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Article

Amenability Modulo an Ideal of Second Duals of Semigroup Algebras

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Abstract: The aim of this paper is to investigate the amenability modulo an ideal of Banach algebras with emphasis on applications to homological algebras. In doing so, we show that amenability modulo an ideal of A^{**} implies amenability modulo an ideal of A. Finally, for a large class of semigroups, we prove that $l^1(S)^{**}$ is amenable modulo I_σ^{**} if and only if an appropriate group homomorphic image of S is finite where I_σ is the closed ideal induced by the least group congruence σ.

Keywords: amenability modulo an ideal; semigroup algebra; group congruence

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1. Introduction

The concept of amenability modulo an ideal of Banach algebras was initiated by the first author and Amini (2014) [1]. They stated a version of Johnson's theorem for a large class of semigroups, including inverse semigroups, E-inversive semigroup and E-inversive E-semigroups; $l^1(S)$ is amenable modulo I_σ if and only if S is amenable. Some characterization of amenability modulo an ideal of Banach algebras and their hereditary properties were investigated in [12,13]. A characterization of amenability of Banach algebras with applications to homological algebra were studied by P. C. Curtis and R. J. Loy (1989) [4]. Amenability of the second dual of a Banach algebra was considered by Gourdeau (1997) [8]; if a Banach algebra A^{**} is amenable then A is amenable, a fact which is also proved (using a different proof) by Ghahramani, Loy and Willis (1996) [6]. Then, it was shown that for a locally compact group G, if $L^1(G)^{**}$ is amenable then G is finite [6]. Whereas we are not aware of a similar result for semigroups, in general.

In section two of this paper, we give a brief review of amenability modulo an ideal of Banach algebras, the necessary preliminaries of homological algebra and amenability modulo an ideal of Banach algebras in terms of short exact sequences. Thence, we show that for a Banach algebra A and a closed ideal I of A such that $I^2 = I$, if A^{**} is amenable modulo I^{**}, then A is amenable modulo I.

In section three, we show that amenability modulo an ideal of the second dual of a semigroup algebra implies amenability modulo an ideal of the semigroup algebra. Then we prove that $l^1(S)^{**}$ is amenable modulo I_σ^{**} if and only if S/σ is finite where I_σ^{**} is the induced closed ideal by the least group congruence σ.

2. Structure of amanability modulo an ideal

In this section, we first express the concept of amenability modulo an ideal of Banach algebras and some of their characterizations in terms of virtuals and diagonals. To see details, the reader may refer to [1,12,13].

Let A be a Banach algebra and X be a Banach A-bimodule. A bounded linear mapping $D : A \to X$ is called a derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$, for all $a, b \in A$. A derivation D is called inner if there exists $x \in X$ such that $D = ad_x$, where $ad_x : A \to X$ is defined by $ad_x(a) = a \cdot x - x \cdot a$. For any Banach $A-$bimodule X, its dual X^* is naturally equipped with a Banach $A-$bimodule structure via

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$\langle x, a.f\rangle = \langle xa, f\rangle, \langle x, f.a\rangle = \langle ax, f\rangle$, for all $a \in A, x \in X, f \in X^*$. The set of all bounded derivations from A into X is denoted by $Z^1(A;X)$, the set of all inner derivations from A to X is denoted by $N^1(A;X)$, and the quotient space $H^1(A;X) = \frac{Z^1(A;X)}{N^1(A;X)}$ is called the first cohomology group of A with coefficients in X. A Banach algebra A is called amenable if $H^1(A;X^*) = \{0\}$ for every Banach $A-$bimodule X [14].

Definition 2.1. ([1, Definition 1]) Let I be a closed ideal of A. A Banach algebra A is amenable modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$, and every derivation D from A into X^* there is $\phi \in X^*$ such that $D = ad_\phi$ on the set theoretic difference $A \backslash I := \{a \in A : a \notin I\}$.

Theorem 2.2. *([1, Theorem 1] Let I be a closed ideal of A.*
i) If A/I is amenable and $I^2 = I$ then A is amenable modulo I.
ii) If A is amenable modulo I then A/I is amenable.
iii) If A is amenable modulo I and I is amenable, then A is amenable.

