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Article

The Novelty of Infinite Series for the Complete Elliptic Integral of the First Kind

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Abstract: According to the fact that the low convergence level on the complete elliptic integral of the first kind for the modulus which having values approach to one. In this paper we propose novelty of the complete elliptic integral consists of the new infinite series. We apply the scheme of iteration by substituting the common modulus with own modulus function into the new infinite series. We obtained so many new exact formulas of the complete elliptic integral derived from this method correspond to the number of iteration order. On the other hand, it has been also obtained a lot of new modulus functions rather than common used previously. The calculation results show that the number of significant figures of the new infinite series of the complete elliptic integral of the first kind is increased more and more. It means more fastly convergent would be obtained comparison values with the previous infinite series.

Keywords: integral form; infinite series; modulus function; transformation function

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1. Introduction

The complete elliptic integral of the first kind $K(k)$ is one of three elliptic integrals that getting a lot of attentions. It is not only used by mathematicians but also by engineers. On the development of scientifics for instance, the complete elliptic integral of the first kind are commonly used by Glasser [1] in studying a wide variety of problems involving three dimensional lattices, for creating Pi formula via Arithmetic Geometric Mean popularized by Salamin [2], and Borwein, et.al [3], for building analytical solution of the nonlinear pendulum performed by Karlheinz [4], as the basis for generalizing incomplete elliptic integral of the first kind [5], as the basis of development hypergeometric series [6], etc. Whereas in the fields of application, it is widely used in the design of electromagnetic devices, namely as basic function in conformal mapping which is mathematical tool for solving Electromagnetic problems [7],[8],[9], as mathematical model for designing parallel plate capacitor used by Palmer [10], curved patch capacitor [11], and micro-strip [12] that encountered in the fields of communication especially on antennas application and detectors, etc. The first kind $K(k)$ can be used to obtain the complete elliptic integral of the second kind $E(k)$, because both of these functions having relationship of ordinary differential equation [13], and Legendre relation [14]. However, $K(k)$ can be calculated in several ways, that is by using power series, Fourier series, theta functions, and Landen transformations. The first three methods are only convenient and useful for small k (approaching zero), unfortunately they are not convergent for the value of large k (approaching one). On the other hand, the Landen transformations are rapidly convergent, but are non-trivial to be applied [15]. Therefore enhancement of convergence level of the $K(k)$ which consists of large k remains interesting to be considered.

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In this early assignment, we focus to enhance the convergence level of the complete elliptic integral of the first kind $K(k)$ by transforming the value of modulus k into an appropriate modulus functions to produce transformation functions. From the literature review that we have conducted, there are two well known examples of such modulus transformation, namely $k \to \frac{ik}{k'}$ and $k \to \frac{1-k'}{1+k'}$ that respectively are as the generating transformation functions of $K(k) = \frac{1}{k'}K\left(\frac{ik}{k'}\right)$ and $K(k) = \frac{2}{1+k'}K\left(\frac{1-k'}{1+k'}\right)$, in which $i = \sqrt{-1}$, where $k' = \sqrt{1-k^2}$ is the complementary of modulus k [16]. Nevertheles, it is necessarry to find the other forms of transformation function that provide higher degree of convergence level. For this purpose we perform modification to the original integral form of $K(k)$ to obtain its new infinite series. Further, from this new infinite series will be known the new transformation function, which is useful for determining the modulus function. The modulus function of k will be useful to enhance the level of $K(k)$ convergence through employing the other scheme of iteration beyond, that has been applied on previous work as mentioned in Borwein's book[17].

