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Type II Bivariate Generalized Power Series Poisson Distribution and its Applications in Risk Analysis

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Abstract

In this paper we consider type II bivariate generalized power series Poisson distribution as a compound Poisson distribution with bivariate generalized power series compounding distribution. We obtain some properties, p.m.f and conditional distributions. In addition we also give a brief discussion about the multivariate extension of this case. Then we introduce type II bivariate generalized power series Poisson process and consider a bivariate risk model with type II bivariate generalized power series Poisson model as the counting process. For this model we derive distribution of the time to ruin and bounds for the probability of ruin. We obtain partial integro-differential equation for the ruin probabilities and express its bivariate transform through two univariate boundary transforms, where one of the initial capitals is fixed at zero.

Keywords: bivariate generalized power series distribution; ruin probability; aggregate claims distribution

1. Introduction

Bivariate discrete random variables taking non-negative integer values, have received considerable attention in the literature. The type II bivariate Polya - Aeppli distribution was introduced by Minkova and Balakrishnan(2014). Kostadinova and Minkova(2014) applied bivariate Poisson negative binomial distribution to bivariate risk processes. Furthermore Kostadinova(2015) introduced a bivariate risk model in which distribution of claim counting process is the bivariate Polya-Aeppli distribution. In the literature it has been found that bivariate compound Poisson distributions are very flexible and can be used efficiently in bivariate risk modeling. With this as motivation, different bivariate compound Poisson distributions have been constructed.

The family of bivariate generalized power series distribution is basically used for counting paired events. It contains many important families like bivariate Poisson, bivariate binomial, bivariate negative binomial and bivariate

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logarithmic series distributions. The PMF of bivariate generalized power series distribution is given by

$$P(i,j) = \frac{a_{i,j}\theta_1^i\theta_2^j}{g(\theta_1,\theta_2)}, \quad (i,j) \in \mathcal{T}, \tag{1}$$

where $g(\theta_1,\theta_2) = \sum_{i,j} a_{i,j}\theta_1^i\theta_2^j, \;\; a_{i,j} \geq 0$ and $\mathcal{T}$ is a subset of cartesian product of the set of nonnegative integers.

In this paper we consider compound Poisson distribution with bivariate generalized power series compounding distribution. The bivariate type II Polya-Aeppli distribution and the bivariate Poisson negative binomial distribution are the special cases of bivariate generalized power series Poisson distribution.

For the univariate case, where $X_1, X_2, X_3...$ are independent and identically distributed random variables, independent of N_1 and N_1 has a Poisson distribution with parameter λ, denoted by $N_1 \sim P_o(\lambda)$. Suppose that $X_1, X_2, X_3...$ follow generalized power series distribution with PGF

$$P(s) = \frac{g(\theta s)}{g(\theta)},$$

where $g(\theta) = \sum_i a_i\theta^i, \quad a_i \geq 0.$

Now consider the random sum

$$N = X_1 + X_2 + ... + X_{N_1},$$

The distribution of N is called generalized power series Poisson distribution.

The PGF of the random variable N is given by

$$\Psi_N(s) = e^{-\lambda(1-P(s))} = e^{-\lambda(1-\frac{g(\theta s)}{g(\theta)})}.$$

Then the corresponding PMF is given by

$$\begin{aligned} P(N=m) &= e^{-\lambda}, \quad m = 0 \\ &= e^{-\lambda}\theta^m \sum_{j=1}^{\infty} \frac{C_m(j)(\frac{\lambda}{g(\theta)})^j}{j!}, \quad m = 1, 2, \ldots, \end{aligned}$$

where

$$C_m(j) = \sum_{k_1+k_2+...+k_j=m} a(k_1), a(k_2)...a(k_j), \quad \text{If } (k_1, k_2, ..., k_j) \text{ is the ordered j-tuple of}$$

positive integers in the range set of the random variable X which sum up to m.

This paper is organised as follows . In section 2 the joint probability mass function, correlation coefficient and covariance of type II bivariate generalized power series Poisson distribution are derived. In section 3 marginal distribution and conditional distribution of type II bivariate generalized power series Poisson distribution are given.

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A multivariate extension of the generalized power series Poisson distribution and its properties are discussed in section 4. Bivariate counting processes with type II bivariate generalized power series Poisson distribution is introduced in section 5. In section 6 we consider type II bivariate generalized power series Poisson risk model and derive the distributions of bivariate aggregate claims and sum of aggregate claims of two classes. Section 7 presents three types of ruin probabilities and an expression for ruin probabilities for a type II bivariate generalized power series Poisson risk model is derived. In addition, the bounds for the ruin probabilities are developed. In section 8 a system of partial integro differential equation for the ruin probabilities is developed and the Laplace transform is derived. Section 8 deals with multivariate generalized power series Poisson risk model and the ruin probabilities for the defined risk model.

2. Bivariate Generalized Power Series Poisson Distribution

Let us consider the sequence $(X_i, Y_i),\ i = 1, 2, ...$ of independent and identically distributed random variables, distributed as (X, Y).

Define

$$N_1 = X_1 + X_2 + ... + X_N \text{ and } N_2 = Y_1 + Y_2 + ... + Y_N,$$

where N is independent of the compounding random vector (X, Y) and has a Poisson distribution with parameter λ.

Suppose that (X, Y) has a bivariate generalized power series distribution with PGF

$$P(s_1, s_2) = \frac{g(\theta_1 s_1, \theta_2 s_2)}{g(\theta_1, \theta_2)}.$$

Then, the joint PGF of the bivariate random vector (N_1, N_2) is given by

$$\Psi(s_1, s_2) = e^{-\lambda(1-P(s_1,s_2))} = e^{-\lambda(1-\frac{g(\theta_1 s_1, \theta_2 s_2)}{g(\theta_1,\theta_2)})}. \tag{2}$$

The PGF in (2) can be written as

$$\Psi(s_1, s_2) = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^k}{k!} \sum_{i,j} C_{ij}(k)(\theta_1 s_1)^i (\theta_2 s_2)^j \ , \tag{3}$$

where

$C_{ij}(k) = \sum_{\substack{i_1+i_2+\cdots+i_k=i \\ j_1+j_2+\cdots+j_k=j}} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_k j_k}$, if $(i_1, i_2, \cdots, i_k)$ is the ordered k tuple of elements in the range of X which sum up to i and $(j_1, j_2, \cdots j_k)$ is the ordered k tuple of elements in the range of Y which sum up to j .

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Differentiation in (3) leads to the following derivatives

$$\Psi^{(i,j)}(s_1,s_2) = e^{-\lambda}\sum_{k=1}^{\infty}\frac{\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^k}{k!}\sum_{l\geq i, m\geq j} C_{l,m}(k)l^{(i)}m^{(j)}\theta_1^l\theta_2^m s_1^{l-i}s_2^{m-j}, \quad (4)$$

where $l^{(i)} = \frac{l!}{(l-i)!}, m^{(j)} = \frac{m!}{(m-j)!}$ and $\Psi^{(i,j)}(s_1,s_2) = \frac{\partial^{i+j}}{\partial s_1^i \partial s_2^j}\Psi(s_1,s_2)$.