First, we investigate the structure of amenability modulo an ideal of Banach algebras in terms of virtual diagonals and approximate diagonals. Some results are given in [12], but here we revise and improve them. Let A be a Banach algebra and I be a closed ideal of A. Then $\frac{A}{I}$ with module actions $a.\bar{b} := \overline{ab}$, and $\bar{b}.a = \overline{ba}$ is a Banach A-bimodule where $\bar{a}$ is the image of a in $\frac{A}{I}$. Also there is a canonical $A-$bimodule structure on $\frac{A}{I}\hat{\otimes}A$ defined by the linear extension of $a.(\bar{b} \otimes c) := \overline{ab} \otimes b$, and $(\bar{b} \otimes c).a := (\bar{b} \otimes ca)$, $(a, b, c \in A)$.

By the diagonal operator we mean the bounded linear operator defined by the linear extension of $\pi : (\frac{A}{I}\hat{\otimes}A) \to \frac{A}{I}$ by $\pi(\bar{b} \otimes c) = \overline{bc}$. Clearly, π is a $A-$bimodule homomorphism.

Definition 2.3. (i) By a virtual diagonal modulo I, we mean an element $M \in (\frac{A}{I}\hat{\otimes}A)^{**}$ such that;

$$a \cdot \pi^{**}(M) - \bar{a} = 0\,(a \in A), \text{ and } a \cdot M - M \cdot a = 0, \,(a \in A \setminus I),$$

(ii) an approximate diagonal modulo I, we mean a bounded net $(m_\alpha)_\alpha \subset (\frac{A}{I}\hat{\otimes}A)$ such that;

$$a.\pi(m_\alpha) - \bar{a} \to 0\,(a \in A), \text{ and } a.m_\alpha - m_\alpha.a \to 0, \,(a \in A \setminus I),$$

(iii) a diagonal modulo I, we mean an element $m \in (\frac{A}{I}\hat{\otimes}A)$ such that;

$$a.\pi(m) - \bar{a} = 0\,(a \in A), \text{ and } a.m - m.a = 0, \,(a \in A \setminus I).$$

We recall that a bounded net $\{u_\alpha\}_\alpha \subseteq A$ is called approximate identity modulo I if $\lim_\alpha u_\alpha \cdot a = \lim_\alpha a \cdot u_\alpha = a$, $(a \in A \setminus I)$. If A is amenable modulo I then A has an approximate identity modulo I [12, Theorem 4]. Using the classical method [12,14], with appropriate modifications, we have the following result;

Theorem 2.4. *The following conditions are equivalent;*
(i) A is amenable modulo I,
(ii) There is an approximate diagonal modulo I,
(iii) There is a virtual diagonal modulo I.

Let A be Banach algebra and X, Y, Z be $A-$bimodules. A sequence $\sum : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is called exact if f is one-to-one, $Im(g) = Z$, and $Im(f) = Ker(g)$. Also, the exact sequence $\sum$ is called admissible if there is a bounded linear map $F : Y \to X$ such $Ff = I$ on X and the exact sequence $\sum$ splits if there is a Banach A-module $F : Y \to X$ such that $Ff = I$ on X.

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Let A be a Banach algebra and I be a closed ideal of A. Suppose that $\pi : \frac{A}{I}\hat{\otimes}A \to \frac{A}{I}$ is the diagonal operator, $i : K = ker\pi \to \frac{A}{I}\hat{\otimes}A$ is the inclusion map, then $\prod : 0 \to K \xrightarrow{i} \frac{A}{I}\hat{\otimes}A \xrightarrow{\pi} \frac{A}{I} \to 0$ is a sequence of $A-$bimodules. Also, $(\frac{A}{I})^*$ and $(\frac{A}{I}\hat{\otimes}A)^*$ are clearly Banach A-bimodules where the module actions are defined by;

$$\langle \bar{a}, b.f\rangle := \langle \overline{ab}, f\rangle, \text{and } \langle \bar{a}, f.b\rangle := \langle \overline{ba}, f\rangle, \ \ (a,b \in A, f \in (\frac{A}{I})^*)$$

,

$$\langle \bar{b}\otimes c, a.\phi\rangle := \langle \bar{b}\otimes ca, \phi\rangle, \text{and } \langle \bar{b}\otimes c, \phi.a\rangle := \langle \overline{ab}\otimes c, \phi\rangle, \ \ (a,b,c \in A, \phi \in (\frac{A}{I}\hat{\otimes}A)^*),$$

so the dual sequence $\prod^* : 0 \to (\frac{A}{I})^* \xrightarrow{\pi^*} (\frac{A}{I}\hat{\otimes}A)^* \xrightarrow{i^*} K^* \to 0$ is a sequence of $A-$bimodules.

