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Article

Local Polynomial Smoother for Solving Bagley-Torvik Fractional Differential Equations

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Abstract: Local polynomial smoother (LPS) is a weighted local least-squares nonparametric method. It provides a local Taylor series fit of the data at any location and can be directly used in a differential equation to provide a numerical scheme. In this article, we introduce this new nonparametric idea based on local polynomial smoother, for acquiring the numerical solution of the Bagley-Torvik fractional-order differential equations. Furthermore, this paper will present a numerical comparison with some methods, such as legendre operational matrix and pseudo-spectral method. The efficiency and accuracy of the LPS method are demonstrated by two numerical examples.

Keywords: numerical solution; · Bagley-Torvik fractional differential equations; · local polynomial smoother

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1 Introduction

The fractional differential equations appear more and more frequently in different research areas and engineering applications. Phenomena in electromagnetics, acoustics, viscoelasticity, and electrochemistry and material science are also described by differential equations of fractional order. The solution of the differential equation containing fractional derivative is much involved. For example, fractional-order derivatives have been successfully used to model damping forces with memory effect or to describe state feedback controllers [1-4]. Consequently, an effective and easy to use method for solving such equations is needed. it was undoubtedly the book written by Oldham and Spanier [3] which played an outstanding role in the development of this subject.

The Bagley-Torvik equation with 1/2-order derivative or 3/2-order derivative describes motion of real physical systems, an immersed plate in a Newtonian fluid and a gas in a fluid, respectively [1, 2, 4, 5]. The Bagley-Torvik equation has been numerically solved by using, Adomian decomposition method [6], the generalized Taylor collocation method [7], the fractional linear multistep methods, a predictor-corrector method of Adams type [8, 9], Haar wavelet operational matrix [10]. Besides, during the last decades, several methods have been employed to solve fractional differential equations. Diethelm et al. [12] and Ford et al. [13] have reviewed some of the existing methods and explained their respective strengths and weaknesses. There are some further methods, such as operational method [14], homotopy analysis method [15], differential transform method [16-18] and other methods [11,19-23].

Recently, Su and Yan [24], have used local polynomial smoother method for the numerical solution of integro-differential equations. Later, Su and Yan [25, 26], manage to solve ordinary differential equations and partial differential equations respectively by using this method. Numerical results demonstrate that local polynomial fitting method is more accurate, simply and efficient. In this study, we consider the following initial value problem in the case of the inhomogeneous Bagley-Torvik equation, then compared with other methods:

$$P(t)D^2y(t) + R(t)D^{\frac{3}{2}}y(t) + S(t)y(t) = f(t), 0 \leq t \leq b \tag{1}$$

with the boundary conditions:

$$y(a) = \gamma_0, y(b) = \gamma_1. \tag{2}$$

where $y^{(0)}(t) = y(t)$ and $y(t)$ is an unknown function, $P(t)$, $R(t)$, $S(t)$, and $f(t)$ are functions defined on the interval $a \leq t \leq b$, and b, γ_0 and γ_1 are real constants. The

main destination of this paper is to explore a fast algorithm for numerical solution of Bagley-Torvik FDEs (1) using a new nonparametric method, local polynomial smoother (LPS). In Section 2, we present the definition of fractional-order integration and fractional-order differentiation. In Section 3, local polynomial smoother method are described and their main thinking are reviewed. In Section 4, three parameters selections, the kernel K and the bandwidth h, are discussed in LPS method. In Section 5, we employ local polynomial Smoother method for solving of the Bagley-Torvik FDEs (1). Finally, in Section 7, we present some numerical tests to validate the theoretical results and we make comparisons of the methods discussed in the article.

2 Basic definitions

We first give the definition of fractional-order integration and fractional-order differentiation [17]. For the concept of fractional derivative, we will adopt Caputos definition, which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems.

Definition 2.1 A real function $f(x),\ x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\ \infty]$. Clearly, $C_\mu < C_\beta$ if $\beta < \mu$.

Definition 2.2 A function $f(x),\ x > 0$, is said to be in the space $C_\mu^M,\ \mu,\ m \in N \cup \{0\}$ if $f^{(m)} \in C_\mu$.

Definition 2.3 The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function, $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha \geq 0, \quad x > 0$$
$$J^0 f(x) = f(x).$$

The properties of the operator J^α can be found in [34, 35]; we mention only the following:

For $f \in C_\mu,\ \mu \geq -1,\ \alpha, \beta \geq 0$ and $\gamma > -1$:

$1^0 :\ J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$

$2^0 :\ J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$

$3^0 : J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena using fractional differential equations. Therefore, we will introduce a modified fractional differential operator D_*^α proposed by Caputos work on the theory of viscoelasticity [39].

