Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1

Peer-reviewed version available at *Publications de l'Institut Mathematique (Beograd)* **2019**, *106*, 125-133; doi:10.2298/PIM1920125Q

Article

SOME INTEGRAL TRANSFORMS OF THE GENERALIZED k-MITTAG-LEFFLER FUNCTION

FENG QI

Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province 454010, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City 300387, China

KOTTAKKARAN SOOPPY NISAR

Department of Mathematics, College of Arts and Science, Prince Sattam bin Abdulaziz University, Wadi Al dawaser, Riyadh region 11991, Saudi Arabia

Abstract. In the paper, the authors generalize the notion "k-Mittag-Leffler function", establish some integral transforms of the generalized k-Mittag-Leffler function, and derive several special and known conclusions in terms of the generalized Wright function and the generalized k-Wright function.

1. Preliminaries

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ denote respectively the sets of complex numbers, real numbers, non-negative numbers, positive numbers, non-positive integers, and positive integers.

The Pochhammer symbol $(\lambda)_\nu$ can be defined for $\lambda, \nu \in \mathbb{C}$ by $(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$, where

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-$$

is called the classical gamma function and its reciprocal $\frac{1}{\Gamma}$ is analytic on the whole complex plane $\mathbb{C}$. See [14, Chapter 5], [16, Section 1], and [24, Section 1.1]. In particular, when $\nu \in \{0\} \cup \mathbb{N}$, the quantity

$$(\lambda)_n = \begin{cases} 1, & \nu = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbb{N} \end{cases}$$

is called the rising factorial. See [17] and closely-related references therein.

E-mail addresses: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com, ksnisar1@gmail.com, n.sooppy@psau.edu.sa.

2010 *Mathematics Subject Classification.* Primary 33E12; Secondary 33C20, 44A20, 44A30, 65R10.

Key words and phrases. integral transform; generalized k-Mittag-Leffler function; generalized Wright function; generalized k-Wright function; gamma function.

This paper was typeset using $\mathcal{A}_\mathcal{M}\mathcal{S}$-LaTeX.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1

Peer-reviewed version available at Publications de l'Institut Mathematique (Beograd) 2019, 106, 125-133; doi:10.2298/PIM1920125Q

The k-Pochhammer symbol $(\lambda)_{n,k}$ was defined in [2] for $\lambda,\nu\in\mathbb{C}$ and $k\in\mathbb{R}$ by

$$(\lambda)_{\nu,k}=\frac{\Gamma_k(\lambda+\nu k)}{\Gamma_k(\lambda)}, \tag{1.1}$$

where

$$\Gamma_k(z)=k^{z/k-1}\Gamma\biggl(\frac{z}{k}\biggr) \tag{1.2}$$

is called the k-gamma function. In particular,

$$(\lambda)_{n,k}=\begin{cases}1, & n=0;\\ \lambda(\lambda+k)\cdots(\lambda+(n-1)k), & n\in\mathbb{N}.\end{cases}$$

In 1903, Mittag-Leffler, a Swedish mathematician, introduced and investigated in [12, 13] the so-called Mittag-Leffler function

$$E_\alpha(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\alpha n+1)}$$

for $z\in\mathbb{C}$ and $\alpha\in\mathbb{R}_0^+$. In 1905, Wiman [25] generalized $E_\alpha(z)$ as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\alpha n+\beta)},$$

where $z\in\mathbb{C}$, $\alpha,\beta\in\mathbb{C}$, and $\Re(\alpha),\Re(\beta)>0$. In 1971, Prabhakar [15] introduced the function

$$E_{\alpha,\beta}^\gamma(z)=\sum_{n=0}^\infty\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$$

for $z\in\mathbb{C}$, $\alpha,\beta,\gamma\in\mathbb{C}$, and $\Re(\alpha),\Re(\beta),\Re(\gamma)>0$. In 2012, Dorrego and Cerutti [3] introduced the k-Mittag-Leffler function

$$E_{k,\alpha,\beta}^\gamma(z)=\sum_{n=0}^\infty\frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!},$$