Setting $s_1 = s_2 = 1$ in (4),we obtain the $(i,j)^{th}$ factorial moments of (N_1,N_2)

$$EN_1(N_1-1)\cdots(N_1-i+1)N_2(N_2-1)\cdots(N_2-j+1)$$
$$= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^k}{k!}\sum_{l\geq i, m\geq j} C_{lm}(k)l^{(i)}m^{(j)}\theta_1^l\theta_2^m.$$

2.1. Covariance and Correlation

The means are given by

$\mu_1 = E(N_1) = \lambda\theta_1\frac{\partial}{\partial\theta_1}\log g(\theta_1,\theta_2)$ and $\mu_2 = E(N_2) = \lambda\theta_2\frac{\partial}{\partial\theta_2}\log g(\theta_1,\theta_2)$.

Similarly the variances are obtained as

$\text{Var}\,(N_1) = \theta_1\frac{\partial}{\partial\theta_1}\mu_1 + \frac{1}{\lambda}\mu_1^2, \text{Var}\,(N_2) = \theta_2\frac{\partial}{\partial\theta_2}\mu_2 + \frac{1}{\lambda}\mu_2^2.$

From (2), we obtain

$$\frac{\partial^2}{\partial s_1\partial s_2}\Psi(s_1,s_2) = \left(\frac{\lambda}{g(\theta_1,\theta_2)}\frac{\partial^2}{\partial s_1\partial s_2}g(\theta_1 s_1,\theta_2 s_2) + \left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^2\frac{\partial}{\partial s_1}g(\theta_1 s_1,\theta_2 s_2)\right.$$
$$\left.\frac{\partial}{\partial s_2}g(\theta_1 s_1,\theta_2 s_2)\right)\Psi(s_1,s_2), \quad (5)$$

Setting $s_1 = s_2 = 1$ in (5), we easily obtain

$$EN_1N_2 = \theta_2\frac{\partial}{\partial\theta_2}\mu_1 + \left(1+\frac{1}{\lambda}\right)\mu_1\mu_2.$$

The covariance and correlation between N_1 and N_2 are respectively given by

$$\text{Cov}\,(N_1,N_2) = \theta_2\frac{\partial}{\partial\theta_2}\mu_1 + \frac{1}{\lambda}\mu_1\mu_2$$
$$\text{Corr}\,(N_1,N_2) = \frac{\theta_2\frac{\partial}{\partial\theta_2}\mu_1 + \frac{1}{\lambda}\mu_1\mu_2}{\sqrt{\prod_{i=1}^2\left(\theta_i\frac{\partial}{\partial\theta_i}\mu_i + \frac{1}{\lambda}\mu_i^2\right)}}.$$

2.2. Joint Probability Mass Function

The joint probability mass function of (N_1,N_2) is obtained by expanding the PGF, $\Psi(s_1,s_2)$ in powers of s_1 and s_2.

Let $f(i,j) = P(N_1 = i, N_2 = j), i = j = 0,1,2...$ denote the joint PMF of (N_1,N_2).

On the other hand, from Johnson et al., it is known that

$$f(i,j) = \frac{\Psi^{(i,j)}(s_1,s_2)}{i!j!}\Big/ s_1 = s_2 = 0.$$

Using the PGF in (2) and the derivatives in (4) we obtain the joint PMF of (N_1, N_2) and is given by

$$f(0,0) = e^{-\lambda\left(1-\frac{a_{0,0}}{g(\theta_1,\theta_2)}\right)},$$

$$f(i,j) = e^{-\lambda}\theta_1^i\theta_2^j\sum_{k=1}^{\infty}\frac{C_{ij}(k)\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^k}{k!}, \quad i,j=0,1,\cdots,(i,j)\neq(0,0)\,,$$

where $C_{ij}(k)=\sum_{\substack{i_1+i_2+\cdots+i_k=i\\ j_1+j_2+\cdots+j_k=j}} a_{i_1,j_1}a_{i_2,j_2}\cdots a_{i_k,j_k}$, if $(i_1,i_2,\cdots,i_k)$ is the ordered k tuple of elements in the range of X which sums up to i and $(j_1,j_2,\cdots j_k)$ is the ordered k tuple of elements in the range of Y which sums up to j .

3. Marginal Distributions

The PGFs of the marginal compounding distributions are given by

$P_1(s_1) = P(s_1,1) = \frac{g(\theta_1 s_1,\theta_2)}{g(\theta_1,\theta_2)} = \frac{\sum_i b_i(\theta_1 s_1)^i}{\sum_i b_i\theta_1^i}$, where $b_i=\sum_j a_{ij}\theta_2^j$, is independent of θ_1,

$P_2(s_2) = P(1,s_2) = \frac{g(\theta_1,\theta_2 s_2)}{g(\theta_1,\theta_2)} = \frac{\sum_j c_j(\theta_2 s_2)^j}{\sum_j c_j\theta_2^j}$, where $c_j=\sum_i a_{ij}\theta_1^i$, is independent of θ_2.

Therefore the above marginal PGFs can be written in the form

$$P_1(s_1)=\frac{h_1(\theta_1 s_1)}{h_1(\theta_1)},$$
$$P_2(s_2)=\frac{h_2(\theta_2 s_2)}{h_2(\theta_2)}. \tag{6}$$

From which it follows that the random variable X has a generalized power series distribution with series function $h_1(\theta_1)=\sum_i b_i\theta_1^i$, where $b_i=\sum_j a_{ij}\theta_2^j$ and θ_2 are treated as constants.

Analogously, Y has a generalized power series distribution with series function $h_2(\theta_2)=\sum_j c_j\theta_2^j$, where $c_j=\sum_i a_{ij}\theta_1^i$ and θ_1 are treated as constants.

Then , from (2) and (6), we obtain the corresponding marginal PGFs of N_1 and N_2

$$\Psi_{N_1}(s_1)=\Psi(s_1,1)=e^{-\lambda(1-h_1(\theta_1))},$$
$$\Psi_{N_2}(s_2)=\Psi(1,s_2)=e^{-\lambda(1-h_2(\theta_2))}. \tag{7}$$

The corresponding marginal distributions of N_1 and N_2 are easily obtained from (7),respectively, to be

$$P(N_1=m)=e^{-\lambda},\quad m=0$$
$$=e^{-\lambda}\theta_1^m\sum_{j=1}^{\infty}\frac{C_m(j)(\frac{\lambda}{h_1(\theta_1)})^j}{j!},\quad m=1,2,\ldots,$$

where

$$C_m(j)=\sum_{k_1+k_2+\ldots+k_j=m} b_{k_1},b_{k_2}...b_{k_j},\quad \text{If } (k_1,k_2,...,k_j) \text{ is the ordered j-tuple of}$$

positive integers in the range set of the random variable X which sum up to m.

and

$$P(N_2 = m) = e^{-\lambda}, \quad m = 0$$
$$= e^{-\lambda}\theta_2^m \sum_{j=1}^{\infty} \frac{D_m(j)(\frac{\lambda}{h_2(\theta_2)})^j}{j!}, \quad m = 1, 2, \ldots,$$

where

$$D_m(j) = \sum_{k_1+k_2+...+k_j=m} c_{k_1}, c_{k_2}...c_{k_j}, \quad \text{If } (k_1, k_2, ..., k_j) \text{ is the ordered j-tuple of}$$

positive integers in the range set of the random variable Y which sum up to m.