Lemma 2.5. *If A has an identity, the sequence $\prod$ is admissible. If A has a bounded left or right approximate identity, $\prod^*$ is admissible.*

Proof. Let A has an identity then $\phi : \frac{A}{I} \to \frac{A}{I}\hat{\otimes}A$ by $\phi(\bar{a}) = \bar{a}\otimes 1$ is the required right inverse for π, and ϕ^* is the required left inverse for π^*. Let now A has a bounded left approximate identity (e_j) and $u \in (\frac{A}{I}\otimes A)^{**}$ be a w^*-limit point of $(\bar{e_j}\otimes e_j)$, so by passing to a sub-net we may assume $\langle h, u\rangle = \lim_j\langle \bar{e_j}\otimes e_j, h\rangle (h \in (\frac{A}{I}\otimes A)^*)$. Set $\psi : (\frac{A}{I}\otimes A)^* \to (\frac{A}{I})^*$ by $\langle \bar{a}, \psi(h)\rangle = \langle \bar{a}.h, u\rangle (\bar{a} \in \frac{A}{I}, h \in (\frac{A}{I}\hat{\otimes}A)^*)$. It is not far to see that ψ is well-defined. Now for every $f \in (\frac{A}{I}\hat{\otimes}A)^*, \bar{a} \in \frac{A}{I}$ we have;

$$\begin{aligned}\langle \bar{a}, \psi\pi^* f\rangle = \langle a.\pi^*(f), u\rangle &= \lim_\alpha\langle \bar{e_\alpha}\otimes e_\alpha, a.\pi^*(f)\rangle \\ &= \lim_\alpha\langle \pi(\bar{e_\alpha}\otimes e_\alpha.a), f\rangle \\ &= \lim_\alpha\langle \bar{e_\alpha}^2.\bar{a}, f\rangle \\ &= \langle \bar{a}, f\rangle,\end{aligned}$$

hence $\psi\pi^* = id_{(\frac{A}{I})^*}$ as required. □

Theorem 2.6. *Let A be Banach algebra and I be a closed ideal of A. Then the following assertions hold.*
(i) If A is amenable modulo I then the exact sequence $\prod^*$ splits.
(ii) If A has a bounded approximate identity and the exact sequence $\prod^*$ is split then A is amenable modulo I.

Proof. (i) Suppose that A is amenable modulo I and M is a virtual diagonal modulo I for A. Set $\theta^* : (\frac{A}{I}\hat{\otimes}A)^* \to (\frac{A}{I})^*$ by $\langle \bar{a}, \theta^*(f)\rangle = \langle f.\bar{a}, M\rangle$ $(f \in (\frac{A}{I}\hat{\otimes}A)^*, \bar{a} \in \frac{A}{I})$. Let $b \in A\backslash I$, then;

$$\begin{aligned}\langle \bar{a}, \theta^*(b.f)\rangle &= \langle (b.f).\bar{a}, M\rangle = \langle b.(f.\bar{a}), M\rangle \\ &= \langle f.\bar{a}, M.b\rangle = \langle f.\bar{a}, b.M\rangle \\ &= \langle f.(\bar{a}.b), M\rangle = \langle (\bar{a}.b), \theta^*(f)\rangle \\ &= \langle \bar{a}, b.\theta^*(f)\rangle,\end{aligned}$$

so θ^* is an $A-$bimodule morphism. To see $\theta^*\pi^* = I$, let $f \in (\frac{A}{I})^*$, then

$$\begin{aligned}\langle \bar{a}, \theta^*\pi^* f\rangle &= \langle (\pi^* f).\bar{a}, M\rangle = \langle \pi^*(f.\bar{a}), M\rangle \\ &= \langle f.\bar{a}, \pi^{**}M\rangle = \langle f, \bar{a}.\pi^{**}M\rangle \\ &= \langle f, a.\pi^{**}M\rangle = \langle f, \bar{a}\rangle\end{aligned}$$