where $\alpha,\beta,\gamma\in\mathbb{C}$, $k\in\mathbb{R}$, $\Re(\alpha),\Re(\beta),\Re(\gamma)>0$, and $(\gamma)_{n,k}$ is the k-Pochhammer symbol. In 2012, a generalization of the k-Mittag-Leffler function

$$GE_{k,\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^\infty\frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!}$$

for $\alpha,\beta,\gamma\in\mathbb{C}$, $k\in\mathbb{R}$, $\Re(\alpha),\Re(\beta),\Re(\gamma)>0$, and $q\in(0,1)\cup\mathbb{N}$ was introduced and studied in [6]. For more information on generalizations of the Mittag-Leffler function, please refer to the papers [1, 9, 19, 20, 21] and closely-related references therein.

In this paper, we consider a more general generalization

$$E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(z)=\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{(\delta)_n}, \tag{1.3}$$

where $\alpha,\beta,\gamma,\delta,\tau\in\mathbb{C}$, $k\in\mathbb{R}$, $\Re(\alpha),\Re(\beta)>0$, and $\delta\ne0,-1,-2,\dotsc$. It is clear that $E_{k,\alpha,\beta,1}^{\gamma,\tau}(z)=GE_{k,\alpha,\beta}^{\gamma,\tau}(z)$ and $E_{k,\alpha,\beta,1}^{\gamma,1}(z)=E_{k,\alpha,\beta}^{\gamma}(z)$.

It is well known [14, 18] that the generalized hypergeometric function can be defined by

$${}_pF_q[(\alpha_p);(\beta_q);z]=\sum_{n=0}^\infty\frac{\Pi_{j=1}^p(\alpha_j)_n}{\Pi_{j=1}^q(\beta_j)_n}\frac{z^n}{n!}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1

Peer-reviewed version available at *Publications de l'Institut Mathematique (Beograd)* **2019**, *106*, 125-133; doi:10.2298/PIM1920125Q

for $|z| < 1$ and $p \le q$ with $p = q + 1$ and that the generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is given by the series

$$ {}_p\Psi_q(z) = {}_p\Psi_q[(a_i, \alpha_i)_{1,p}; (b_j, \beta_j)_{1,q}; z] = \frac{\prod_{j=1}^{q} \Gamma(\beta_j)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!} \tag{1.4} $$

for $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}$ with $1 \le i \le p$ and $1 \le j \le q$. Asymptotic behavior of the function ${}_p\Psi_q(z)$ for large values of argument of $z \in \mathbb{C}$ were studied in [5, 26, 27] under the condition $\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1$. If putting $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ in (1.4), then

$$ \begin{aligned} {}_p\Psi_q(z) &= {}_p\Psi_q[(a_1, 1), \dots, (a_p, 1); (b_1, 1), \dots, (b_q, 1); z] \\ &= \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} {}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z]. \end{aligned} $$

The generalized k-Wright function was introduced in [7] as

$$ {}_p\Psi_q^k(z) = {}_p\Psi_q^k[(a_i, \alpha_i)_{1,p}; (b_j, \beta_j)_{1,q}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + \alpha_i n)}{\prod_{j=1}^{q} \Gamma_k(b_j + \beta_j n)} \frac{z^n}{n!} $$

for $k \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R} \setminus \{0\}$, and $a_i + \alpha_i n, b_j + \beta_j n \in \mathbb{C} \setminus k\mathbb{Z}^-$ with $1 \le p$ and $1 \le j \le q$.

The Euler transform of a function $f(z)$ is defined by

$$ B\{f(z); \alpha, \beta\} = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} f(z) \, \mathrm{d}\, z \tag{1.5} $$

for $\alpha, \beta \in \mathbb{C}$ and $\Re(\alpha), \Re(\beta) > 0$. The Laplace transform of a function $f(t)$ is defined as

$$ F(s) = L\{f(t); s\} = \int_0^{\infty} e^{-st} f(t) \, \mathrm{d}\, t, \quad \Re(s) > 0. $$