55 Then from it follows that marginal distributions of N_1 and N_2 belongs to univariate generalized power series Poisson distribution.

3.1. Conditional Distribution

From Johnson et al.(1997), the conditional P.G.F. of N_2 given N_1 written $\Psi_{N_2/N_1=k}(s_2)$, is

$$\Psi_{N_2/N_1=k}(s) = \frac{\Psi^{(k,0)}(0, s_2)}{\Psi^{(k,0)}(0, 1)}, \tag{8}$$

where $\Psi^{(k,0)}(s_1, s_2) = \frac{\partial^k}{\partial s_1^k}\Psi(s_1, s_2)$.

Substituting $(i, j) = (k, 0)$ and $s_1 = 0$ in (4), we get

$$\Psi^{(k,0)}(0, s_2) = e^{-\lambda} \sum_{m=1}^{\infty} \sum_{j} \frac{\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^m}{m!} C_{kj}(m)k!\theta_1^k(\theta_2 s_2)^j. \tag{9}$$

60 Using (8) and (9) we obtain

$$\Psi_{N_2/N_1=k}(s_2) = \frac{\sum_{m=0}^{\infty} \sum_j \frac{\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^m}{m!} C_{kj}(m)\theta_1^k(\theta_2 s_2)^j}{\sum_{m=1}^{\infty} \sum_j \frac{\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^m}{m!} C_{kj}(m)\theta_1^k\theta_2^j}. \tag{10}$$

For $k = 0$, we get

$$\Psi_{N_2/N_1=0}(s_2) = \frac{\Psi(0, s_2)}{\Psi(0, 1)}$$
$$= e^{-\lambda \frac{g(0,\theta_2)}{g(\theta_1,\theta_2)}} \left[1 - \frac{g(0, \theta_2 s_2)}{g(\theta_1, \theta_2)}\right].$$

It follows immediately that the conditional mean is

$$E[N_2/N_1 = k] = \frac{\sum_{m=1}^{\infty} \sum \frac{\left(\frac{\lambda}{f(\theta_1,\theta_2)}\right)^m}{m!} C_{kj}(m)j\theta_2^j}{\sum_{m=1}^{\infty} \sum \frac{\left(\frac{\lambda}{f(\theta_1,\theta_2)}\right)^m}{m!} C_{kj}(m)\theta_2^j}.$$

In particular

$$E(N_2/N_1 = 0) = \lambda t\theta_2 \frac{\frac{\partial}{\partial\theta_2} g(0, \theta_2)}{g(\theta_1, \theta_2)}.$$

4. Multivariate Extension

Let $X = (X_1, \cdots X_k)$ be a k-dimensional random vector of generalized power series random variables. The PGF of X is given by

$$P(s_1, s_2, \cdots, s_k) = \frac{g(\theta_1 s_1, \theta_2 s_2, \cdots, \theta_k s_k)}{g(\theta_1, \theta_2, \cdots, \theta_k)}.$$

Define

$$N_i = X_{i1} + X_{i2} + ... + X_{iN}, i = 1, 2, ..., k,$$

where N is independent of compounding random vector X and has a Poisson distribution with parameter λ. Then, the joint PGF of $(N_1, N_2, \cdots, N_k)$ is given by

$$\Psi(s_1, s_2, \cdots, s_k) = e^{-\lambda(1-P(s_1,s_2,\cdots s_k))} = e^{-\lambda\left(1-\frac{g(\theta_1 s_1,\theta_2 s_2,\cdots,\theta_k s_k)}{g(\theta_1,\theta_2,\cdots,\theta_k)}\right)}. \quad (11)$$

65 The PGFs of the marginal compounding distributions are given by

$$P_{X_i}(s_i) = P(1, \cdots, s_i, \cdots, 1) = \frac{g(\theta_1, \cdots, \theta_i s_i, \cdots \theta_k)}{g(\theta_1, \cdots, \theta_k)} = \frac{h_i(\theta_i s_i)}{h_i(\theta_i)}, i = 1, 2, \cdots k, \quad (12)$$

Where $h_i(\theta_i) = \sum_{xi} b_i(x_i)\theta_i^{x_i}$ and $b_i(x_i) = \sum_{x_1\cdots x_{i-1}x_{i+1}\cdots x_k} a_{x_1 x_2\cdots x_k}\theta_1^{x_1}\cdots\theta_{i-1}^{x_i-1}\theta_{i+1}^{x_i+1}\cdots\theta_k^{x_k}$.
Therefore from (12) it follows that the random variable X_i has a generalized power series distribution with series function $h_i(\theta_i) = \sum_{xi} b_i(x_i)\theta_i^{x_i}$ as expanded in powers of θ_i, other θ's treated as constants.
The marginal PGFs of $N_i, i = 1, 2, \ldots k$ are obtained from (11) and (12), and are given by

$$\Psi_{N_i}(s_i) = \Psi(1, \ldots, s_i, \ldots, 1) = e^{-\lambda\left(1-\frac{h_i(\theta_i s_i)}{h_i(\theta_i)}\right)}, i = 1, 2, \ldots, k.$$

Then from it follows that $N_i, i = 1, 2, \ldots k$ belongs to univariate generalized power series Poisson distribution. The corresponding marginal P.M.Fs are given by

$$P(N_i = m) = \begin{cases} e^{-\lambda}, & m = 0 \\ e^{-\lambda}\theta_i^m \sum_{j=1}^{\infty} \frac{\left(\frac{\lambda}{h_i(\theta_i)}\right)^j}{j} C_{im}(j), & m = 1, 2, \ldots, \end{cases}$$

where $C_{im}(j) = \sum_{k_1+k_2\cdots+k_j=m} b_i(k_1)b_i(k_2)\ldots b_i(k_j)$ If $(k_1, k_2, \ldots, k_j)$ is ordered j-tuple of positive integers in the range set of X_i which sum up to m.