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(ii) Let θ^* be an $A-$bimodule morphism such that $\theta^*\pi^* = I$ and (e_α) be the approximate identity of A. By passing to a sub-net we can suppose that $E \in (\frac{A}{I}\hat{\otimes}A)^{**}$ is w^*-accumulation point of $(\bar{e_\alpha}\hat{\otimes}e_\alpha)$. We show that $M = \theta^{**}\pi^{**}E$ is a virtual diagonal modulo I for A. Let $a \in A\backslash I, f \in (\frac{A}{I} \otimes A)^*$, then;

$$\begin{aligned}\langle f, a.M\rangle &= \langle f, a.\theta^{**}\pi^{**}E\rangle = \langle f.a, \theta^{**}\pi^{**}E\rangle = \langle \pi^*\theta^* f.a, E\rangle \\ &= \lim\langle \theta^* f, a.e_\alpha^2\rangle = \lim\langle \theta^* f, e_\alpha^2.a\rangle \\ &= ... = \langle f, M.a\rangle\end{aligned}$$

Also,

$$\begin{aligned}\langle f, a.\pi^{**}M\rangle &= \langle f, a.\pi^{**}\theta^{**}\pi^{**}E\rangle = \langle \theta^*\pi^*(f.a), \pi^{**}(E)\rangle \\ &= \langle (f.a), \pi^{**}(E)\rangle = \lim\langle f, a.\bar{e_\alpha}^2\rangle \\ &= \lim\langle f, \bar{a}.\bar{e_\alpha}^2\rangle = \langle f, \bar{a}\rangle\end{aligned}$$

Hence M is a virtual diagonal modulo I for A, so A is amenable modulo I [12]. □

Suppose that X, Z are $A-$bimodules, then the space of all bounded linear operators $T : Z \to Y$ which is denoted by $B(X;Y)$ is an $A-$bimodule where the module actions are defined by $\langle z, a.T\rangle = a.\langle z, T\rangle$, and $\langle z, T.a\rangle = \langle a.z, T\rangle$ $(T \in B(Z;X), z \in Z, a \in A)$. Also, $Z\hat{\otimes}X$ is an $A-$bimodule with the module actions $a.(z \otimes x) = (a.z) \otimes x$, $(z \otimes x).a = z \otimes (x.a)$. Now, the map $T : (Z\hat{\otimes}X)^* \to B(Z;X^*)$ given by $T : \phi \mapsto T_\phi$, where $\langle x, T_\phi(z)\rangle = \langle z \otimes x, \phi\rangle$ is an isometric $A-$module morphism and $B(Z;X^*) \simeq (Z\hat{\otimes}X)^*$[4].

Definition 2.7. Let A be a Banach algebra, I be closed ideal of A and X, Y, Z be Banach $A-$bimodules. An exact (admissible) sequence $\sum : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact (admissible) sequence modulo I if $X \cdot I = I \cdot X = Y \cdot I = I \cdot Y = Z \cdot I = I \cdot Z = 0$.

Theorem 2.8. *Let A be a Banach algebra and $\sum : 0 \to X^* \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an admissible sequence modulo I. If A is amenable modulo I then $\sum$ splits.*

Proof. Since $\sum$ is an admissible modulo I, there exists $G : Z \to Y$ such that $gG = I$. Put $D : A \to B(Z;Y)$ by $D(a) = a.G - G.a$. Then D is a non zero bounded derivation on A and $g(D(a)(z)) = 0$ so $D(A) \subseteq B(Z; kerg) = B(Z; Imf)$. Thus $f^{-1}D : A \to B(Z;X^*) \simeq (Z\hat{\otimes}X)^*$ is a bounded derivation. Clearly that $(Z\hat{\otimes}X)^* \cdot I = I \cdot (Z\hat{\otimes}X)^* = 0$. Since A is amenable modulo I, there exists $\psi \in B(Z;X^*)$ such that $f^{-1}D(a) = a.\psi - \psi.a$ $(a \in A\backslash I)$. Let $\hat{G} = G - f\psi$, then $\hat{G}$ is an $A-$module morphism and $g\hat{G} = I$ on Z. □

At the end of this section, we would like to discuss the connection between amenability modulo an ideal of Banach algebras and their closed ideals. Following Curtis and Loy [4, Theorem 3.7], if A is a amenable Banach algebra, $J \neq 0$ is a closed left, right, or two sided ideal in A, then J has a bounded right, left, or two-sided approximate identity if and only if J^* is complemented in A^*. In the following, we state the same result for amenable modulo an ideal Banach algebras.