The Fourier transform of a function $u = u(t) \in S(\mathbb{R})$ is defined by

$$ \widehat{u} = \Im[u](w) = \int_{\mathbb{R}} u(t) e^{iwt} \, \mathrm{d}\, t, $$

where $S(\mathbb{R})$ denotes the Shwartzian space of rapidly decreasing test functions on $\mathbb{R}$. The fractional Fourier transform of order α for $0 \le \alpha \le 1$ was defined in [10] by

$$ \hat{u}_\alpha(w) = \Im_\alpha[u](w) = \int_{\mathbb{R}} e^{iw^{1/\alpha} t} u(t) \, \mathrm{d}\, t \tag{1.6} $$

for $u \in \Phi(\mathbb{R})$, where

$$ \Phi(\mathbb{R}) = \{\varphi \in S(\mathbb{R}) : \hat{\varphi} \in V(\mathbb{R})\} $$

denotes the Lizorkin space of functions and

$$ V(\mathbb{R}) = \{v \in S(\mathbb{R}) : v^{(n)}(0) = 0, n = 0, 1, 2, \dots\}. $$

When $\alpha = 1$, the quantity (1.6) reduces to the Fourier transform; when $w > 0$, the transform (1.6) reduces to the fractional Fourier transform.

The aim of this paper is to present the Euler, Laplace, Whittaker, and Fractional Fourier transforms of the generalized k-Mittag-Leffler function (1.3). From these conclusions, we can derive some known and new results.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1

Peer-reviewed version available at Publications de l'Institut Mathematique (Beograd) 2019, 106, 125-133; doi:10.2298/PIM1920125Q

2. Main results

Now we are in a position to state and prove our main results.

Theorem 1. *If $k\in\mathbb{R}$, $\alpha,\beta,\gamma,a,b,\sigma\in\mathbb{C}$, $\Re(\alpha),\Re(\beta)>0$, $\delta\ne0,-1,-2,\dots$, and $q>0$, then*

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma)\,\mathrm{d}\,z \\ =\frac{k^{1-\beta/k}\Gamma(b)\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})}{}_3\Psi_3\biggl[\biggl(\frac{\gamma}{k},\tau\biggr),(a,\sigma),(1,1);\biggl(\frac{\beta}{k},\frac{\alpha}{k}\biggr),(a+b,\sigma),(\delta,1);k^{\tau-\alpha/k}x\biggr]. \quad (2.1)$$

Proof. Denote the left-hand side of the equation (2.1) by $\mathcal{I}_1$. By definition of the generalized k-Mittag-Leffler function and (1.5), we have

$$\mathcal{I}_1=\int_0^1 z^{a-1}(1-z)^{b-1}E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma)\,\mathrm{d}\,z=\int_0^1 z^{a-1}(1-z)^{b-1}\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{(xz^\sigma)^n}{(\delta)_n}\,\mathrm{d}\,z.$$

By interchanging the order of the integration and summation, we obtain

$$\mathcal{I}_1=\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{x^n}{(\delta)_n}\int_0^1 z^{a+\sigma n-1}(1-z)^{b-1}\,\mathrm{d}\,z=\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{x^n}{(\delta)_n}\frac{\Gamma(a+\sigma n)\Gamma(b)}{\Gamma(a+b+\sigma n)}.$$

From (1.1) and (1.2), we acquire

$$\mathcal{I}_1=\frac{k^{1-\beta/k}\Gamma(b)\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})}\sum_{n=0}^\infty\frac{\Gamma(\frac{\gamma}{k}+n\tau)\Gamma(a+\sigma n)\Gamma(\delta)}{\Gamma(\frac{\beta}{k}+\frac{\alpha}{k}n)\Gamma(a+b+\sigma n)\Gamma(\delta+n)}\frac{k^{n\tau}}{k^{\alpha n/k}}.$$

In view of (1.4), we arrive at the desired result. □

Remark 2.1. Taking $\delta=1$ in Theorem 1 gives [23, Eq. (24)] which reads that

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)\,\mathrm{d}\,z=\frac{k^{1-\beta/k}\Gamma(b)}{\Gamma(\frac{\gamma}{k})}{}_2\Psi_2\biggl[\biggl(\frac{\gamma}{k},\tau\biggr),(a,\sigma);\biggl(\frac{\beta}{k},\frac{\alpha}{k}\biggr),(a+b,\sigma);k^{\tau-\alpha/k}x\biggr].$$