I.1 Formulation of the New Infinite Series of the Complete Elliptic Integral of the First Kind

In order to obtain the new infinite series version of the $K(k)$ we firstly recall the definition of the complete elliptic integral of the first kind as appear on mathematical text books authorized respectively by Carlson [16], Borwein [17], and Boas [18],

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \quad , k \in (0,1) \tag{1}$$

We call Eqn.(1) as the original complete elliptic integral of the first kind which its infinite series is in the following form as appears in Eqn.(1.3.6) on Borwein's book [17],

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}, \tag{2}$$

where for the first five terms (the highest term corresponds to $n = 4$) as following,

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{8}\right)^2 k^4 + \left(\frac{5}{16}\right)^2 k^6 + \left(\frac{35}{128}\right)^2 k^8 + \cdots\right]. \tag{3}$$

The fact that the double factorial of (2n-1) can be represented as following,

$$(2n-1)!! = \frac{(2n)!}{2^n n!}, \tag{4}$$

then the infinite series in Eqn.(2) can be written in the following form,

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n} n!^2}\right]^2 k^{2n}, \tag{5}$$

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In formulating the new version of such $K(k)$ infinite series, we firstly modify the integral form in Eqn.(1) by varying the angle θ into the double angle 2θ through relationship of the following trigonometry identity,

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta). \tag{6}$$

By substituting Eqn.(6) into Eqn.(1) gives,

$$K(k)_N = \frac{1}{\sqrt{1-\frac{k^2}{2}}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{\cos 2\theta}{1-\frac{2}{k^2}}}}. \tag{7}$$

where the subscript N is included to distinguish from its original integral form. The new version of the elliptic integral is also in infinite series form, i.e,

$$K(k)_N = \frac{\pi}{2\sqrt{1-\frac{k^2}{2}}}\sum_{n=0}^{\infty}\frac{(4n)!}{\left(2^{3i}n!\right)^2(2n)!}\left(\frac{1}{1-\frac{2}{k^2}}\right)^{2n}, \tag{8}$$

or, it can be written as,

$$K(k)_N = \frac{\pi}{2\sqrt{1-\frac{k^2}{2}}}\sum_{n=0}^{\infty}\frac{(4n-1)!!}{\left(2^{2n}n!\right)^2}\left(\frac{1}{1-\frac{2}{k^2}}\right)^{2n}. \tag{9}$$

On both Eqn.(8) and Eqn.(9) above, we have employed the following relationship of double factorial and factorial in Eqn.(4) by replacing n with $2n$, namely,

$$(4n-1)!! = \frac{(4n)!}{2^{2n}(2n)!}, \tag{10}$$

Other form of the new version of the complete elliptic integral of the first kind is of form,

$$K(k)_N = \frac{1}{\sqrt{1-\frac{k^2}{2}}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\frac{1}{2}\left(\frac{k}{\sqrt{1-\frac{k^2}{2}}}\right)^2 \cos 2\theta}}, \tag{11}$$

that having infinite series form,

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$$K(k)_N = \frac{\pi}{2\sqrt{1-\frac{k^2}{2}}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{\left(2^{3n} n!\right)^2} \left(\frac{k}{\sqrt{1-\frac{k^2}{2}}} \right)^{4n}, \tag{12}$$

where for the first four terms as following,

$$K(k)_N = \frac{\pi}{2\sqrt{1-\frac{k^2}{2}}} \left\{ 1 + \frac{1\cdot 3}{8^2}\left(\frac{k}{\sqrt{1-\frac{k^2}{2}}} \right)^4 + \frac{1\cdot 3\cdot 5\cdot 7}{128^2}\left(\frac{k}{\sqrt{1-\frac{k^2}{2}}} \right)^8 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{3072^2}\left(\frac{k}{\sqrt{1-\frac{k^2}{2}}} \right)^{12} + \cdots, \right. \tag{13}$$

which can further be simplified to,

$$K(k)_N = \frac{\pi}{2\sqrt{1-\frac{k^2}{2}}} \left\{ 1 + \frac{1}{2}\cdot\frac{3}{8}\left(\frac{1}{1-\frac{2}{k^2}} \right)^2 + \frac{3}{8}\cdot\frac{35}{128}\left(\frac{1}{1-\frac{2}{k^2}} \right)^4 + \frac{5}{16}\cdot\frac{231}{1024}\left(\frac{1}{1-\frac{2}{k^2}} \right)^6 + \cdots \right. \tag{14}$$

It appears that Eqn.(14) equals to Eqn.(8) and/or with Eqn.(9).

