 0.$$

The Fourier transform of a function $u = u(t) \in S(\mathbb{R})$ is defined by

$$\hat{u} = \mathfrak{F}[u](w) = \int_{\mathbb{R}} u(t) e^{iwt} dt,$$

where $S(\mathbb{R})$ denotes the Schwartzian space of rapidly decreasing test functions on \mathbb{R} . The fractional Fourier transform of order α for $0 \leq \alpha \leq 1$ was defined in [10] by

$$\hat{u}_\alpha(w) = \mathfrak{F}_\alpha[u](w) = \int_{\mathbb{R}} e^{iw^{1/\alpha} t} u(t) dt \quad (1.6)$$

for $u \in \Phi(\mathbb{R})$, where

$$\Phi(\mathbb{R}) = \{\varphi \in S(\mathbb{R}) : \hat{\varphi} \in V(\mathbb{R})\}$$

denotes the Lizorkin space of functions and

$$V(\mathbb{R}) = \{v \in S(\mathbb{R}) : v^{(n)}(0) = 0, n = 0, 1, 2, \dots\}.$$

When $\alpha = 1$, the quantity (1.6) reduces to the Fourier transform; when $w > 0$, the transform (1.6) reduces to the fractional Fourier transform.

The aim of this paper is to present the Euler, Laplace, Whittaker, and Fractional Fourier transforms of the generalized k -Mittag-Leffler function (1.3). From these conclusions, we can derive some known and new results.

2. MAIN RESULTS

Now we are in a position to state and prove our main results.

Theorem 1. *If $k \in \mathbb{R}$, $\alpha, \beta, \gamma, a, b, \sigma \in \mathbb{C}$, $\Re(\alpha), \Re(\beta) > 0$, $\delta \neq 0, -1, -2, \dots$, and $q > 0$, then*

$$\int_0^1 z^{\alpha-1}(1-z)^{b-1} E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma) dz = \frac{k^{1-\beta/k}\Gamma(b)\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})} {}_3\Psi_3\left[\left(\frac{\gamma}{k}, \tau\right), (a, \sigma), (1, 1); \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (a+b, \sigma), (\delta, 1); k^{\tau-\alpha/k}x\right]. \quad (2.1)$$

Proof. Denote the left-hand side of the equation (2.1) by \mathcal{I}_1 . By definition of the generalized k -Mittag-Leffler function and (1.5), we have

$$\mathcal{I}_1 = \int_0^1 z^{\alpha-1}(1-z)^{b-1} E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma) dz = \int_0^1 z^{\alpha-1}(1-z)^{b-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{(xz^\sigma)^n}{(\delta)_n} dz.$$

By interchanging the order of the integration and summation, we obtain

$$\mathcal{I}_1 = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{x^n}{(\delta)_n} \int_0^1 z^{\alpha+\sigma n-1}(1-z)^{b-1} dz = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{x^n}{(\delta)_n} \frac{\Gamma(a + \sigma n)\Gamma(b)}{\Gamma(a + b + \sigma n)}.$$

From (1.1) and (1.2), we acquire

$$\mathcal{I}_1 = \frac{k^{1-\beta/k}\Gamma(b)\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\gamma}{k} + n\tau)\Gamma(a + \sigma n)\Gamma(\delta)}{\Gamma(\frac{\beta}{k} + \frac{\alpha}{k}n)\Gamma(a + b + \sigma n)\Gamma(\delta + n)} \frac{k^{n\tau}}{k^{\alpha n/k}}.$$

In view of (1.4), we arrive at the desired result. □

Remark 2.1. Taking $\delta = 1$ in Theorem 1 gives [23, Eq. (24)] which reads that

$$\int_0^1 z^{\alpha-1}(1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz = \frac{k^{1-\beta/k}\Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\Psi_2\left[\left(\frac{\gamma}{k}, \tau\right), (a, \sigma); \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (a+b, \sigma); k^{\tau-\alpha/k}x\right].$$

Setting $\delta = 1$ and $\tau = q > 0$ in Theorem 1 leads to [23, Eq. (25)] which states that

$$\int_0^1 z^{\alpha-1}(1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{k^{1-\beta/k}\Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\Psi_2\left[\left(\frac{\gamma}{k}, q\right), (a, \sigma); \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (a+b, \sigma); k^{q-\alpha/k}x\right].$$