Setting $\delta=1$ and $\tau=q>0$ in Theorem 1 leads to [23, Eq. (25)] which states that

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma)\,\mathrm{d}\,z=\frac{k^{1-\beta/k}\Gamma(b)}{\Gamma(\frac{\gamma}{k})}{}_2\Psi_2\biggl[\biggl(\frac{\gamma}{k},q\biggr),(a,\sigma);\biggl(\frac{\beta}{k},\frac{\alpha}{k}\biggr),(a+b,\sigma);k^{q-\alpha/k}x\biggr].$$

Further letting $k=1$ in the above equation derives [23, Eq. (26)] which formulates that

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{\alpha,\beta}^{\gamma,q}(xz^\sigma)\,\mathrm{d}\,z=\frac{\Gamma(b)}{\Gamma(\gamma)}{}_2\Psi_2[(\gamma,q),(a,\sigma);(\beta,\alpha),(a+b,\sigma);x].$$

Theorem 2. *If $k\in\mathbb{R}$, $\alpha,\beta,\gamma,a,\sigma\in\mathbb{C}$, $\Re(\alpha),\Re(\beta),\Re(s)>0$, $\tau\in\mathbb{C}$, $\left|\frac{x}{s^\sigma}\right|<1$, and $\delta\ne0,-1,-2,\dots$, then*

$$\int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(xz^\sigma)\,\mathrm{d}\,z=\frac{k^{1-\beta/k}\Gamma(\delta)}{s^a\Gamma(\frac{\gamma}{k})}{}_3\Psi_2\biggl[\biggl(\frac{\gamma}{k},\tau\biggr),(a,\sigma),(1,1);\biggl(\frac{\beta}{k},\frac{\alpha}{k}\biggr),(\delta,1);\frac{xk^{\tau-\alpha/k}}{s^\sigma}\biggr]. \quad (2.2)$$

Proof. Denote the left-hand side of (2.2) by $\mathcal{I}_2$. Applying definition of the generalized k-Mittag-Leffler function results in

$$\mathcal{I}_2=\int_0^\infty z^{a-1}e^{-sz}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)\,\mathrm{d}\,z=\int_0^\infty z^{a-1}e^{-sz}\sum_{n=0}^\infty\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{(xz^\sigma)^n}{(\delta)_n}\,\mathrm{d}\,z.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1

Peer-reviewed version available at Publications de l'Institut Mathematique (Beograd) 2019, 106, 125-133; doi:10.2298/PIM1920125Q

Interchanging the order of the integration and summation leads to

$$\mathcal{I}_2=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{x^n}{(\delta)_n}\int_0^{\infty}z^{a+\sigma n-1}e^{-sz}\,\mathrm{d}\,z.$$

In view of definition of the Laplace transform, we have

$$\mathcal{I}_2=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{x^n}{(\delta)_n}\frac{\Gamma(\sigma n+a)}{s^{\sigma n+a}}.$$

Utilizing (1.1) and (1.2) derives the required result. □

Remark 2.2. If setting $\delta=1$ in Theorem 2, then we deduce [23, Eq. (27)] which formulates that

$$\int_0^{\infty}z^{a-1}e^{-sz}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^{\sigma})\,\mathrm{d}\,z=\frac{k^{1-\beta/k}s^{-a}}{\Gamma(\frac{\gamma}{k})}{}_2\Psi_1\left[\left(\frac{\gamma}{k},\tau\right),(a,\sigma);\left(\frac{\beta}{k},\frac{\alpha}{k}\right);\frac{xk^{\tau-\alpha/k}}{s^{\sigma}}\right].$$

If taking $\tau=q>0$, $k=1$, and $\delta=1$, then we acquire [23, Eq. (29)] which reads that

$$\int_0^{\infty}z^{a-1}e^{-sz}E_{\alpha,\beta}^{\gamma,q}(xz^{\sigma})\,\mathrm{d}\,z=\frac{s^{-a}}{\Gamma(\gamma)}{}_2\Psi_1\left[(\gamma,q),(a,\sigma);(\beta,\alpha);\frac{x}{s^{\sigma}}\right].$$