I.2 Formulation of New Transformation Function for the Complete Elliptic integral of the First Kind

Before performing the step formulation for finding the new transformation function of $K(k)$ and/or $K(k)_N$, it is necessary to show that really both original and new version of the complete elliptic integral of first kind are the same. Both integrals are different only in the convergence level of its infinite series. Of course $K(k)_N$ will reduce to $K(k)$ when 2θ is varied back into θ. Nevertheles, because $\cos(2\theta)$ has two definitions, then varying the cosine of such 2θ must be performed one by one. Begin by introducing the following variable,

$$x = \frac{1}{\sqrt{1-\frac{k^2}{2}}}, \tag{15}$$

so that $K(k)_N$ in Eqn.(11) can be written in the form,

$$K(k)_N = x\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\frac{1}{2}(kx)^2\cos 2\theta}}. \tag{16}$$

Into Eqn.(16), we firstly subtitute the following cosine of 2θ,

$$\cos 2\theta = 1 - 2\sin^2\theta, \tag{17}$$

that giving the following integral form,

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$$K(k)_N = A\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k_{1N}{}^2\sin^2\theta}}, \tag{18}$$

where

$$A = \frac{x}{\sqrt{1+\frac{1}{2}(kx)^2}} = \frac{1}{\sqrt{1-\frac{k^2}{2}}}\frac{1}{\sqrt{1+\frac{k^2}{2}\frac{1}{1-\frac{k^2}{2}}}} = 1, \tag{19}$$

and

$$k_{1N} = \frac{kx}{\sqrt{1+\frac{1}{2}(kx)^2}} = k. \tag{20}$$

With the above values of $A=1$ and $k_{1N}=k$, it appears that Eqn.(18) has verified the equality of

$$K(k)_N = K(k). \tag{21}$$

After using the following cosine of 2θ

$$\cos 2\theta = 2\cos^2\theta - 1, \tag{22}$$

then $K(k)_N$ in Eqn.(11) can be written as,

$$K(k)_N = B\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k_{2N}{}^2\cos^2\theta}}, \tag{23}$$

where,

$$B = \frac{x}{\sqrt{1-\frac{1}{2}(kx)^2}} = \frac{1}{\sqrt{1-\frac{k^2}{2}}}\frac{1}{\sqrt{1-\frac{k^2}{2}\frac{1}{1-\frac{k^2}{2}}}} = \frac{1}{\sqrt{1-k^2}} = \frac{1}{k'}, \tag{24}$$

and

$$k_{2N} = \frac{ikx}{\sqrt{1-\frac{1}{2}(kx)^2}} = \frac{ik}{k'}. \tag{25}$$

Further, by using both values of B and k_{2N} above then Eqn.(23) becomes,

$$K(k)_N = \frac{1}{k'}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\left(\frac{ik}{k'}\right)^2\cos^2\theta}}. \tag{26}$$

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Eqn.(26) indicates that there is the other form of the original of the complete elliptic integral of first kind $K(k)$ aside from its original form on Eqn.(1), namely in the form,

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\cos^2\theta}}. \tag{27}$$

By involving the new definition of $K(k)$ then from Eqn.(26) we obtain the following transformation function,

$$K(k)_N=\frac{1}{k'}K\left(\frac{ik}{k'}\right). \tag{28}$$

Due to an equality in Eqn.(21), then from the Eqn.(28) it can also be formed the following transformation function,

$$K(k)=\frac{1}{k'}K\left(\frac{ik}{k'}\right), \tag{29}$$

Ones can also verify Eqn.(21) and Eqn.(27) form Eqn.(11). Applying $\cos 2\theta$ from the Eqn.(17) will give transformation function in Eqn.(21), while applying $\cos 2\theta$ from the Eqn.(22) produces transformation function in Eqn.(28). The equality of $K(k)_N$ and $K(k)$ also presents the equality of transformation function in Eqn.(29) that giving,

$$K(k)_N=\frac{1}{k'}K\left(\frac{ik}{k'}\right)_N. \tag{30}$$

Also, the following transformation function,

$$K(k)=\frac{2}{1+k'}K\left(\frac{1-k'}{1+k'}\right), \tag{31}$$

gives,

$$K(k)_N=\frac{2}{1+k'}K\left(\frac{1-k'}{1+k'}\right)_N. \tag{32}$$

Nevertheles, due to the difference between both infinite series of $K(k)$ and $K(k)_N$, then the convergence of Eqn.(29) and Eqn.(31) also differ from Eqn.(30) and Eqn.(32).