Lemma 2.9. *Let A be Banach algebra and I, J be closed ideals of A such that $J \supsetneqq I$. Let A be amenable modulo I, then $\frac{J}{I}$ has a bounded approximate identity if and only if $(\frac{J}{I})^\perp$ is complemented in $(\frac{A}{I})^*$.*

Proof. Since A is amenable modulo I, $\frac{A}{I}$ is amenable (by Theorem 2.2). It is clear that $\frac{J}{I}$ is a closed ideal of $\frac{A}{I}$, so $\frac{J}{I}$ has a bounded approximate identity if and only if $(\frac{J}{I})^\perp$ is complemented in $(\frac{A}{I})^*$ (by [4, Theorem 3.7]). □

Theorem 2.10. *Let A be a Banach algebra, I be a closed ideal of A with $I^2 = I$ and $J \supsetneqq I$ be a closed ideal of A. If A is amenable modulo I and $\frac{J}{I}$ has a bounded approximate identity, then J is amenable modulo I.*

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Proof. Since A is amenable modulo I, $\frac{A}{I}$ is amenable (by Theorem 2.2). Since $\frac{J}{I}$ has a bounded approximate identity, $\frac{J}{I}$ is amenable [11]. Thus J is amenable modulo I (by Theorem 2.2 again). $\square$

3. Amenability modulo an ideal of $l^1(S)^{**}$

Following Arens [2], for a Banach algebra A there are two algebra multiplications on the second dual of A which extend multiplication on A. It is shown that A^{**} is a Banach algebra under two Arens products [5]. Amenability of second conjugate Banach algebras are considered in [6]; for a Banach algebra A, amenability of A^{**} necessitates amenability of A. In the following we present the similar result for amenable modulo an ideal Banach algebras.

Theorem 3.1. *Let A be a Banach algebra, I be closed ideal of A such that $I^2 = I$. If A^{**} is amenable modulo I^{**}, then A is amenable modulo I.*

Proof. Since A^{**} is amenable modulo I^{**}, $\frac{A^{**}}{I^{**}} \simeq (\frac{A}{I})^{**}$ is amenable(by Theorem 2.2). Thus $\frac{A}{I}$ is amenable (by [6, Theorem 1.8]). Since $I^2 = I$, amenability of $\frac{A}{I}$ implies that A is amenable modulo I (by Theorem 2.2). $\square$

We apply the obtained results to characterize amenability of second dual of semigroup algebras. First of all, we have to recall some basic properties of semigroup theory to which we shall refer to, for full details, see [3,10]. A semigroup S is called an *E-semigroup* if the set of all idempotents $E(S)$ forms a sub-semigroup of S, S is called *E-inversive* if for all $x \in S$, there exists $y \in S$ such that $xy \in E(S)$. A semigroup S is called *regular*, if $V(a) = \{x \in S : \ a = axa, x = xax\} \neq \phi$ for every $a \in S$, S is called an *inverse semigroup* if the inverse of each element is unique, S is called a *semilattice* if is a commutative and idempotent semigroup, and S is called *eventually inverse* if every element of S has some power that is regular and $E(S)$ is a semilattice.