If taking $k=q=1$ and $\delta=1$ in the above equation reduces to

$$\int_0^{\infty}z^{a-1}e^{-sz}E_{\alpha,\beta}^{\gamma}(xz^{\sigma})\,\mathrm{d}\,z=\frac{s^{-a}}{\Gamma(\gamma)}{}_2\Psi_1\left[(\gamma,1),(a,\sigma);(\beta,\alpha);\frac{x}{s^{\sigma}}\right]$$

which is the main result in [22].

Recall that

$$\int_0^{\infty}t^{v-1}e^{-t/2}W_{\lambda,\mu}(t)\,\mathrm{d}\,t=\frac{\Gamma(\frac12+\mu+v)\Gamma(\frac12-\mu+v)}{\Gamma(1-\lambda+v)},\quad \Re(v\pm\mu)>-\frac12, \tag{2.3}$$

where the Whittaker function

$$W_{\lambda,\mu}(t)=\frac{\Gamma(-2\mu)}{\Gamma(\frac12-\mu-\lambda)}M_{\lambda,\mu}(t)+\frac{\Gamma(2\mu)}{\Gamma(\frac12+\mu-\lambda)}M_{\lambda,-\mu}(t)$$

and

$$M_{\lambda,\mu}(t)=z^{\mu+1/2}e^{-t/2}{}_1F_1\left(\frac12+\mu+v;2\mu+1;t\right)$$

are given in [4, 11].

Theorem 3. *If* $k\in\mathbb{R}$, $\alpha,\beta,\gamma,\delta,\tau,\eta\in\mathbb{C}$, $\Re(\alpha),\Re(\beta),\Re(\rho)>0$, $\delta\neq0,-1,-2,\dots$, *and* $\Re(\rho\pm\mu)>-\frac12$, *then*

$$\begin{aligned}\int_0^{\infty}t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(wt^{\eta})\,\mathrm{d}\,t&=\frac{k^{1-\beta/k}p^{-\rho}\Gamma(\delta)}{\Gamma(\frac{\gamma}{k})}\\&\times{}_3\Psi_3\left[\left(\frac{\gamma}{k},\tau\right),\left(\frac12\pm\mu+\rho,\eta\right),(1,1);\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\lambda+\rho,\eta),(\delta,1);\frac{wk^{\tau-\alpha/k}}{p^{\eta}}\right].\end{aligned}$$

Proof. Letting $pt=v$, interchanging the integration and summation, and using the formula for the Whittaker transform (2.3) yields

$$\begin{aligned}&\int_0^{\infty}t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(wt^{\eta})\,\mathrm{d}\,t\\&=\int_0^{\infty}e^{-v/2}\left(\frac{v}{p}\right)^{\rho-1}W_{\lambda,\mu}(v)\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\delta)_n}\left(\frac{v}{p}\right)^{\delta n}\frac1p\,\mathrm{d}\,v\end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1
Peer-reviewed version available at Publications de l'Institut Mathematique (Beograd) 2019, 106, 125-133; doi:10.2298/PIM1920125Q

$$
\begin{aligned}
&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)}\frac{w^n}{(\delta)_n}\int_0^\infty e^{-v/2}\left(\frac{v}{p}\right)^{\rho-1}\left(\frac{v}{p}\right)^{\delta n}W_{\lambda,\mu}(v)\frac{1}{p}\,\mathrm{d}\,v\\
&=\frac{1}{p^\rho}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n}\left(\frac{w}{p\delta}\right)^n\int_0^\infty e^{-v/2}v^{\delta n+\rho-1}W_{\lambda,\mu}(v)\,\mathrm{d}\,v\\
&=\frac{1}{p^\rho}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}\Gamma(\frac12+\mu+\delta n+\rho)\Gamma(\frac12-\mu+\delta n+\rho)}{\Gamma_k(\alpha n+\beta)(\delta)_n\Gamma(1-\lambda+\delta n+\rho)}\left(\frac{w}{p\delta}\right)^n.
\end{aligned}
$$