4.1. Joint Probability Mass Function

The P.G.F. in (11) can be written as

$$\Psi(s_1, s_2, \ldots, s_k) = e^{-\lambda}\sum_{j=1}^{\infty}\sum_{i_1,i_2,\ldots i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\theta_2,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2,\ldots i_k}(j)(\theta_1 s_1)^{i_1}(\theta_2 s_2)^{i_2}\ldots(\theta_k s_k)^{i_k}. \quad (13)$$

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Differentiation in (13) leads to the following derivatives

$$\Psi^{(r_1,r_2,\dots,r_k)}(s_1,\dots,s_k) = e^{-\lambda}\sum_{j=1}^{\infty}\sum_{i_1\geq r_1,\dots,i_k\geq r_k}\frac{\left(\frac{\lambda}{g(\theta_1,\theta_2,\dots,\theta_k)}\right)^j}{j!}C_{i_1,i_2,\dots,i_k}(j)i_1^{(r_1)}\theta_1^{i_1}s_1^{i_1-r_1}\dots i_k^{(r_k)}\theta_k^{i_k}s_k^{i_k-r_k}, \quad (14)$$

where $i_j^{(r_j)} = \frac{i_j!}{(i_j-r_j)!}, j = 1,2,\dots,k$ and $\Psi^{(r_1,r_2,\dots,r_k)}(s_1,\dots,s_k) = \frac{\partial^{r_1+r_2+\dots+r_k}}{\partial s_1^{r_1}\partial s_2^{r_2},\dots\partial s_k^{r_k}}\Psi(s_1,s_2,\dots,s_k)$. From Johnson et al.(1997), it is known that for $r_1\dots,r_k = 0,1\dots,$ and $(r_1,r_2,\dots,r_k)\neq(0,\dots,0)$,

$$f(r_1,r_2,\dots,r_k) = \frac{\Psi^{(r_1,r_2,\dots,r_k)}(s_1,\dots,s_k)}{r_1!\dots r_k!}\Big/ s_1 = \dots = s_k = 0.$$

Denote by $f(i_1,\dots,i_k) = P(N_1 = i_1,\dots,N_k = i_k), i_1\dots i_k = 0,1,2,\dots$ the joint PMF of $(N_1,N_2,\dots,N_k)$ and is given by

$$f(0,\dots,0) = e^{-\lambda\left(1-\frac{a_{0,\dots,0}}{g(\theta_1,\dots,\theta_k)}\right)},$$

$$f(i_1,i_2,\dots,i_k) = e^{-\lambda}\sum_{j=1}^{\infty}\frac{\left(\frac{\lambda}{g(\theta_1,\dots,\theta_k)}\right)^j}{j!}C_{i_1,i_2,\dots,i_k}(j)\theta_1^{i_1}\theta_2^{i_2}\dots\theta_k^{i_k},$$

$$i_1,i_2,\dots,i_k = 0,1,\dots,(i_1,i_2,\dots,i_k)\neq(0,0,\dots0),$$

where

$$C_{i_1,i_2,\dots,i_k}(j)\sum_{\substack{i_{11}+i_{12},+\dots,+i_{1j}=i_1\\ \vdots \\ i_{k1}+i_{k2}+\dots+i_{kj}=i_k}} a(i_{11},i_{21},\dots,i_{k1})a(i_{12},i_{22},\dots,i_{k2})\dots a(i_{1j},i_{2j},\dots,i_{kj}).$$

If $(i_{l1},i_{l2},\dots,i_{lj})$ is the ordered j tuple of elements in the range set of X_l which sum up to $i_l, l = 1,2,\dots,k$. Setting $s_1 = s_2 = \dots = s_k = 1$ in (14), we obtain the joint factorial moment of $(N_1,N_2,\dots,N_k)$.

$$E[N_1(N_1-1)\dots(N_1-r_1+1)\dots N_k(N_{k-1})\dots(N_k-r_{k+1})]$$
$$= e^{-\lambda}\sum_{j=1}^{\infty}\sum_{i_1\geq r_1,\dots,i_k\geq r_k}\frac{\left(\frac{\lambda}{g(\theta_1,\dots\theta_k)}\right)^j}{j!}C_{i_1,i_2,\dots,i_k}(j)i_1^{(r_1)}\theta_1^{i_1}\dots i_k^{(r_k)}\theta_k^{i_k}.$$

4.2. Conditional Distribution

From Johnson et al.(1997),the conditional PGF of $(N_2\dots N_k)$, given N_1 written $\Psi_{N_2\dots N_k/N_1=i_1}(s_2,\dots,s_k)$, is

$$\Psi_{N_2\dots N_k/N_1=i_1}(s_2,\dots,s_k) = \frac{\Psi^{(i_1,0,\dots,0)}(0,s_2,\dots,s_k)}{\Psi^{(i_1,0\dots,0)}(0,1,\dots,1)}, \quad (15)$$

where

$$\Psi^{(i_1,0,\dots,0)}(s_1,\dots,s_k) = \frac{\partial^{i_1}\Psi(s_1,\dots,s_k)}{\partial s_1^{i_1}}.$$

Substituting $(r_1, r_2, \ldots, r_k) = (i_1, 0, \ldots, 0)$ and $s_1 = 0$ in (14), we get

$$\Psi^{(i_1,0,\ldots,0)}(0, s_2, \ldots, s_k) = e^{-\lambda} \sum_{j=1}^{\infty} \sum_{i_2,\ldots,i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2\ldots,i_k}(j) i_1! \theta_1^{i_1} (\theta_2 s_2)^{i_2} \ldots (\theta_k s_k)^{i_k}. \quad (16)$$

Using (15) and (16) we obtain

$$\Psi_{N_2\ldots N_k/N_1=i_1}(s_2, \ldots, s_k) = \frac{\sum_{j=1}^{\infty} \sum_{i_2,\ldots,i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2\ldots,i_k}(j) i_1! \theta_1^{i_1} (\theta_2 s_2)^{i_2} \ldots (\theta_k s_k)^{i_k}}{\sum_{j=1}^{\infty} \sum_{i_2,\ldots,i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2\ldots,i_k}(j) i_1! \theta_1^{i_1} \theta_2^{i_2} \ldots \theta_k^{i_k}}.$$

For $i_1 = 0$, we get

$$\begin{aligned}\Psi_{N_2\ldots N_k/N_1=0}(s_2, \ldots, s_k) &= \frac{\Psi(0, s_2, \ldots, s_k)}{\Psi(0, 1, \ldots, 1)} \\ &= e^{-\lambda \frac{g(0,\theta_2,\ldots,\theta_k)}{g(\theta_1,\theta_2,\ldots,\theta_k)}} \left(1 - \frac{g(0, \theta_2 s_2, \ldots, \theta_k s_k)}{g(\theta_1, \theta_2, \ldots, \theta_k)}\right).\end{aligned}$$

The conditional PGF of N_k given $(N_1, N_2, \cdots, N_{k-1})$ is

$$\begin{aligned}\Psi_{N_k/N_1=i_1,\ldots,N_{k-1}=i_{k-1}}(s_k) &= \frac{\Psi^{(i_1,\ldots,i_{k-1},0)}(0, \ldots, 0, s_k)}{\Psi^{(i_1,\ldots,i_{k-1},0)}(0, \ldots, 0, 1)} \\ &= \frac{\sum_{j=1}^{\infty} \sum_{i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2\ldots,i_k}(j) i_1! i_2! \ldots i_k! \theta_1^{i_1} \theta_2^{i_2} \ldots \theta_{k-1}^{i_{k-1}} (\theta_k s_k)^{i_k}}{\sum_{j=1}^{\infty} \sum_{i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2\ldots,i_k}(j) i_1! i_2! \ldots i_{k-1}! \theta_1^{i_1} \theta_2^{i_2} \ldots \theta_k^{i_k}}.\end{aligned}$$