3 Local polynomial smoother

Local polynomial smoother technique was first proposed by Fan.J [27-29]. And, this method was also mentioned and used by Su [30] . In order to describe the basic ideas of the LPS, firstly, we introduce the mathematical thoughts of local polynomial smoother. This idea were detailed mentioned in [27-29]. The basic idea is to form a locally optimal Taylor series to the data by minimizing the L_2 norm of the residual between the polynomial model and the data. Since the form of Smoother function is not specified, so the data points with long distance from t_0 provides little information to $y(t_0)$. Therefore, we can only use the local data points around t_0. We suppose that $y(t)$ has p+1 derivative at t_0, by the Taylor expansion , for point t ,located in the neighborhood of this pointt_0, we can use the p-order multivariate polynomials to locally approximate $y(t)$, and the surrounding local point of t_0, we model $y(t)$ as Taylor series expansion around any pint t_0 is:

$$y(t) \approx \sum_{j=0}^{p} \beta_j (t - t_0)^j. \tag{3}$$

where parameter β_j depends on t_0, so called local parameter. Obviously, the local parameter $\beta_j = {}^{y^{(j)}(t_0)}\!/_{j!}$ fits the local model with local data and it can be minimized,

$$\sum_{i=1}^{n} \left[Y_i - \sum_{j=0}^{p} \beta_j (t_i - t_0)^j \right]^2 K(\frac{t_i - t_0}{h}) \tag{4}$$

where h controls the size of the bandwidth of local area. Using matrix notation to represent the local polynomial smoother is more convenient. Below is the design matrix corresponding to (3) with t and Y :

$$X = \begin{pmatrix} 1 & t_1 - t_0 & \cdots & (t_1 - t_0)^p \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n - t_0 & \cdots & (t_n - t_0)^p \end{pmatrix}, Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}. \tag{5}$$

The weighted least squares problem (3) can be written as

$$\min (Y - X\beta)^T W(Y - X\beta),$$

here,

$$W = diag\left(K_h\left(t_1 - t_0\right), \cdots, K_h\left(t_n - t_0\right)\right), K_h = \frac{1}{h}K(\frac{\cdot}{h})$$

so the solution vector is

$$\beta = \left(X^T W X\right)^{-1} X^T W Y.$$

Furthermore, we can get the estimation $\hat{y}(t_0)$,

$$\hat{y}(t_0) = e_1(X^T W X)^{-1} X^T W Y, \tag{6}$$

where e_1 is a column vector (the same size of β) with the first element equal to 1, and the rest equal to zero, that is, $e_1 = (1, 0, \cdots, 0)_{1\times(p+1)}$. The selection of K does not influence the results much. We selected the quartic kernel as follows

$$K(u) = \begin{cases} \frac{3}{4}(1-u^2)_+, & if \quad |u| \le 1; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

4 LPS for Bagley-Torvik equation

In this section, the LPS method for solving (1) is outlined. As we know, datas Y_i at $t = t_i$ in fractional differential equation are unknown, consequently, instead of estimating data at a specific point t_i, we want to find a best match of some unknown data Y_i, i=1,2,....We could vary Taylor series residual density (4) at t_i with respect to these unknown data. Let equation(8) be an approximate solution of (1):

$$y(t) \approx \sum_{j=0}^{p} \beta_j (t - t_0)^j. \tag{8}$$

where, $t_1 = a < t_2 < \cdots < t_n = b$. The conventional collocation method can be expressed $t_i = a + \frac{b-a}{n-1}(i-1), \quad i = 1, \cdots, n$. We can adjust the value of parameter n flexibly, for example n=30 ,50, 70. It is required that the approximate solution (8)

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satisfies the (1-2) at the point $t = t_i$. Putting expression (8) into equation (1), we can obtain expression as follows:

$$\begin{aligned} &P(t) \cdot \sum_{j=2}^{p} j(j-1)\beta_j (t-t_0)^{j-2} + R(t)\frac{1}{\Gamma(2-\frac{3}{2})}\int_0^t (t-s)^{2-\frac{3}{2}-1}\cdot \\ &\sum_{j=2}^{p} j(j-1)\beta_j (s-t_0)^{j-2} ds + S(t)\sum_{j=0}^{p}\beta_j (t-t_0)^j \approx f(t), \end{aligned} \tag{9}$$