In view of (1.1) and (1.2), we find the desired result. □

Remark 2.3. Taking $\delta=1$ in Theorem 3 gives [23, Eq. (30)] which reads that

$$
\begin{aligned}
&\int_0^1 t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\eta)\,\mathrm{d}\,t\\
&\qquad=\frac{k^{1-\beta/k}p^{-\rho}}{\Gamma(\frac{\gamma}{k})}{}_2\Psi_2\left[\left(\frac{\gamma}{k},\tau\right),\left(\frac12\pm\mu+\rho,\eta\right);\left(\frac{\beta}{k},\frac{\alpha}{k}\right),(1-\lambda+\rho,\eta);\frac{wk^{\tau-\alpha/k}}{p^\eta}\right].
\end{aligned}
$$

Setting $\tau=q>0$, $k=1$, and $\delta=1$ results in [23, Eq. (32)] which states that

$$
\begin{aligned}
&\int_0^1 t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{\alpha,\beta}^{\gamma,q}(wt^\eta)\,\mathrm{d}\,t\\
&\qquad=\frac{1}{p^\rho\Gamma(\gamma)}{}_2\Psi_2\left[(\gamma,\tau),\left(\frac12\pm\mu+\rho,\eta\right);(\beta,\alpha),(1-\lambda+\rho,\eta);\frac{w}{p^\eta}\right].
\end{aligned}
$$

Letting $q=1$ in the above equation derives a result given in [22].

Theorem 4. *If $k\in\mathbb{R}$, $\alpha,\beta,\gamma,a,b,\sigma\in\mathbb{C}$, $\Re(\alpha),\Re(\beta)>0$, $\tau\in\mathbb{C}$, and $\delta\ne0,-1,-2,\dots$, then*

$$
\Im_\sigma[E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t)](w)=\frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})}\sum_{n=0}^{\infty}(-1)^n\frac{n!k^{(\tau-\beta/k)n}\Gamma(\frac{\gamma}{k}+n\tau)i^{n-1}w^{-(n+1)/\sigma}}{(\delta)_n\Gamma(\frac{\alpha}{k}n+\frac{\beta}{k})}.
$$

Proof. Using definitions of the generalized k-Mittag-Leffler function and the fractional Fourier transform and interchanging the integration and summation give

$$
\Im_\sigma[E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t)](w)=\int_0^1\exp(iw^{1/\sigma}t)E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t)\,\mathrm{d}\,t=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n}\int_R\exp(iw^{1/\sigma}t)t^n\,\mathrm{d}\,t.
$$

Letting $iw^{1/\sigma}t=-\eta$ reduces to

$$
\begin{aligned}
\Im_\sigma[E_{k,\alpha,\beta,\delta}^{\gamma,\tau}(t)](w)&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n}\int_{-\infty}^0\exp(-\eta)\left(\frac{-\eta}{iw^{1/\sigma}}\right)^n\left(\frac{-\,\mathrm{d}\,\eta}{iw^{1/\sigma}}\right)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(\alpha n+\beta)(\delta)_n i^{n+1}w^{(n+1)/\sigma}(-1)^n}\int_0^\infty e^{-\eta}\eta^n\,\mathrm{d}\,\eta\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}i^{-n-1}w^{-(n+1)/\sigma}(-1)^n n!}{\Gamma_k(\alpha n+\beta)(\delta)_n}
\end{aligned}
$$

Further using the formulas (1.1) and (1.2) arrives at the required result. □

Remark 2.4. If taking $\delta=1$ in Theorem 4, then the equation

$$
\Im_\sigma[E_{k,\alpha,\beta}^{\gamma,\tau}(t)](w)=\frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})}\sum_{n=0}^{\infty}(-1)^n\frac{k^{(\tau-\beta/k)n}\Gamma(\frac{\gamma}{k}+n\tau)i^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha}{k}n+\frac{\beta}{k})}.
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1

Peer-reviewed version available at Publications de l'Institut Mathematique (Beograd) 2019, 106, 125-133; doi:10.2298/PIM1920125Q

in [23, Eq. (33)] follows readily.