For $i_1 = i_2 = \cdots = i_{k-1} = 0$, we get

$$\begin{aligned}\Psi_{N_k/N_1=N_2=\cdots=N_{k-1}=0} &= \frac{\Psi(0, \ldots, 0, s_k)}{\Psi(0, \ldots, 0, 1)} \\ &= e^{-\lambda \frac{g(0,\ldots,0,\theta_k)}{g(\theta_1,\ldots,\theta_k)}} \left(1 - \frac{g(0, \ldots, 0, \theta_k s_k)}{g(\theta_1, \ldots, \theta_k)}\right).\end{aligned}$$

It follows immediately that the conditional mean is

$$\begin{aligned}&E(N_k/N_1 = i_1, N_2 = i_2, \ldots, N_{k-1} = i_{k-1}) \\ &= \frac{\sum_{j=1}^{\infty} \sum_{i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2\ldots,i_k}(j) i_1! \ldots i_{k-1}! \theta_1^{i_1} \ldots \theta_{k-1}^{i_{k-1}} (\theta_k s_k)^{i_k}}{\sum_{j=1}^{\infty} \sum_{i_k} \frac{\left(\frac{\lambda}{g(\theta_1,\ldots,\theta_k)}\right)^j}{j!} C_{i_1,i_2\ldots,i_k}(j) i_1! i_2! \ldots i_{k-1}! \theta_1^{i_1} (\theta_2)^{i_2} \ldots \theta_k^{i_k}}.\end{aligned}$$

In particular

$$E(N_k/N_1 = \cdots = N_{k-1} = 0) = \frac{\lambda t \theta_k \frac{\partial}{\partial \theta_k} g(0, \ldots, 0, \theta_k)}{g(\theta_1, \ldots, \theta_k)}.$$

75 ## 5. Type II bivariate Generalized Power Series Poisson Process

Consider a compound Poisson process with bivariate generalized power series compounding distribution, given in (1). The resulting bivariate counting process $(N_1(t), N_2(t))$ has type II bivariate generalized power series Poisson distribution with parameters $\lambda t, \theta_1$ *and* θ_2. i.e,

$$f(0,0) = e^{-\lambda t\left(1-\frac{a_{0,0}}{g(\theta_1,\theta_2)}\right)},$$

$$f(i,j) = e^{-\lambda t}\theta_1^i\theta_2^j\sum_{k=1}^{\infty}\frac{C_{ij}(k)\left(\frac{\lambda}{g(\theta_1,\theta_2)}\right)^k}{k!}, \quad i,j=0,1,\cdots,(i,j)\neq(0,0)\ ,$$

where $C_{ij}(k) = \sum_{\substack{i_1+i_2+\cdots+i_k=i \\ j_1+j_2+\cdots+j_k=j}} a_{i_1,j_1}a_{l_2,j_2}\cdots a_{i_k,j_k}$, if $(i_1,i_2,\cdots,i_k)$ is the ordered k tuple of elements in the range of X which sums up to i and $(j_1,j_2,\cdots j_k)$ is the ordered k tuple of elements in the range of Y which sums up to j .

To express $\{(N_1(t),N_2(t)), t\geq 0\}$ is type II bivariate generalized power series Poisson process with parameters $\lambda t, \theta_1$ *and* θ_2 we use the notation $(N_1(t),N_2(t)) \sim BGPS_{II}(\lambda t,\theta_1,\theta_2)$.

Remark: 1. *1.In the case of* $g(\theta_1,\theta_2) = (1-\theta_1-\theta_2)^{-r}, \theta_1=\alpha, \theta_2=\beta$ *and* $a(i,j) = \binom{i+j}{j}\binom{r+i+j-1}{i+j}$, *the type II bivariate generalized power series Poisson process coincides with bivariate Poisson negative binomial process; see Kostadinova and Minkova(2014).*

2.In the case of $g(\theta_1,\theta_2) = (1-\theta_1-\theta_2)^{-1}, \theta_1=\alpha, \theta_2=\beta$ *and* $a(i,j) = \binom{i+j}{j}\binom{r+i+j-1}{i+j}$, *the type II bivariate generalized power series Poisson process coincides with type II bivariate Polya-Aeppli process; see Kostadinova(2015).*

6. Type II Bivariate Generalized Power Series Poisson Risk Model

Let us assume that there are two kinds of claims X_{1i} and X_{2i} belonging to two classes. We will investigate a two dimensional model

$$\begin{aligned} U_1(t) &= u_1 + c_1 t - S_1(t), \\ U_2(t) &= u_2 + c_2 t - S_2(t), \end{aligned} \tag{17}$$

where $u_i, i=1,2$, is the initial capital, $c_i > 0, i=1,2$, is the constant premium income per unit time, $N_i(t)$ is the number of claims up to time t, X_{ik} is the size of the k^{th} claim and $S_i(t) = \sum_{j=1}^{N_i(t)} X_{ij}, i=1,2$ is the aggregate claims amount for i^{th} class.

For fixed $i=1,2,\ \{X_{ik}\}_{k\geq 1}$ are independent and identically distributed (i.i.d) nonnegative random variable with distribution function $F_i(X_i)$ such that $F_i(0)=0$ and finite mean $\mu_i = E(X_i)$. Assume that $\{N_i(t), t\geq 0\}$, $\{(X_{1k},X_{2k})\}_{k\geq 1}$ are mutually independent and $\{(X_{1k},X_{2k})\}_{k\geq 1}$ is a sequence of i.i.d bivariate random vectors with joint distribution function $F(x_1,x_2)$. Here we assume that bivariate counting process $\{(N_1(t),N_2(t)), t\geq 0\}$ has a type II bivariate generalized power series Poisson process and will call the process the bivariate generalized power series Poisson risk model.

Now we consider the sum of both risk process (17), the joint capital for the two classes is given by:

$$U(t) = U_1(t) + U_2(t) = u + ct - S(t),$$

where $u = u_1 + u_2, c = c_1 + c_2$ and $S(t) = S_1(t) + S_2(t)$.