In order to satisfy unique solution theorem of integral-differential equation, we should take initial conditions (2) into account. Consequently, putting (8) to initial conditions (2), then we can acquire the approximate expression (11) as follows:

$$\sum_{j=0}^{p}\beta_j (a-t_0)^j \approx \gamma_0, \quad \sum_{j=0}^{p}\beta_j (b-t_0)^j \approx \gamma_1 \tag{10}$$

Consequently, according to the deduction process of equation (4), we can obtain easily the minimization function of problems (1-2), that is to say, the minimization function (Taylor series residual density) can be varied as (11):

$$\begin{aligned} &\left[\gamma_0 - \sum_{j=0}^{p}\beta_j (a-t_0)^j\right]^2 \cdot K_h (0-t_0)^j + \sum_{i=2}^{p}\{f(t_i) - P(t_i)\cdot \\ &\sum_{j=2}^{p} j(j-1)\beta_j (t_i-t_0)^{j-2} - R(t_i)\frac{1}{\Gamma(2-\frac{3}{2})}\int_0^{t_i} [(t_i-s)^{2-\frac{3}{2}-1}\cdot \\ &\sum_{j=2}^{p} j(j-1)\beta_j (s-t_0)^{j-2}]ds - S(t_i)\sum_{j=0}^{p}\beta_j (t-t_0)^j\}\cdot K_h (t_i-t_0)^j \\ &+ \left[\gamma_1 - \sum_{j=0}^{p}\beta_j (1-t_0)^j\right]^2 \cdot K_h (b-t_0)^j \end{aligned} \tag{11}$$

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Combination equations (9), (10) with (11), we can acquire the system which can be expressed (12):

$$\begin{cases} i=1, \quad A_{i,j_1}=(a-t_0)^{j_1} \quad j_1=0,\cdots,p \quad y(1)=\gamma_0, \\ i=2,\cdots,n-1, \quad B_{i,j_2}=P(t_i)\cdot j_2(j_2-1)(t_i-t_0)^{j_2-2} \quad j_2=2,\cdots,p \\ \qquad\qquad C_{i,j_3}=R(t_i)\dfrac{1}{\Gamma(2-\frac{3}{2})}\displaystyle\int_0^{t_i}[(t_i-s)^{2-\frac{3}{2}-1}\cdot j_3(j_3-1)\cdot \\ \qquad\qquad\qquad \beta_{j_3}(s-t_0)^{j_3-2}]ds, \qquad j_3=2,\cdots,p \\ \qquad D_{i,j_4}=S(t_i)\displaystyle\sum_{j_4=0}^{p}\beta_j(t_i-t_0)^{j_4}\} \quad j_4=0,\cdots,p \\ \qquad y(i)=f(t_i), \\ i=n, \quad G_{i,j_5}=(b-t_0)^{j} \quad j_5=0,\cdots,p \quad y(n)=\gamma_1, \end{cases} \tag{12}$$

then, the matrix form(5) can be written as follows by using (13-14).

$$\chi=\begin{bmatrix} A_{10} & A_{11} & A_{12} & \cdots & \cdots & A_{1p} \\ D_{20} & D_{21} & B_{22}+C_{22}+D_{22} & B_{23}+C_{23}+D_{23} & \cdots & B_{2p}+C_{2p}+D_{2p} \\ D_{30} & D_{31} & B_{33}+C_{33}+D_{33} & B_{34}+C_{34}+D_{34} & \cdots & B_{3p}+C_{3p}+D_{3p} \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ D_{n-1,0} & D_{n-1,1} & B_{n-1,3}+C_{n-1,3}+D_{n-1,3} & \cdots & \cdots & B_{n-1,p}+C_{n-1,p}+D_{n-1,p} \\ G_{n0} & G_{n1} & G_{n2} & \cdots & \cdots & G_{np} \end{bmatrix} \tag{13}$$

$$\Upsilon=\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(n-1) \\ y(n) \end{bmatrix}=\begin{bmatrix} \gamma_0 \\ f(t_i) \\ \vdots \\ f(t_{n-1}) \\ \gamma_n \end{bmatrix} \tag{14}$$

$$\varpi = \begin{bmatrix} K_h(a-x) & & & & & & \\ & K_h(X_2-x) & & & & & \\ & & K_h(X_3-x) & & & & \\ & & & K_h(X_4-x) & & & \\ & & & & \ddots & & \\ & & & & & K_h(X_{n-1}-x) & \\ & & & & & & K_h(b-x) \end{bmatrix} \tag{15}$$

where

$$K_h = \frac{1}{h}K(\frac{\cdot}{h})$$

Putting (14-15) to (6),

$$\hat{y}(t_0) = e_1(\chi^T \varpi \chi)^{-1} \chi^T \varpi \Upsilon, \tag{16}$$

where $e_1 = (1, 0, 0, \cdots, 0)^T{}_{p+1,1}$. In this paper, we choose the optimal weight function

$$K(u) = \begin{cases} \frac{3}{4}(1-u^2)_+, & if \quad |u| \leq 1; \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Then estimated set of coefficients β_i can be obtained by solving matrix system solution, therefore, approximate-value $\hat{y}(t_0)$ at point t_0 can be obtained. Repeating the above steps, we can acquire $\hat{y}(t_i)$ at point t_i, $i = 1, \cdots, n$, where n can be selected any positive integer.