In order to obtain the new transformation function of $K(k)_N$, we explore the right side of Eqn.(32) by exerting the change of modulus $k\rightarrow\frac{1-k'}{1+k'}$ into $K(k)_N$ in Eqn.(7), so we find,

$$K\left(\frac{1-k'}{1+k'}\right)_N=\frac{\sqrt{2}(1+k')}{\sqrt{2(1+k')^2-(1-k')^2}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{(1-k')^2\cos 2\theta}{(1-k')^2-2(1+k')^2}}}, \tag{33}$$

after applying the following identity,

$$(1+k')^2 = 4k' + (1-k')^2, \tag{34}$$

then Eqn.(33) becomes,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \frac{\sqrt{2}(1+k')}{\sqrt{(1+k')^2+4k'}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\dfrac{(1-k')^2 \cos 2\theta}{(1+k')^2+4k'}}}. \tag{35}$$

In addition, applying the cosine of 2θ from Eqn.(17), then the Eqn.(35) can be simplified as,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\left(\dfrac{1-k'}{1+k'}\right)^2 \sin^2\theta}}. \tag{36}$$

Like previously explanation, from the Eqn.(36) appears that applying the cosine of 2θ as in Eqn.(21) only gives an equality,i.e,

$$K\left(\frac{1-k'}{1+k'}\right)_N = K\left(\frac{1-k'}{1+k'}\right), \tag{37}$$

while applying the cosine of 2θ from Eqn.(22) gives,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \frac{1+k'}{2\sqrt{k'}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\left(\dfrac{1-k}{i2\sqrt{k'}}\right)^2 \cos^2\theta}}, \tag{38}$$

So, by using Eqn.(27) then Eqn.(38) produces the following transformation function,

$$K\left(\frac{1-k'}{1+k'}\right)_N = \frac{1+k'}{2\sqrt{k'}} K\left(\frac{1-k'}{i2\sqrt{k'}}\right). \tag{39}$$

Similarly, by noticing Eqn.(32), then we obtain,

$$K(k)_N = \frac{1}{\sqrt{k'}} K\left(\frac{1-k'}{i2\sqrt{k'}}\right), \tag{40}$$

Finally, we obtain a new transformation function in the following form,

$$K(k)_N = \frac{1}{\sqrt{k'}} K\left(\frac{1-k'}{i2\sqrt{k'}}\right)_N. \tag{41}$$

I.3 Enhancement the Level of Convergence of the Complete Elliptic Integral of the First Kind by Applying the Scheme of Iteration to Its New Transformation Function

As mentioned previously that the infinite series of the complete elliptic integral of the first kind is slowly convergent. To enhance the level of its convergence, we implement the scheme of iteration

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to the transformation functions of $K(k)$. Here, we just involving two transformation functions in Eqn.(30) and Eqn.(41). Starting with Eqn.(30), after exerting the change of modulus $k \rightarrow \frac{ik}{k'}$ into Eqn.(8) to forms $K\left(\frac{ik}{k'}\right)_N$ so we obtain,

$$K_1(k)_N = \frac{1}{k'} \frac{\pi}{2\sqrt{1-\frac{1}{2}\left(\frac{ik}{k'}\right)^2}} \sum_{n=0}^{\infty} \frac{(4n)!}{\left(2^{3n} n!\right)^2 (2n)!} \left(\frac{1}{1-2\left(\frac{k'}{ik}\right)^2}\right)^{2n}, \tag{42}$$

substituting the complementary modulus $k' = \sqrt{1-k^2}$ into Eqn.(42), then we have,

$$K_1(k)_N = \frac{\sqrt{2}\pi}{2\sqrt{2-k^2}} \sum_{n=0}^{\infty} \frac{(4n)!}{\left(2^{3n} n!\right)^2 (2n)!} \left(\frac{k^2}{2-k^2}\right)^{2n} \tag{43}$$