Central problem in risk theory is the modeling of the probability distribution for the aggregate claims. The aggregate claims distribution is mainly used to compute ruin probabilities. Hesselarger(1996) introduced recursive formulas for the joint distribution of the bivariate aggregate claims random variables. Clark and Homer(2003) used Fast Forier Transformation(FFT) to compute bivariate aggregate claims distribution. Here we derive bivariate aggregate claims distribution from type II bivariate generalized power series Poisson risk model via convolution. Let $H(u_1, u_2, t)$ denotes the joint cumulative distribution function of bivariate aggregate claims, $(S_1(t), S_2(t))$ and and $F^{*n}(x)$ is the n-fold convolution of claim amount distribution which can be calculated recursively as

$$F^{*n}(x) = \int_0^x F^{*n-1}(x-y)f(y)dy.$$

The joint CDF of aggregate claims is given by

$$\begin{aligned} H_{(S_1(t),S_2(t))}(x,y,t) &= P(S_1(t) \le x, S_2(t) \le y) \\ &= \sum_{i,j=0}^{\infty} P(N_1(t) = i, N_2(t) = j) F_1^{*i}(x) F_2^{*j}(y) \\ &= e^{-\lambda t}\left(1 - \frac{a_{0,0}}{g(\theta_1,\theta_2)}\right) \\ &+ e^{-\lambda t} \sum_{i,k=1}^{\infty} \frac{C_{i,0}(k)\theta_1^i \left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^k}{k!} F_1^{*i}(x) \\ &+ e^{-\lambda t} \sum_{j,k=1}^{\infty} \frac{C_{0,j}(k)\theta_2^j \left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^k}{k!} F_2^{*j}(y) \\ &+ e^{-\lambda t} \sum_{i,j,k=1}^{\infty} C_{i,j}(k)\theta_1^i\theta_2^j \frac{\left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^k}{k!} F_1^{*i}(x) F_2^{*j}(y). \end{aligned} \tag{18}$$

Let $N(t) = N_1(t) + N_2(t)$ denotes the total number of claims happened in both classes.

Then the PMF of $N(t)$ is given by

$$P(N(t) = k) = \begin{cases} e^{-\lambda t}(1 - \frac{a_{0,0}}{g(\theta_1,\theta_2)}), & k = 0 \\ e^{-\lambda t} \sum_{j=1}^{\infty} \sum_{i=0}^{k} \frac{\left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^j}{j!} C_{i,k-i}(j)\theta_1^i\theta_2^{k-i}, & k = 1, 2, \cdots \end{cases}$$

Now we consider the sum of aggregate claims of two classes

$$S(t) = S_1(t) + S_2(t)$$

Case 1:two classes have different claim size distribution

In this case

$$S(t) = \sum_{j=1}^{N_1(t)} X_{1j} + \sum_{j=1}^{N_2(t)} X_{2j}$$

and the corresponding CDF G(u) is given by

$$\begin{aligned} G(u) &= P(S(t) \leq x) \\ &= \sum_{i,j=0}^{\infty} P(N_1(t)=i, N_2(t)=j) F_1^{*i} * F_2^{*j}(x) , \\ &= e^{-\lambda t\left(1-\frac{a_{0,0}}{g(\theta_1,\theta_2)}\right)} + e^{-\lambda t} \sum_{i,k=1}^{\infty} \frac{C_{i,0}(k)\theta_1^i \left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^k}{k!} F_1^{*i}(x) \\ &+ e^{-\lambda t} \sum_{j,k=1}^{\infty} \frac{C_{0,j}(k)\theta_2^j \left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^k}{k!} F_2^{*j}(x) + e^{-\lambda t} \sum_{i,j,k=1}^{\infty} C_{i,j}(k)\theta_1^i\theta_2^j \frac{\left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^k}{k!} F_1^{*i} * F_2^{*j}(x). \end{aligned} \tag{19}$$

Case 2:two classes have the same claim size distribution

In this case

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)},$$

where $N(t) = N_1(t) + N_2(t)$.

Denote by $G(x)$ the CDF of $S(t)$ and is given by

$$\begin{aligned} G(x) &= P(S(t) \leq x) \\ &= \sum_{i=0}^{\infty} P(N(t)=i) F^{*i}(x) , \\ &= e^{-\lambda t\left(1-\frac{a_{0,0}}{g(\theta_1,\theta_2)}\right)} + e^{-\lambda t} \sum_{i,j=1}^{\infty} \sum_{r=0}^{i} \frac{\left(\frac{\lambda t}{g(\theta_1,\theta_2)}\right)^j}{j!} C_{r,i-r}(j)\theta_1^r\theta_2^{i-r} F^{*i}(x). \end{aligned} \tag{20}$$

7. Ruin probabilities

Ruin theory for the bivariate risk model has been extensively considered in the literature. It has been found that ruin probabilities are often fundamental interest in risk management purpose. Chan et al.(2003) discussed various ruin concept for bivariate risk process.

The time of ruin for the i^{th} class $(i = 1, 2)$ is defined by

$$\tau_i = \inf\{t \geq 0; U_i(t) < 0\},$$

and the corresponding probability of ruin is

$$\Psi_i(u_i) = P(\tau_i < \infty / U_i(0) = u_i).$$

If for each i, the process $U_i(t) \geq 0$ for all $t \geq 0$ (no ruin occurs), we indicae this by writing $\tau_i = \infty$.

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Here we consider three kinds of ruin time as follows. The first one is $\tau_{\max}(u_1, u_2) = \inf\{t \geq 0/ \max(U_1(t), U_2(t)) < 0\}$, representing the first time when both $U_1(t)$ and $U_2(t)$ became negative, whereas the second one is $\tau_{\min}(u_1, u_2) = \inf\{t \geq 0/ \min(U_1(t), U_2(t)) < 0\}$, representing the first time when either $U_1(t)$ or $U_2(t)$ became negative, and last one is $\tau_{\text{sum}} = \inf\{t \geq 0/U(t) < 0\}$, representing the time when the joint capital for the two classes $U(t)$ became negative. The associated ruin probabilities will be respectively denoted by $\Psi_{\max}(u_1, u_2)$, $\Psi_{\min}(u_1, u_2)$ and $\Psi_{sum}(u_1, u_2)$. First we derive the expression for the ruin probability $\Psi_{\max}(u_1, u_2)$

$$\begin{aligned}
\Psi_{\max}(u_1, u_2) &= P(\tau_{\max} < \infty/U_1(0) = u_1, U_2(0) = u_2) \\
&= P(\max(U_1(t), U_2(t)) < 0) \\
&= P(U_1(t) < 0, U_2(t) < 0) \\
&= P(S_1(t) > u_1 + c_1 t, S_2(t)) > u_2 + c_2 t) \\
&= \bar{H}(u_1 + c_1 t, u_2 + c_2 t).
\end{aligned}$$

where $\bar{H}(u_1, u_2)$ is the joint survival function of $(S_1(t), S_2(t))$.

Next we consider the expression for the ruin probability $\Psi_{\min}(u_1, u_2)$

$$\begin{aligned}
\Psi_{\min}(u_1, u_2) &= P(\tau_{\min} < \infty/U_1(0) = u_1, U_2(0) = u_2) \\
&= P(\min(U_1(t), U_2(t)) < 0) \\
&= 1 - P(\min(U_1(t), U_2(t)) > 0) \\
&= 1 - P(U_1(t) > 0, U_2(t)) > 0) \\
&= 1 - P(S_1(t) < u_1 + c_1 t, S_2(t) < u_2 + c_2 t) \\
&= 1 - H(u_1 + c_1 t, u_2 + c_2 t) ,
\end{aligned}$$

where $H(u_1, u_2)$ is the joint CDF of $(S_1(t), S_2(t))$ given by (18).