A congruence ρ on semigroup S is called a group congruence if S/ρ is group. The existence of the least group congruence on semigroups have also been investigated by various authors, see [7,9,15]. If the least group congruence on semigroup S exists, we denote it by σ. In general we know that there is no the least group congruence on semigroups. For example, Consider the semigroup of positive integers $(\mathbb{N}, +)$ (with respect to addition). It is not hard to see that every group congruence on $\mathbb{N}$ is of the form $\{(p, q) \in \mathbb{N} \times \mathbb{N} : kn = p - q, \text{for some } k \in \mathbb{Z}\}$, but $\mathbb{N}$ has not the least group congruence. Let ρ be a group congruence on semigroup S, by the induced ideal I_ρ, we mean an ideal in the semigroup algebra $l^1(S)$ generated by the set $\{\delta_s - \delta_t : \ s, t \in S \ \text{with} \ (s, t) \in \rho\}$. We recall the following Lemma of [1]

Lemma 3.2. *(i) ([1, Lemma 1]) Let S be a semigroup and ρ be a group congruence on S, then $l^1(S/\rho) \simeq l^1(S)/I_\rho$, where I_ρ is a closed ideal of $l^1(S)$.*

(ii) ([1, Lemma 2]) Let S be an E-inversive semigroup with commuting idempotents and σ be the least group congruence on S, then $l^1(S/\sigma) \simeq l^1(S)/I_\sigma$ where I_σ is a closed ideal of $l^1(S)$ and $I_\sigma^2 = I_\sigma$.

Theorem 3.3. *Let S be a semigroup, ρ be a group congruence on S such that I_ρ has an approximate identity. If $l^1(S)^{**}$ is amenable modulo I_ρ^{**}, then $l^1(S)$ is amenable modulo I_ρ.*

Proof. It is obvious that for a group congruence ρ on semigroup S, I_ρ is a closed ideal of $l^1(S)$ so I_ρ^{**} is a closed ideal of $l^1(S)^{**}$.Since I_ρ has an approximate identity, $I_\rho^2 = I_\rho$. Using Theorem 3.1, $l^1(S)$ is amenable modulo I_ρ. $\square$

If S is an E-inversive E-semigroup such that $E(S)$ is commutative, then the relation $\sigma^* = \{(a, b) \in S \times S \,|\, ea = fb \ \text{for some} \ e, f \in E_S\}$ is the least group congruence on S [7,15].

Theorem 3.4. *Let S be an E-inversive E-semigroup with commuting idempotents and σ^* be the least group congruence on S. If $l^1(S)^{**}$ is amenable modulo $I_{\sigma^*}^{**}$, then $l^1(S)$ is amenable modulo I_{σ^*}.*

Proof. Since σ^* is the least group congruence, $l^1(S/\sigma^*) \simeq l^1(S)/I_{\sigma^*}$ and $I^2_{\sigma^*} = I_{\sigma^*}$ (by Lemma 3.2). Now if $l^1(S)^{**}$ is amenable modulo $I^{**}_{\sigma^*}$, then $\frac{l^1(S)^{**}}{I^{**}_{\sigma^*}} \simeq (\frac{l^1(S)}{I_{\sigma^*}})^{**}$ is amenable (by Theorem 2.2). Thus $\frac{l^1(S)}{I_{\sigma^*}}$ is amenable. Since $I^2_{\sigma^*} = I_{\sigma^*}$, $l^1(S)$ is amenable modulo I_{σ^*}. $\square$

It is shown that if S is an *eventually semigroup* then the relation $\sigma' = \{(s,t) : es = et, \text{ for some } e \in E(S)\}$ is the least group congruence on S [7,9].

Corollary 3.5. *If S be an eventually inverse semigroup such that $l^1(S)^{**}$ is amenable modulo $I^{**}_{\sigma'}$, then $l^1(S)$ is amenable modulo $I_{\sigma'}$.*

Following Ghahramani, Loy and Willis, [6, Theorem 1.3], if G is a locally compact group or a discrete weakly cancellative semigroup then $l^1(G)^{**}$ is amenable if and only if G is finite. In the next theorem we show that the same remains true for a large class of semigroups.

Theorem 3.6. *If S satisfies one of the following statements:*

(i) S is an E-inversive semigroup with commuting idempotents;
(ii) S be an eventually inverse semigroup;

and σ is the least group congruence on S, then $(l^1(S))^{**}$ is amenable modulo I^{**}_{σ} if and only if S/σ is finite.