Due to Eqn.(43) can reduce to Eqn.(8), we conclude that the scheme of iteration by the change modulus $k \rightarrow \frac{ik}{k'}$ can not be used to enhance the level of convergence of the complete elliptic integral of the first kind. Therefore, implemetation of the iteration scheme is now focused on Eqn.(41),

$$K_m(k)_N = \frac{1}{\sqrt{k'}} K_{m-1}\left(\frac{1-k'}{i2\sqrt{k'}}\right)_N, \quad m = 1,2,3\cdots \tag{44}$$

here m is the step of iteration, whereas $K_0(k)_N$ is the infinite series of the new version of elliptic integral in Eqn.(8) and/or Eqn.(9). But for simplicity we choose the form of infinite series of Eqn.(9), where for the first iteration ($m = 1$), we obtain

$$K_1(k)_N = \frac{1}{\sqrt{k'}} K_0\left(\frac{1-k'}{i2\sqrt{k'}}\right)_N, \tag{45}$$

After exerting the change of modulus $k \rightarrow \frac{1-k'}{i2\sqrt{k'}}$ into on Eqn.(9), we obtain the following infinite series of $K_1(k)$, namely:

$$K_1(k)_N = \frac{\sqrt{2}\pi}{\sqrt{1+k'^2+6k'}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{\left(2^{2n} n!\right)^2} \left(\frac{\left(\sqrt{1-k'}\right)^2}{\sqrt{1+k'^2+6k'}}\right)^{4n}. \tag{46}$$

Further, for the second iteration ($m = 2$) we obtain,

$$K_2(k)_N = \frac{1}{\sqrt{k'}} K_1\left(\frac{1-k'}{i2\sqrt{k'}}\right)_N. \tag{47}$$

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However, before applying the change of modulus $k \to \frac{1-k'}{i2\sqrt{k'}}$ into $K_1(k)_N$ on Eqn.(46), we must substitute $k' = \sqrt{1-k^2}$ so that Eqn.(47) forms the following infinite series, namely;

$$K_2(k)_N = \frac{1}{\sqrt{k'}} \frac{\sqrt{2}\pi}{\sqrt{2-k^2+6\sqrt{1-k^2}}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{\left(2^{2n} n!\right)^2} \left(\frac{1-\sqrt{1-k^2}}{\sqrt{2-k^2+6\sqrt{1-k^2}}} \right)^{4n}, \tag{48}$$

Finally, the change of modulus $k \to \frac{1-k'}{i2\sqrt{k'}}$ into Eqn.(48) gives,

$$K_2(k)_N = \frac{2\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}}} \sum_{n=0}^{\infty} \frac{(4n-1)!!}{\left(2^{2n} n!\right)^2} \left(\frac{\left(1-\sqrt{k'}\right)^2}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}}} \right)^{4n}. \tag{49}$$

The same procedure to the second iteration, for the third iteration ($m=3$), we obtain

$$K_3(k)_N = \frac{1}{\sqrt{k'}} K_2\left(\frac{1-k'}{i2\sqrt{k'}} \right)_N \tag{50}$$

and we obtain,

$$\begin{aligned} K_3(k)_N &= \frac{4\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}+12\left(1+\sqrt{k'}\right)^2\sqrt{2(1+k')\sqrt{k'}}}} \\ &\times \sum_{n=0}^{\infty} \frac{(4n-1)!!}{\left(2^{2n} n!\right)^2} \left(\frac{\left(\sqrt{1+k'}-\sqrt{2\sqrt{k'}}\right)^2}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}+12\left(1+\sqrt{k'}\right)^2\sqrt{2(1+k')\sqrt{k'}}}} \right)^{4n} \end{aligned} \tag{51}$$