Further letting $k = 1$ in the above equation derives [23, Eq. (26)] which formulates that

$$\int_0^1 z^{\alpha-1}(1-z)^{b-1} E_{\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{\Gamma(b)}{\Gamma(\gamma)} {}_2\Psi_2[(\gamma, q), (a, \sigma); (\beta, \alpha), (a+b, \sigma); x].$$

Theorem 2. *If $k \in \mathbb{R}$, $\alpha, \beta, \gamma, a, \sigma \in \mathbb{C}$, $\Re(\alpha), \Re(\beta), \Re(s) > 0$, $\tau \in \mathbb{C}$, $|\frac{x}{s^\sigma}| < 1$, and $\delta \neq 0, -1, -2, \dots$, then*

$$\int_0^\infty z^{\alpha-1} e^{-sz} E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma) dz = \frac{k^{1-\beta/k}\Gamma(\delta)}{s^\alpha \Gamma(\frac{\gamma}{k})} {}_3\Psi_2\left[\left(\frac{\gamma}{k}, \tau\right), (a, \sigma), (1, 1); \left(\frac{\beta}{k}, \frac{\alpha}{k}\right), (\delta, 1); \frac{xk^{\tau-\alpha/k}}{s^\sigma}\right]. \quad (2.2)$$

Proof. Denote the left-hand side of (2.2) by \mathcal{I}_2 . Applying definition of the generalized k -Mittag-Leffler function results in

$$\mathcal{I}_2 = \int_0^\infty z^{\alpha-1} e^{-sz} E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma) dz = \int_0^\infty z^{\alpha-1} e^{-sz} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{(xz^\sigma)^n}{(\delta)_n} dz.$$

Interchanging the order of the integration and summation leads to

$$\mathcal{I}_2 = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{x^n}{(\delta)_n} \int_0^{\infty} z^{a+\sigma n-1} e^{-sz} dz.$$

In view of definition of the Laplace transform, we have

$$\mathcal{I}_2 = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{x^n}{(\delta)_n} \frac{\Gamma(\sigma n + a)}{s^{\sigma n + a}}.$$

Utilizing (1.1) and (1.2) derives the required result. \square

Remark 2.2. If setting $\delta = 1$ in Theorem 2, then we deduce [23, Eq. (27)] which formulates that

$$\int_0^{\infty} z^{a-1} e^{-sz} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz = \frac{k^{1-\beta/k} s^{-a}}{\Gamma(\frac{\gamma}{k})} {}_2\Psi_1 \left[\left(\frac{\gamma}{k}, \tau \right), (a, \sigma); \left(\frac{\beta}{k}, \frac{\alpha}{k} \right); \frac{xk^{\tau-\alpha/k}}{s^\sigma} \right].$$

If taking $\tau = q > 0$, $k = 1$, and $\delta = 1$, then we acquire [23, Eq. (29)] which reads that

$$\int_0^{\infty} z^{a-1} e^{-sz} E_{\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{s^{-a}}{\Gamma(\gamma)} {}_2\Psi_1 \left[(\gamma, q), (a, \sigma); (\beta, \alpha); \frac{x}{s^\sigma} \right].$$

If taking $k = q = 1$ and $\delta = 1$ in the above equation reduces to

$$\int_0^{\infty} z^{a-1} e^{-sz} E_{\alpha,\beta}^{\gamma}(xz^\sigma) dz = \frac{s^{-a}}{\Gamma(\gamma)} {}_2\Psi_1 \left[(\gamma, 1), (a, \sigma); (\beta, \alpha); \frac{x}{s^\sigma} \right]$$

which is the main result in [22].

Recall that

$$\int_0^{\infty} t^{v-1} e^{-t/2} W_{\lambda,\mu}(t) dt = \frac{\Gamma(\frac{1}{2} + \mu + v) \Gamma(\frac{1}{2} - \mu + v)}{\Gamma(1 - \lambda + v)}, \quad \Re(v \pm \mu) > -\frac{1}{2}, \quad (2.3)$$

where the Whittaker function

$$W_{\lambda,\mu}(t) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \lambda)} M_{\lambda,\mu}(t) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \lambda)} M_{\lambda,-\mu}(t)$$

and

$$M_{\lambda,\mu}(t) = z^{\mu+1/2} e^{-t/2} {}_1F_1 \left(\frac{1}{2} + \mu + v; 2\mu + 1; t \right)$$

are given in [4, 11].