5 Numerical examples

In this section, in order to illustrate the description above and to show the efficiency of the mentioned method for solving equation (1-2), we give the values of the exact solution $y(t)$, In tables and figures, the approximate solution $\hat{y}(t)$, the absolute error function $e_{AE}(t) = |y(t) - \hat{y}(t)|$ at the selected points of the given interval. We have performed all numerical computations in MATLAB.
Example 1 ([34]) First, we consider the Bagley-Torvik equation

$$y^{(2)}(t) + y^{(3/2)}(t) + y(t) = 1 + t, \quad 0 \leq t \leq 1$$

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with the boundary conditions $y(0) = 1$ and $y(1) = 2$, and the exact solution $y(t) = 1+t$, certainly, $P(t) = R(t) = S(t) = 1$, $f(t) = 1+t$. To simplify the procedures, giving n=11, p=1, h=0.5, we calculate the estimator $\hat{y}(t_i)$ i=1, 2, ...,11 at point $t = t_i$, such as $\hat{y}(t_0)$

$$\hat{y}(t_0) = e_1(\chi^T \varpi \chi)^{-1} \chi^T \varpi \Upsilon,$$

$$\chi = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -0.9 & 5.052 & -12.407 \\ 1 & -0.8 & 5.8107 & -13.556 \\ 1 & -0.7 & 6.3733 & -13.863 \\ 1 & -0.6 & 6.844 & -13.681 \\ 1 & -0.5 & 7.2633 & -13.152 \\ 1 & -0.4 & 7.6517 & -12.349 \\ 1 & -0.3 & 8.0218 & -11.318 \\ 1 & -0.2 & 8.3813 & -10.806 \\ 1 & -0.1 & 8.736 & -8.6722 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\varpi = \begin{bmatrix} 0.75 & & & & & & & & & & \\ & 0.72 & & & & & & & & & \\ & & 0.63 & & & & & & & & \\ & & & 0.48 & & & & & & & \\ & & & & 0.27 & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & 0 & & \\ & & & & & & & & & 0 & \end{bmatrix} \quad \Upsilon = \begin{bmatrix} 1 \\ 1.1 \\ 1.2 \\ 1.3 \\ 1.4 \\ 1.5 \\ 1.6 \\ 1.7 \\ 1.8 \\ 1.9 \\ 2 \end{bmatrix}$$

Fig 1 shows local polynomial smoother for example 1 given parameters $n = 51$, p=0, h=0.1.

Example 2 ([35])
Let us consider Bagley-Torvik equation

$$y^{(3/2)}(x) + y(x) = \frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x, \ 0 \le x \le 1$$

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Table 1 Comparison of the absolutes for the numerical results and absolute errors at point t_i= 0.1: 0.1: 1.0 for Example 1.

t_i	Absolute errors for n=50, p=1, h=0.2 LPS method	Absolute errors for n=100, p=1, h=0.15 LPS method	Absolute errors for m=8 $\alpha = 0.85$ with Legendre operational matrix [32]
0.1	2.2×10^{-11}	6.7×10^{-13}	8.0×10^{-4}
0.2	4.0×10^{-11}	1.2×10^{-12}	1.2×10^{-3}
0.3	2.5×10^{-12}	1.5×10^{-15}	6.6×10^{-4}
0.4	4.6×10^{-13}	5.3×10^{-15}	8.0×10^{-4}
0.5	2.0×10^{-13}	3.0×10^{-16}	7.5×10^{-4}
0.6	1.3×10^{-12}	5.2×10^{-14}	5.9×10^{-4}
0.7	5.2×10^{-12}	6.0×10^{-12}	7.6×10^{-4}
0.8	6.8×10^{-10}	3.7×10^{-14}	1.8×10^{-4}
0.9	8.5×10^{-13}	4.1×10^{-15}	6.2×10^{-4}
1.0	5.0×10^{-11}	8.3×10^{-13}	1.5×10^{-4}

with the boundary conditions y(0)=0 and y(1)=0 and the exact solution $y(x) = x^2 - x$. In here, $f(x) = \frac{2x^{1/2}}{\Gamma(3/2)} + x^2 - x$, $P(x) = 0$, $R(x) = 1$ and $S(x)$=1.