By similarly way for the fourth iteration ($m=4$), we obtain

$$K_4(k)_N = \frac{1}{\sqrt{k'}} K_3\left(\frac{1-k'}{i2\sqrt{k'}} \right)_N \tag{52}$$

that giving the following infinite series,

$$\begin{aligned} K_4(k)_N &= \frac{8\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}+12\left(1+\sqrt{k'}\right)^2\sqrt{2(1+k')\sqrt{k'}}+12\left(\sqrt{1+k'}+\sqrt{2\sqrt{k'}}\right)^2\left(1+\sqrt{k'}\right)\sqrt{2\sqrt{2(1+k')\sqrt{k'}}}}} \\ &\times \sum_{n=0}^{\infty} \frac{(4n-1)!!}{\left(2^{2n} n!\right)^2} \left(\frac{\left(1+\sqrt{k'}-\sqrt{2\sqrt{2(1+k')\sqrt{k'}}}\right)^2}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}+12\left(1+\sqrt{k'}\right)^2\sqrt{2(1+k')\sqrt{k'}}+12\left(\sqrt{1+k'}+\sqrt{2\sqrt{k'}}\right)^2\left(1+\sqrt{k'}\right)\sqrt{2\sqrt{2(1+k')\sqrt{k'}}}}} \right)^{4n} \end{aligned} \tag{53}$$

After performing the simplification of algebra processes, the fourth exact formulas of $K_m(k)_N$ infinite series above can be expressed in each transformation function, namely,

$$K_1(k)_N = \frac{1}{\sqrt{k'}} K(k_1)_N, \quad k_1 = \frac{1-k'}{i2\sqrt{k'}} \tag{54}$$

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$$K_2(k)_N = \frac{2K(k_2)_N}{\sqrt{(1+k')\sqrt{4k'}}},\ k_2 = \frac{\left(1-\sqrt{k'}\right)^2}{i2\sqrt{(1+k')\sqrt{4k'}}} \tag{55}$$

$$K_3(k)_N = \frac{4K(k_3)_N}{\left(1+\sqrt{k'}\right)\sqrt{2\sqrt{(1+k')\sqrt{4k'}}}},\ k_3 = \frac{\left(\sqrt{1+k'}-\sqrt[4]{4k'}\right)^2}{i2\left(1+\sqrt{k'}\right)\sqrt{2\sqrt{(1+k')\sqrt{4k'}}}} \tag{56}$$

and

$$\begin{aligned} K_4(k)_N &= \frac{8K(k_4)_N}{\left(\sqrt{1+k'}+\sqrt[4]{4k'}\right)\sqrt{2\left(1+\sqrt{k'}\right)\sqrt[4]{4(1+k')\sqrt{4k'}}}}, \\ k_4 &= \frac{\left(1+\sqrt{k'}-\sqrt[4]{4(1+k')\sqrt{4k'}}\right)^2}{i2\left(\sqrt{1+k'}+\sqrt[4]{4k'}\right)\sqrt{2\left(1+\sqrt{k'}\right)\sqrt[4]{4(1+k')\sqrt{4k'}}}} \end{aligned} \tag{57}$$

where k_1, k_2, k_3, and k_4 are the corresponding modulus functions, that we call as own modulus function.

2. Discussion

The discussion about the enhance of the convergence level of the complete elliptic integral of the first kind here is focused to give some comments to achievement the number of significant figures obtained after performing all calculations by using the facilities of integral, summation, and evaluation of function that available on MapleV-Soft. Beginning by presenting the calculations results of the significant figures of infinite series of the original $K(k)$ in Eqn.(2) as shown in Table.1.

Table.1 Significant figures of infinite series of the original $K(k)$ for the number of terms multiple of ten

ℓ	$k = 1/10$	$k = 9/10$
0	1.570796326…	1.570796326…
10	1.574745562…	2.262667579…
20	1.574745562…	2.279280028…
30	1.574745562…	2.280439683…
40	1.574745562…	2.280538812…

Here, ℓ denotes the highest term in each infinite series of $K(\ell)$. After comparing the numerical values of the original integral form in Eqn.(1) i.e, $K\left(\frac{1}{10}\right) = 1.574745561517356\cdots$ and $K\left(\frac{9}{10}\right) = 2.280549138422770\cdots$, the number of significant figures for the modulus $k = \frac{9}{10}$. that are too little and slow for the number of terms multiple of ten comparing with the achievements of $k = \frac{1}{10}$. It has verified the statement in [14] that power series of the complete elliptic integral of the first kind is slowly convergent for the value of modulus k approaches one. Further, to verify our statement above that really the exact values of the original elliptic integral in Eqn.(1) and both of its new version in Eqn.(7) and Eqn.(11) are the same, we present the results of calculation in Table.2 below. Here we truncate numerical value of all calculations only until 16 significant figures.