Finally we derive the expression for the ruin probability $\Psi_{\text{sum}}(u_1, u_2)$

$$\begin{aligned}
\Psi_{\text{sum}}(u_1, u_2) &= P(\tau_{\text{sum}} < \infty/U_1(0) = u_1, U_2(0) = u_2) \\
&= P(U(t) < 0) \\
&= P(S(t) > u + ct) \\
&= \overline{G}(u + ct)
\end{aligned}$$

where $\overline{G}(x)$ is the survival function of $S(t)$.

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7.1. Bounds for Ruin Probability

Most of the papers in the literature of bivariate risk theory are concerned with ruin probabilities. Exact solutions of these probabilities are rarely available, and existing result are mostly in the form of bounds. Chan et al.(2003) , Cai and Li(2005) and Yuen et al.(2006) derived bounds for the ultimate ruin probability $\Psi_{\min}(u_1, u_2)$. Simple bounds for $\Psi_{\max}(u_1, u_2)$ was given by Cai and Li(2005, 2007).

The lower and upper bounds on $\Psi_{\max}(u_1, u_2)$ and $\Psi_{\min}(u_1, u_2)$ are respectively described by the following inequalities.

$$\begin{aligned} \Psi_1(u_1)\,\Psi_2(u_2) \leq \Psi_{\max}(u_1, u_2) \leq \min(\Psi_1(u_1),\, \Psi_2(u_2)) \\ \max(\Psi_1(u_1),\, \Psi_2(u_2)) \leq \Psi_{\min}(u_1, u_2) \leq \Psi_1(u_1) + \Psi_2(u_2) - \Psi_1(u_1)\Psi_2(u_2), \end{aligned} \tag{21}$$

where the final expression in the second equation is exactly the ruin probability in the case where $\{U_1(t)\}_{t\geq 0}$ and $\{U_2(t)\}_{t\geq 0}$ are independent.

If there is no initial capitals ($u_1 = u_2 = 0$), then the above relations becomes

$$\begin{aligned} \Psi_1(0)\,\Psi_2(0) \leq \Psi_{\max}(0, 0) \leq \min(\Psi_1(0),\, \Psi_2(0)) \\ \max(\Psi_1(0),\, \Psi_2(0) \leq \Psi_{\min}(0, 0) \leq \Psi_1(0) + \Psi_2(0) - \Psi_1(0)\Psi_2(0) \end{aligned} \tag{22}$$

In the case of univariate generalized power series Poisson risk model the ruin probabilities are given by

$$\Psi_i(0) = \frac{\lambda \mu_i h_i'(\theta_i)}{c_i h_i(\theta_1)} \quad i = 1, 2. \tag{23}$$

Using the equations (22) and (23) we can obtain bounds for the ruin probabilities $\Psi_{\max}(0, 0)$ and $\Psi_{\min}(0, 0)$ for the type II bivariate generalized power series Poisson risk model and are given by

$$\frac{\lambda^2 \mu_1 \mu_2 h_1'(\theta_1) h_2'(\theta_2)}{c_1 c_2 h_1(\theta_1) h_2(\theta_2)} \leq \Psi_{\max}(0, 0) \leq \min\left(\frac{\lambda \mu_1 h_1'(\theta_1)}{c_1 h_1(\theta_1)}, \frac{\lambda \mu_2 h_2'(\theta_2)}{c_2 h_2(\theta_2)}\right).$$

$$\max\left(\frac{\lambda \mu_1 h_1'(\theta_1)}{c_1 h_1(\theta_1)}, \frac{\lambda \mu_2 h_2'(\theta_2)}{c_2 h_2(\theta_2)}\right) \leq \Psi_{\min}(0, 0) \leq \frac{\lambda \mu_1 h_1'(\theta_1)}{c_1 h_1(\theta_1)} + \frac{\lambda \mu_2 h_2'(\theta_2)}{c_2 h_2(\theta_2)} - \frac{\lambda^2 \mu_1 \mu_2 h_1'(\theta_1) h_2'(\theta_2)}{c_1 c_2 h_1(\theta_1) h_2(\theta_2)}.$$

Moreover, we have

$$\Psi_{\min}(u_1, u_2) = \Psi_1(u_1) + \Psi_2(u_2) - \Psi_{\max}(u_1, u_2).$$

In the case of no initial capital above relation is

$$\Psi_{\min}(0, 0) = \Psi_1(0) + \Psi_2(0) - \Psi_{\max}(0, 0).$$

and hence,we obtain

$$\Psi_{\min}(0, 0) = \frac{\lambda \mu_1 h_1'(\theta_1)}{c_1 h_1(\theta_1)} + \frac{\lambda \mu_2 h'(\theta_2)}{c_2 h_2(\theta_2)} - \Psi_{\max}(0, 0).$$

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8. Two Dimensional Integro Differential Equation

In this section we will derive partial integro differential equation for the bivariate survival probability for the bivariate surplus process (17) defined in section 6.

Define the infinite time joint survival probability

$$\Phi(u_1, u_2) = P(U_1(t) \geq 0, U_2(t) \geq 0; \text{ for all } t \geq 0).$$

and infinite time joint ruin probability is $\Psi(u_1, u_2) = 1 - \Phi(u_1, u_2)$.

In a small time interval $(0, h]$, there are following possible cases: no claim, one claim from class 1 and no claim from class 2, no claim from class 1 and one claim from class 2, one or more than one claims from each class. It follows that

$$\begin{aligned}
\Phi(u_1, u_2) = & \left(1 - \lambda h \left(1 - \frac{a_{0,0}}{g(\theta_1, \theta_2)}\right) + o(h)\right) \Phi(u_1 + c_1 h, u_2 + c_2 h) \\
& + \left(\frac{a_{10}\theta_1 h}{g(\theta_1, \theta_2)} + o(h)\right) \int_0^{u_1 + c_1 h} \Phi(u_1 + c_1 h - x, u_2 + c_2 h) dF_1(x) \\
& + \left(\frac{a_{01}\theta_2 h}{g(\theta_1, \theta_2)} + o(h)\right) \int_0^{u_2 + c_2 h} \Phi(u_1 + c_1 h, u_2 + c_2 h - y) dF_2(y) \\
& + \left(\sum_{i,j=1}^{\infty} \frac{a_{i,j}\theta_1^i \theta_2^j h}{g(\theta_1, \theta_2)} + o(h)\right) \int_0^{u_1 + c_1 h} \int_0^{u_2 + c_2 h} \Phi(u_1 + c_1 h - x, u_2 + c_2 h - y) dF_1^{*i}(x) dF_2^{*j}(y),
\end{aligned}$$

where $F_i^{*m}(x), i = 1, 2., m = 1, 2, \cdots$ is the distribution function of $X_{i1} + X_{i2} + ... + X_{im}$.