Theorem 3. If $k \in \mathbb{R}$, $\alpha, \beta, \gamma, \delta, \tau, \eta \in \mathbb{C}$, $\Re(\alpha), \Re(\beta), \Re(\rho) > 0$, $\delta \neq 0, -1, -2, \dots$, and $\Re(\rho \pm \mu) > -\frac{1}{2}$, then

$$\int_0^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(wt^\eta) dt = \frac{k^{1-\beta/k} p^{-\rho} \Gamma(\delta)}{\Gamma(\frac{\gamma}{k})} \times {}_3\Psi_3 \left[\left(\frac{\gamma}{k}, \tau \right), \left(\frac{1}{2} \pm \mu + \rho, \eta \right), (1, 1); \left(\frac{\beta}{k}, \frac{\alpha}{k} \right), (1 - \lambda + \rho, \eta), (\delta, 1); \frac{wk^{\tau-\alpha/k}}{p^\eta} \right].$$

Proof. Letting $pt = v$, interchanging the integration and summation, and using the formula for the Whittaker transform (2.3) yields

$$\begin{aligned} & \int_0^{\infty} t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(wt^\eta) dt \\ &= \int_0^{\infty} e^{-v/2} \left(\frac{v}{p} \right)^{\rho-1} W_{\lambda,\mu}(v) \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)} \frac{w^n}{(\delta)_n} \left(\frac{v}{p} \right)^{\delta n} \frac{1}{p} dv \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)(\delta)_n} \frac{w^n}{\delta} \int_0^{\infty} e^{-v/2} \left(\frac{v}{p}\right)^{\rho-1} \left(\frac{v}{p}\right)^{\delta n} W_{\lambda,\mu}(v) \frac{1}{p} dv \\
&= \frac{1}{p^\rho} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)(\delta)_n} \left(\frac{w}{p\delta}\right)^n \int_0^{\infty} e^{-v/2} v^{\delta n + \rho - 1} W_{\lambda,\mu}(v) dv \\
&= \frac{1}{p^\rho} \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} \Gamma(\frac{1}{2} + \mu + \delta n + \rho) \Gamma(\frac{1}{2} - \mu + \delta n + \rho)}{\Gamma_k(\alpha n + \beta)(\delta)_n \Gamma(1 - \lambda + \delta n + \rho)} \left(\frac{w}{p\delta}\right)^n.
\end{aligned}$$

In view of (1.1) and (1.2), we find the desired result. \square

Remark 2.3. Taking $\delta = 1$ in Theorem 3 gives [23, Eq. (30)] which reads that

$$\begin{aligned}
&\int_0^1 t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\eta) dt \\
&= \frac{k^{1-\beta/k} p^{-\rho}}{\Gamma(\frac{\gamma}{k})} {}_2\Psi_2 \left[\left(\frac{\gamma}{k}, \tau \right), \left(\frac{1}{2} \pm \mu + \rho, \eta \right); \left(\frac{\beta}{k}, \frac{\alpha}{k} \right), (1 - \lambda + \rho, \eta); \frac{wk^{\tau-\alpha/k}}{p^\eta} \right].
\end{aligned}$$

Setting $\tau = q > 0$, $k = 1$, and $\delta = 1$ results in [23, Eq. (32)] which states that

$$\begin{aligned}
&\int_0^1 t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{\alpha,\beta}^{\gamma,q}(wt^\eta) dt \\
&= \frac{1}{p^\rho \Gamma(\gamma)} {}_2\Psi_2 \left[(\gamma, \tau), \left(\frac{1}{2} \pm \mu + \rho, \eta \right); (\beta, \alpha), (1 - \lambda + \rho, \eta); \frac{w}{p^\eta} \right].
\end{aligned}$$

Letting $q = 1$ in the above equation derives a result given in [22].

Theorem 4. If $k \in \mathbb{R}$, $\alpha, \beta, \gamma, a, b, \sigma \in \mathbb{C}$, $\Re(\alpha), \Re(\beta) > 0$, $\tau \in \mathbb{C}$, and $\delta \neq 0, -1, -2, \dots$, then

$$\mathfrak{F}_\sigma[E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t)](w) = \frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} (-1)^n \frac{n! k^{(\tau-\beta/k)n} \Gamma(\frac{\gamma}{k} + n\tau) i^{n-1} w^{-(n+1)/\sigma}}{(\delta)_n \Gamma(\frac{\alpha}{k}n + \frac{\beta}{k})}.$$