If setting $\delta=1$ and $\tau=q$ in Theorem 4, then the equation

$$\Im_\sigma[E_{k,\alpha,\beta}^{\gamma,\tau}(t)](w)=\frac{k^{1-\beta/k}}{\Gamma(\frac{\gamma}{k})}\sum_{n=0}^{\infty}(-1)^n\frac{k^{(q-\beta/k)n}\Gamma(\frac{\gamma}{k}+nq)i^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\frac{\alpha}{k}n+\frac{\beta}{k})}$$

in [23, Eq. (34)] can be derived immediately.

If letting $\delta=k=1$ and $\tau=q$ in Theorem 4, then

$$\Im_\sigma[E_{\alpha,\beta}^{\gamma,q}(t)](w)=\frac{1}{\Gamma(\gamma)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\gamma+n\tau)i^{n-1}w^{-(n+1)/\sigma}}{\Gamma(\alpha n+\beta)}$$

in [23, Eq. (34)] can be deduced straightforwardly.

Remark 2.5. In [8, Section 3], the quantity i^k for $k\in\mathbb{N}$ was computed generally by three approaches.

3. Concluding remark

Some integral transforms of the generalized k-Mittag-Leffler function are established and the results are expressed in terms of the generalized Wright function. By taking $\delta=1$ and using the formulas (1.1) and (1.2), we express Theorems 1 to 3 in terms of the generalized k-Wright function as follows.

It is noted that, using the appropriate formulas mentioned in Section 1, one can easily express the Euler integral in terms of the k-Wright function as

$$\int_0^1 z^{a-1}(1-z)^{b-1}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)\,\mathrm{d}\,z=\frac{\Gamma(b)k^b}{\Gamma_k(\gamma)}{}_2\Psi_2^k[(\gamma,\tau k),(ak,\sigma k);(\beta,\alpha),((a+b)k,\sigma k);x].$$

By applying suitable formula for the k-gamma function, Theorem 2 can be expressed in terms of the k-Wright function as

$$\int_0^1 z^{a-1}e^{-sz}E_{k,\alpha,\beta}^{\gamma,\tau}(xz^\sigma)\,\mathrm{d}\,z=\frac{k^{2-\gamma/k}}{(sk)^a\Gamma_k(\gamma)}{}_2\Psi_1^k\left[(\gamma,\tau k),(ak,\sigma k);(\beta,\alpha);\frac{x}{(ks)^\sigma}\right].$$

Theorem 3 can be expressed in terms of the k-Wright function as

$$\begin{aligned}\int_0^1 & t^{\rho-1}e^{-pt/2}W_{\lambda,\mu}(pt)E_{k,\alpha,\beta}^{\gamma,\tau}(wt^\eta)\,\mathrm{d}\,t\\ &=\frac{p^{-\rho}k^{1-\rho-\lambda}}{\Gamma_k(\gamma)}{}_2\Psi_2^k\left[(\gamma,\tau k),\left(\left(\frac{1}{2}\pm\mu+\rho\right)k,\eta k\right);(\beta,\alpha),((1-\lambda+\rho)k,\eta k);\frac{xk^{\tau-\eta-\alpha/k}}{p^\eta}\right].\end{aligned}$$

References

[1] P. Agarwal, F. Qi, M. Chand, and S. Jain, *Certain integrals involving the generalized hypergeometric function and the Laguerre polynomials*, J. Comput. Appl. Math. (2017), in press; Available online at `http://dx.doi.org/10.1016/j.cam.2016.09.034`.

[2] R. Díaz and E. Pariguan, *On hypergeometric functions and k-Pochhammer symbol*, Divulg. Mat. **15** (2007), no. 2, 179–192.

[3] G. A. Dorrego and R. A. Cerutti, *The k-Mittag-Leffler function*, Int. J. Contemp. Math. Sci. **7** (2012), no. 13-16, 705–716.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, Vol. II. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[5] C. Fox, *The asymptotic expansion of generalized hypergeometric functions*, Proc. London. Math. Soc. **27** (1928), no. 4, 389–400.