Proof. $l^1(S)^{**}$ is amenable modulo I^{**}_{σ} if and only if $\frac{(l^1(S))^{**}}{(I_\sigma)^{**}} \simeq (\frac{l^1(S)}{I_\sigma})^{**} \simeq l^1(S/\sigma)^{**}$ is amenable, if and only if S/σ is finite (by [6, Theorem 1.3] and we note that S/σ is a group). $\square$

Example 3.7. (1) Let $S = \{p^m q^n : m, n \geq 0\}$. It is known that S with multiplication operation $(p^{n_1}q^{m_1})(p^{m_2}q^{n_2}) = p^{m_1-n_1+max\{n_1,m_2\}}q^{m_2-n_2+max\{n_1,m_2\}}$, and $(p^n q^m)^* = (p^m q^n)$ is an E-unitary semigroup with $E(S) = \{p^n q^n : n = 0,1,2,...\}$ [3,10]. Set $x\sigma y$ if and only if $ex = ey$ for some $e \in E(S)$. Then σ is the least group congruence on S and $S/\sigma = \mathbb{Z}$. We know that $l^1(S)$ is amenable modulo I_σ whereas $l^1(S)$ is not amenable [1]. Also, since $S/\sigma = \mathbb{Z}$ is infinite, $(l^1(S))^{**}$ is not amenable modulo I^{**}_σ.

(2) Let $S = \mathbb{N}$ be the semigroup of positive integers with operation $n.m = max\{n,m\}$ $(n, m \in \mathbb{N})$. Set $n\sigma m$ if and only if $pn = pm$ for some $p \in E(S)$. Clearly S/σ is the trivial maximum group image of $\mathbb{N}$. Since S/σ is finite, $l^1(S)^{**}$ is amenable modulo I^{**}_σ. We note that $\mathbb{N}$ is infinite weakly cancellative semigroup so $l^1(S)^{**}$ is not amenable [6, Theorem 1.3].

References

1. M. Amini, H. Rahimi, Amenability of semigroups and their algebras modulo a group congruence, Acta Mathematica Hungarica, Vol 144, Issue 2, 407-415, (2014).
2. R. E. Arens, The adjoint of a bilinear operation, Amer. Math. Soc. 2, 839-848, (1951).
3. A. H. Clifford and J. B. Preston, The Algebraic Theory of Semigroups I, American Mathematical Society, Surveys **7**, (1961).
4. P. C. Curtis and R. J. Loy, The structure of amenable Banach algebras, J. London Math. Soc. (2) 40, no. 1, 89–104, (1989).
5. J. Duncan and S. A. R. Hosseiniun, The second dual of a Banach algebra , Proc. Roy. Soc. Edinburgh Sect. A 19, 309–325, MR 81f:46057, (1979).
6. F. Ghahramani, R.J. Loy, G.A. Willis, Amenability and weak amenability of second conjugate Banach algebras, Proc. Amer. Math. Soc. 124, 1489–1497, (1996).
7. R. S. Gigon, Congruences and group congruences on a semigroup, Semigroup Forum **86**, 431-450, (2013).
8. F. Gourdeau, Amenablility and the second dual of a Banach algebras, Studia Math. 125, no.1, 75–80, (1997).
9. G. M. S. Gomes, Group congruences on eventually inverse semigroups, Portugaliae Mathematica **49.4**, 417-428, (1992).
10. J. M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, (1995).

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Peer-reviewed version available at *Mathematics* **2016**, *4*, 55; doi:10.3390/math4030055

11. A. T. M. Lau and R. J. Loy, Amenability of convulotion algebras, Math. Scand. 79, 283-296, (1996).
12. H. Rahimi and E. Tahmasebi, Amenability and Contractibility modulo an ideal of Banach algebras, Abstract and Applied Analysis, 514761, (2014).
13. H. Rahimi and E. Tahmasebi, Hereditary properties of amenability modulo an ideal of Banach algebras, J. Linear Topol. Algebra, Vol 03, No 02, 107- 114,(2014).
14. V. Runde, Lectures on Amenability, vol. 1774 of Lecture Notes in Mathematics, Springer, Berlin, Germany, (2002).
15. M. Siripitukdet and S. Sattayaporn, The least group congruence on E-inversive semigroups and E-inversive E-semigroups, Thai Journal of Mathematics **3**, 163-169, (2005).