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Table.2 The exact value of the original and new version of the complete elliptic integral of the first kind

k	$K(k)$	$K(k)_N$
$\frac{1}{10}$	1.574745561517356…	1.574745561517356…
$\frac{1}{2}$	1.685750354812596…	1.685750354812596…
$\frac{1}{\sqrt{2}}$	1.854074677301372…	1.854074677301372…
$\frac{9}{10}$	2.280549138422770…	2.280549138422770…

However, as shown in the following Table.3, the numerical values of both infinite series $K(k)$ and $K(k)_N$ are still different. Although to reach 16 significant figures are still required so many terms, but it appears that for all of modulus k the number of terms required by the $K(k)_N$ are more little. This fact as a guarantee that the new version of the complete elliptic integral of the first kind is faster to converg than its original version.

Table.3 Highest term ℓ of $K(k)$ and $K(k)_N$ infinite series to reach 16 significant figures

k	ℓ of $K(k)$	ℓ of $K(k)_N$
$\frac{1}{10}$	6	4
$\frac{1}{2}$	24	8
$\frac{1}{\sqrt{2}}$	45	14
$\frac{9}{10}$	150	41

The enhancement convergence level of the complete elliptic integral of the first kind can be traced by noticing the significant figures resulted for each highest term of the original version of the complete integral $K(k)$ on Eqn.(2), $K(k)_N$ of the new version on Eqn.(12), and the iterative version $K_1(k)_N$ on Eqn.(42). The calculation results for the values of modulus $\frac{1}{10}$, $\frac{1}{\sqrt{2}}$, and $\frac{9}{10}$ can be seen in Table.4 below,

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Table.4 Significant figures of the first six terms of the original, new, and iterative version of the complete elliptic integral of the firs kind

	$K(k)$			$K(k)_N$			$K_1(k)_N$		
ℓ	$k=\frac{1}{10}$	$k=\frac{1}{\sqrt{2}}$	$k=\frac{9}{10}$	$k=\frac{1}{10}$	$k=\frac{1}{\sqrt{2}}$	$k=\frac{9}{10}$	$k=\frac{1}{10}$	$k=\frac{1}{\sqrt{2}}$	$k=\frac{9}{10}$
0	3	1	0	5	2	1	12	3	2
1	5	1	0	10	3	2	23	8	5
2	7	2	1	15	3	2	29	10	7
3	9	2	1	20	5	2	41	15	8
4	11	2	1	24	6	2	55	19	10
5	13	3	1	30	7	2	67	23	12

The results of calculation for the three values of modulus k ranging from small until big values as shown in Table.4, confirm again, the significant figures of the new version of the complete elliptic integral of the first kind are more than the significant figures of the original integral form.

In closing this discussion, we present the sequential of approximation formulas obtained by setting the highest of term $\ell = 0$ into all of new infinite series formulas in Eqn.(46), Eqn.(49), Eqn.(51), and Eqn.(53), namely:

$$K_{1,0}(k)_N = \frac{\pi\sqrt{2}}{\sqrt{1+k'^2+6k'}}, \tag{58}$$

$$K_{2,0}(k)_N = \frac{2\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}}}, \tag{59}$$

$$K_{3,0}(k)_N = \frac{4\sqrt{2}\pi}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}+12\left(1+\sqrt{k'}\right)^2\sqrt{2(1+k')\sqrt{k'}}}}, \tag{60}$$

and

$$k'_{4,0}(k)_N = \frac{8\pi\sqrt{2}}{\sqrt{1+k'^2+6k'+12(1+k')\sqrt{k'}+12\left(1+\sqrt{k'}\right)^2\sqrt{2(1+k')\sqrt{k'}}+12\left(\sqrt{1+k'}+\sqrt{2\sqrt{k'}}\right)^2\left(1+\sqrt{k'}\right)\sqrt{2\sqrt{2(1+k')\sqrt{k'}}}}}. \tag{61}$$