Rearranging the terms leads to

$$\begin{aligned}
\frac{\Phi(u_1 + c_1 h, u_2 + c_2 h) - \Phi(u_1, u_2)}{h} = & \lambda \left(1 - \frac{a_{0,0}}{g(\theta_1, \theta_2)}\right) \Phi(u_1 + c_1 h, u_2 + c_2 h) \\
& + \frac{a_{10}\theta_1}{g(\theta_1, \theta_2)} \int_0^{u_1 + c_1 h} \Phi(u_1 + c_1 h - x, u_2 + c_2 h) dF_1(x) \\
& + \frac{a_{01}\theta_2}{g(\theta_1, \theta_2)} \int_0^{u_2 + c_2 h} \Phi(u_1 + c_1 h, u_2 + c_2 h - y) dF_2(y) + \sum_{i,j=1}^{\infty} \frac{a_{i,j}\theta_1^i \theta_2^j}{g(\theta_1, \theta_2)} \\
& \int_0^{u_1 + c_1 h} \int_0^{u_2 + c_2 h} \Phi(u_1 + c_1 h - x, u_2 + c_2 h - y) dF_1^{*i}(x) dF_2^{*j}(y) + o(h).
\end{aligned}$$

As h tends to zero, we get

$$\begin{aligned}
c_1 \frac{\partial}{\partial u_1} \Phi(u_1, u_2) + c_2 \frac{\partial}{\partial u_2} \Phi(u_1, u_2) = & \lambda \left(1 - \frac{a_{0,0}}{g(\theta_1, \theta_2)}\right) \Phi(u_1, u_2) \\
& + \frac{a_{10}\theta_1}{g(\theta_1, \theta_2)} \int_0^{u_1} \Phi(u_1 - x, u_2) dF_1(x) \\
& + \frac{a_{01}\theta_2}{g(\theta_1, \theta_2)} \int_0^{u_2} \Phi(u_1, u_2 - y) dF_2(y) \\
& + \sum_{i,j=1}^{\infty} \frac{a_{i,j}\theta_1^i \theta_2^j}{g(\theta_1, \theta_2)} \int_0^{u_1} \int_0^{u_2} \Phi(u_1 - x, u_2 - y) dF_1^{*i}(x) dF_2^{*j}(y)
\end{aligned} \tag{24}$$

It is difficult to solve this two dimensional integro differential equation.

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8.1. Laplace Transforms of the Survival Probabilities

Having obtained the partial integro differential equations(PIDE) for the survival probabilities $\Phi(u_1, u_2)$ of the surplus process (17), in the following we will derive the Laplace transforms for the survival probabilities.

Firstly, we define the following Laplace transforms

$\check{\Phi}(s_1, u_2) = \int_0^\infty e^{-s_1 u_1}\Phi(u_1, u_2)du,\ \tilde{f}_i(s_i) = \int_0^\infty e^{-s_i x}dF_i(x), i = 1, 2.$ and

$\check{\check{\Phi}}(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-s_1 u_1 - s_2 u_2}\Phi(u_1, u_2)du.$

Taking Laplace transform on both sides of the PIDE (24) with respect to u_1, we get

$$\begin{aligned} c_1(s_1\check{\Phi}(s_1,u_2) - \Phi(0,u_2)) + c_2\frac{\partial}{\partial u_2}\check{\Phi}(s_1,u_2) &= \lambda\left(1 - \frac{a_{0,0}}{g(\theta_1,\theta_2)}\right)\check{\Phi}(s_1,u_2) + \frac{a_{10}\theta_1\tilde{f}_1(s_1)}{g(\theta_1,\theta_2)}\check{\Phi}(s_1,u_2) \\ &+ \frac{a_{01}\theta_2}{g(\theta_1,\theta_2)}\int_0^{u_2}\check{\Phi}(s_1,u_2-y)dF_2(y) \\ &+ \sum_{i,j=1}^{\infty}\frac{a_{i,j}\theta_1^i\theta_2^j\tilde{f}_1^{*i}(s_1)}{g(\theta_1,\theta_2)}\int_0^{u_2}\check{\Phi}(s_1,u_2-y)dF_2^{*j}(y), \end{aligned}$$

On simplification we get

$$\begin{aligned} c_2\frac{\partial}{\partial u_2}\check{\Phi}(s_1,u_2) &= c_1\Phi(0,u_2) + (\lambda\left(1 - \frac{a_{0,0}}{g(\theta_1,\theta_2)}\right) - c_1 s_1 + \frac{a_{10}\theta_1\tilde{f}_1(s_1)}{g(\theta_1,\theta_2)})\check{\Phi}(s_1,u_2) \\ &+ \frac{a_{01}\theta_2}{g(\theta_1,\theta_2)}\int_0^{u_2}\check{\Phi}(s_1,u_2-y)dF_2(y) + \sum_{i,j=1}^{\infty}\frac{a_{i,j}\theta_1^i\theta_2^j\tilde{f}_1^{*i}(s_1)}{g(\theta_1,\theta_2)}\int_0^{u_2}\check{\Phi}(s_1,u_2-y)dF_2^{*j}(y), \end{aligned} \tag{25}$$

Taking Laplace transform on both sides of the PIDE (25) with respect to u_2, we get

$$\begin{aligned} c_2(s_2\check{\check{\Phi}}(s_1,s_2) - \check{\Phi}(s_1,0)) &= c_1\check{\Phi}(0,s_2) + (\lambda\left(1 - \frac{a_{0,0}}{g(\theta_1,\theta_2)}\right) - c_1 s_1 + \frac{a_{10}\theta_1\tilde{f}_1(s_1)}{g(\theta_1,\theta_2)})\check{\check{\Phi}}(s_1,s_2) \\ &+ \frac{a_{01}\theta_2\tilde{f}_2(s_2)}{g(\theta_1,\theta_2)}\check{\check{\Phi}}(s_1,s_2) + \sum_{i,j=1}^{\infty}\frac{a_{i,j}\theta_1^i\theta_2^j\tilde{f}_1^{*i}(s_1)\tilde{f}_2^{*j}(s_2)}{g(\theta_1,\theta_2)}\check{\check{\Phi}}(s_1,s_2) \end{aligned}$$

Finally we get

$$\check{\check{\Phi}}(s_1,s_2) = \frac{c_1 s_1\check{\Phi}(0,s_2) + c_2 s_2\check{\Phi}(s_1,0)}{c_1 s_1 + c_2 s_2 - \lambda\left(1 - \frac{a_{0,0}}{g(\theta_1,\theta_2)}\right) - \frac{a_{10}\theta_1\tilde{f}_1(s_1)}{g(\theta_1,\theta_2)} - \frac{a_{01}\theta_2\tilde{f}_2(s_2)}{g(\theta_1,\theta_2)} - \sum_{i,j=1}^{\infty}\frac{a_{i,j}\theta_1^i\theta_2^j\tilde{f}_1^{*i}(s_1)\tilde{f}_2^{*j}(s_2)}{g(\theta_1,\theta_2)}}$$

9. Conclusion

In this paper we introduced the type II bivariate generalized power series Poisson distribution as a compound Poisson distribution with generalized power series compounding distribution. We have considered the bivariate risk model with type II bivariate generalized power series Poisson distribution as claim number distribution. Three models of ruin and the probabilities of ruin for the type II bivariate generalized power series Poisson risk model are investigated. Also the bounds for ruin probabilities are developed. We obtained PIDE for the survival probability and derived an expression for bivariate Laplace transform of ruin probabilities.

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