Given n=51, p=3, h=0.2, we calculate the estimator $\hat{y}(t_i)$ i=1, 2, ...,51 at point $t = t_i$,

$$\hat{y}(t_0) = e_1(\chi^T \varpi \chi)^{-1} \chi^T \varpi \Upsilon.$$

$$\chi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -0.98 & 1.2796 & -1.8859 \\ 1 & -0.96 & 1.373 & -2.2027 \\ 1 & -0.94 & 1.4364 & -2.4226 \\ 1 & -0.92 & 1.4847 & -2.5915 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -0.08 & 2.171 & -2.5115 \\ 1 & -0.06 & 2.1916 & -2.4508 \\ 1 & -0.04 & 2.2128 & -2.3881 \\ 1 & -0.02 & 2.2345 & -2.3234 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{51\times4}$$

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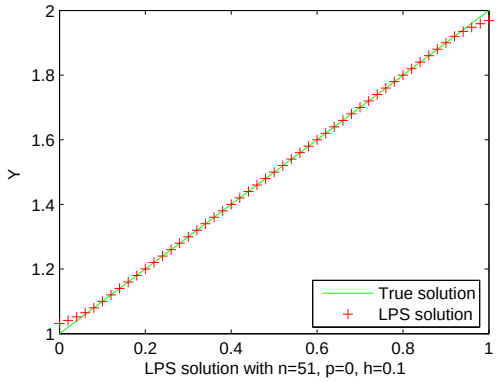

Fig. 1 LPS solution with n=51,p=0,h=0.1 for example 1.

$$\varpi = \begin{bmatrix} 0.75 & & & & & & & & \\ & 0.7425 & & & & & & & \\ & & 0.6825 & & & & & & \\ & & & 0.63 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 3.3307\text{e-}016 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & \ddots \\ & & & & & & & & 0 \end{bmatrix}_{51\times 51} \quad \Upsilon = \begin{bmatrix} 0 \\ 0.29955 \\ 0.41295 \\ 0.56471 \\ \vdots \\ \vdots \\ 2.0509 \\ 2.0911 \\ 2.1316 \\ 2.1728 \\ 2.2145 \\ 0 \end{bmatrix}_{51\times 1}$$

In fig 3, local polynomial smoother approximate analytical solution of example 2 with parameters n=51, p=3, h=0.1 and shows efficiency and high accuracy.

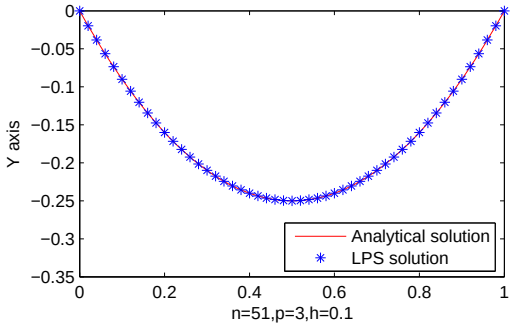

Fig. 2 LPS solution with n=51, p=3, h=0.1 for example 2.

Table 2 Comparison of the absolutes for the numerical results and absolute errors at point t_i= 0.0: 0.2: 1.0 for Example 2.

t_i	Absolute errors for n=51, p=3, h=0.2 LPS method	Absolute errors for N=10 $\alpha = 0.5$ with the pseudospectral method [33]	Absolute errors for h=1/20 $\alpha = 0.5$ with Adams-Bashforth method [34]
0.0	1.05×10^{-13}	5.14×10^{-5}	3.63×10^{-4}
0.2	7.46×10^{-13}	1.55×10^{-4}	4.56×10^{-3}
0.4	6.38×10^{-12}	3.81×10^{-4}	6.10×10^{-4}
0.6	3.27×10^{-11}	2.01×10^{-5}	6.19×10^{-4}
0.8	9.10×10^{-13}	6.32×10^{-4}	1.33×10^{-3}
1.0	7.26×10^{-14}	6.83×10^{-4}	3.42×10^{-3}

6 Conclusions

In this paper, we propose a fast algorithm based on local polynomial smoother which is applied for the numerical solution of Bagley-Torvik fractional differential equations (FDEs). The error result of LPS solution is very small and calculation processes are simple and feasible. Compared with Legendre operational matrix and pseudo-spectral method, LPS methods converges to solutions with fewer number of nodes and possess more accuracy. Moreover, it is more flexible to resolve problems just only adjusting

parameters p, bandwidth h.

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