Comparing with the results of applying the iteration scheme of Eqn.(44) but here we replace $K_{m-1}\left(\frac{1-k'}{i2\sqrt{k'}}\right)_N$ with $K_{m-1}\left(\frac{1-k'}{i2\sqrt{k'}}\right)$, where $K_m(k)$ corresponds to the infinite series of the original complete elliptic integral in Eqn.(2). The sequential approximation formulas for the first term of $K_m(k)$ are obtained in the following forms,

$$K_{1,0}(k) = \frac{\pi}{2\sqrt{k'}}, \tag{62}$$

$$K_{2,0}(k) = \frac{\pi}{\sqrt{(1+k')\sqrt{4k'}}}, \tag{63}$$

$$K_{3,0}(k)=\frac{2\pi}{\left(1+\sqrt{k'}\right)\sqrt{2\sqrt{(1+k')\sqrt{4k'}}}}, \tag{64}$$

and

$$K_{4,0}(k)=\frac{4\pi}{\left(\sqrt{1+k'}+\sqrt[4]{4k'}\right)\sqrt{2\left(1+\sqrt{k'}\right)\sqrt[4]{4(1+k')\sqrt{4k'}}}}. \tag{65}$$

On all of sequential approximation formulas $K_{m,\ell}(k)_N$ in Eqn.(58) until Eqn.(61) and $K_{m,\ell}(k)$ in Eqn.(62) until Eqn.(65) we have put the subscript ℓ to indicate the highest term used in each infinite series. As previously, we set $\ell=0$ which means that all of the sequential formulas contain only one term. Finally we present the comparison of the numerical values of Eqn.(58)-Eqn.(61) and Eqn.(62)-Eqn.(65) are shown in Tabel.5,

Table.5 Significant figures of the sequential approximation formulas of $K_{m,\ell}(k)_N$ and $K_{m,\ell}(k)$ for the first term ($\ell=0$)

k	$m=1$		$m=2$		$m=3$		$m=4$	
	$K_{1,0}(k)_N$	$K_{1,0}(k)$	$K_{2,0}(k)_N$	$K_{2,0}(k)$	$K_{3,0}(k)_N$	$K_{\backslash 3,0}(k)$	$K_{4,0}(k)_N$	$K_{4,0}(k)$
$\frac{1}{10}$	12	6	25	12	50	24	101	51
$\frac{1}{2}$	5	3	13	6	27	13	54	27
$\frac{1}{\sqrt{2}}$	3	2	10	4	21	11	43	21
$\frac{9}{10}$	2	1	6	3	13	7	30	15

Although the number of its significant figures for all of modulus k as shown in Table.4 increase with increasing the number of terms, however we can not specify how much the number enhancement of such significant figures. But from the significant figures of the sequential approximation formulas of the first term of $K_{m,\ell}(k)_N$ as shown in Table.5, it can be known that the ratio between the number of significant figures of two successive sequential approximation formulas is approximately 2, that also holds for $K_{m,\ell}(k)$. Here, it means that the enhancement of convergence level of the complete elliptic integral by applying both iteration schemes $K_m(k)_N=\frac{1}{\sqrt{k'}}K_{m-1}\left(\frac{1-k'}{i2\sqrt{k'}}\right)_N$ for the new complete elliptic integral and $K_m(k)=\frac{1}{\sqrt{k'}}K_{m-1}\left(\frac{1-k'}{i2\sqrt{k'}}\right)$ for the original integral form correspond to the level of quadratic convergence. However, the fact that the number of significant figures of $K_{m,0}(k)_N$ that always twice than $K_{m,0}(k)$ as shown in Table.5 are interesting to be researched further.

3. Conclucions

From explanation and discussion above we take several conclusions. The complete elliptic integral of the first kind can be modified into the new form by varying the argument of angle θ into the double angle 2θ. Applying the scheme of iteration by substituting the common modulus k with the modulus function $\frac{1-k'}{i2\sqrt{k'}}$ into the new infinite series produces so many new exact formulas of the complete elliptic integral correspond to the number of iteration order. On the other hand, from the new transformation functions has been also obtained a lot of new modulus functions rather than common used previously. The calculation results show that the enhancement of the number of significant figures of the new infinite series of the complete elliptic integral of the first kind corresponds to the level of quadratic convergence.

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