[6] K. S. Gehlot, *The generalized K-Mittag-Leffler function*, Int. J. Contemp. Math. Sci. **7** (2012), no. 45-48, 2213–2219.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2016 doi:10.20944/preprints201610.0020.v1

Peer-reviewed version available at *Publications de l'Institut Mathematique (Beograd)* **2019**, *106*, 125-133; doi:10.2298/PIM1920125Q

[7] K. S. Gehlot and J. C. Prajapati, *Fractional calculus of generalized k-wright function*, J. Fract. Calc. Appl. **4** (2013), no. 2, 283–289.
[8] B.-N. Guo and F. Qi, *On the Wallis formula*, Internat. J. Anal. Appl. **8** (2015), no. 1, 30–38.
[9] M. A. Khan and S. Ahmed, *On some properties of the generalized Mittag-Leffler function*, SpringerPlus (2013), **2**:337; Available online at http://dx.doi.org/10.1186/2193-1801-2-337.
[10] Y. F. Luchko, H. Matrínez, and J. J. Trujillo, *Fractional Fourier transform and some of its applications*, Fract. Calc. Appl. Anal. **11** (2008), no. 4, 457–470.
[11] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function*, Springer, New York, 2010; Available online at http://dx.doi.org/10.1007/978-1-4419-0916-9.
[12] M. G. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$, C. R. Acad. Sci. Paris **137**(1903), 554–558.
[13] M. G. Mittag-Leffler, *Sur la representation analytique d'une branche uniforme d'une function monogene*, Acta Math. **29** (1905), 101–181.
[14] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; Available online at http://dlmf.nist.gov/.
[15] T. R. Prabhakar, *A singular integral equation with a generalized Mittag Leffler function in the kernel*, Yokohama Math. J. **19** (1971), 7–15.
[16] F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27**(2013), no. 4, 601–604; Available online at http://dx.doi.org/10.2298/FIL1304601Q.
[17] F. Qi, X.-T. Shi, and F.-F. Liu, *Several identities involving the falling and rising factorials and the Cauchy, Lah, and Stirling numbers*, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.2462.1606.
[18] E. D. Rainville, *Special Functions*, The Macmillan Co., New York, 1960.
[19] T. O. Salim, *Some properties relating to the generalized Mittag-Leffler function*, Adv. Appl. Math. Anal. **4** (2009), no. 1, 21–30.
[20] T. O. Salim and A. W. Faraj, *A generalization of Mittag-Leffler function and integral operator associated with fractional calculus*, J. Fract. Calc. Appl. **3** (2012), no. 5, 1–13.
[21] A. K. Shukla and J. C. Prajapati, *On a generalization of Mittag-Leffler function and its properties*, J. Math. Anal. Appl. **336** (2007), no. 2, 797–811; Available online at http://dx.doi.org/10.1016/j.jmaa.2007.03.018.
[22] R. K. Saxena, *Certain properties of generalized Mittag-Leffler function*, Proceedings of the Third Annual Conference of the Society for Special Functions and their Applications, 75–81, Soc. Spec. Funct. Appl., Chennai, 2002.
[23] R. K. Saxena, J. Daiya, and A. Singh, *Integral transforms of the k-generalized Mittag-Leffler function* $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$, Matematiche (Catania) **69** (2014), no. 2, 7–16; Available online at http://dx.doi.org/10.4418/2014.69.2.2.
[24] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Inc., Amsterdam, 2012; Available online at http://dx.doi.org/10.1016/B978-0-12-385218-2.00001-3.
[25] A. Wiman, *Über den Fundamentalsatz in der Theorie der Funktionen* $E_a(x)$, Acta Math. **29** (1905), 191–201.
[26] E. M. Wright, *The asymptotic expansion of integral functions defined by Taylor series*, Philos. Trans. Roy. Soc. London, Ser. A. **238** (1940), 423–451.
[27] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc. (2) **46** (1940), 389–408.

URL: https://qifeng618.wordpress.com