Proof. Using definitions of the generalized k -Mittag-Leffler function and the fractional Fourier transform and interchanging the integration and summation give

$$\mathfrak{F}_\sigma[E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t)](w) = \int_0^1 \exp(iw^{1/\sigma}t) E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t) dt = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)(\delta)_n} \int_R \exp(iw^{1/\sigma}t) t^n dt.$$

Letting $iw^{1/\sigma}t = -\eta$ reduces to

$$\begin{aligned}
\mathfrak{F}_\sigma[E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t)](w) &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)(\delta)_n} \int_{-\infty}^0 \exp(-\eta) \left(\frac{-\eta}{iw^{1/\sigma}}\right)^n \left(\frac{-d\eta}{iw^{1/\sigma}}\right) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n + \beta)(\delta)_n i^{n+1} w^{(n+1)/\sigma} (-1)^n} \int_0^{\infty} e^{-\eta} \eta^n d\eta \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k} i^{-n-1} w^{-(n+1)/\sigma} (-1)^n n!}{\Gamma_k(\alpha n + \beta)(\delta)_n}
\end{aligned}$$

Further using the formulas (1.1) and (1.2) arrives at the required result. \square

Remark 2.4. If taking $\delta = 1$ in Theorem 4, then the equation

$$\mathfrak{F}_\sigma[E_{k,\alpha,\beta}^{\gamma,\tau}(t)](w) = \frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} (-1)^n \frac{k^{(\tau-\beta/k)n} \Gamma(\frac{\gamma}{k} + n\tau) i^{n-1} w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha}{k}n + \frac{\beta}{k})}.$$

in [23, Eq. (33)] follows readily.

If setting $\delta = 1$ and $\tau = q$ in Theorem 4, then the equation

$$\mathfrak{S}_\sigma[E_{k,\alpha,\beta}^{\gamma,\tau}(t)](w) = \frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} (-1)^n \frac{k^{(q-\beta/k)n} \Gamma(\frac{\gamma}{k} + nq) i^{n-1} w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha}{k}n + \frac{\beta}{k})}$$

in [23, Eq. (34)] can be derived immediately.

If letting $\delta = k = 1$ and $\tau = q$ in Theorem 4, then

$$\mathfrak{S}_\sigma[E_{\alpha,\beta}^{\gamma,q}(t)](w) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\gamma + n\tau) i^{n-1} w^{-(n+1)/\sigma}}{\Gamma(\alpha n + \beta)}$$

in [23, Eq. (34)] can be deduced straightforwardly.

Remark 2.5. In [8, Section 3], the quantity i^k for $k \in \mathbb{N}$ was computed generally by three approaches.

3. CONCLUDING REMARK

Some integral transforms of the generalized k -Mittag-Leffler function are established and the results are expressed in terms of the generalized Wright function. By taking $\delta = 1$ and using the formulas (1.1) and (1.2), we express Theorems 1 to 3 in terms of the generalized k -Wright function as follows.

It is noted that, using the appropriate formulas mentioned in Section 1, one can easily express the Euler integral in terms of the k -Wright function as

$$\int_0^1 z^{a-1} (1-z)^{b-1} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz = \frac{\Gamma(b)k^b}{\Gamma_k(\gamma)} {}_2\Psi_2^k[(\gamma, \tau k), (ak, \sigma k); (\beta, \alpha), ((a+b)k, \sigma k); x].$$

By applying suitable formula for the k -gamma function, Theorem 2 can be expressed in terms of the k -Wright function as

$$\int_0^1 z^{a-1} e^{-sz} E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma) dz = \frac{k^{2-\gamma/k}}{(sk)^a \Gamma_k(\gamma)} {}_2\Psi_1^k[(\gamma, \tau k), (ak, \sigma k); (\beta, \alpha); \frac{x}{(ks)^\sigma}].$$

Theorem 3 can be expressed in terms of the k -Wright function as

$$\begin{aligned} & \int_0^1 t^{\rho-1} e^{-pt/2} W_{\lambda,\mu}(pt) E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\eta) dt \\ &= \frac{p^{-\rho} k^{1-\rho-\lambda}}{\Gamma_k(\gamma)} {}_2\Psi_2^k[(\gamma, \tau k), \left(\left(\frac{1}{2} \pm \mu + \rho\right)k, \eta k\right); (\beta, \alpha), ((1-\lambda+\rho)k, \eta k); \frac{xk^{\tau-\eta-\alpha/k}}{p^\eta}]. \end{aligned}$$

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