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ON SOME FRACTIONAL INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR $r-$PREINVEX FUNCTIONS

ABDULLAH AKKURT AND HÜSEYIN YILDIRIM

[DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, UNIVERSITY OF KAHRAMANMARAŞ SÜTÇÜ İMAM, 46000, KAHRAMANMARAŞ, TURKEY; hyildir@ksu.edu.tr

* Correspondence: abdullahmat@gmail.com

ABSTRACT: In this paper, we have established Hermite-Hadamard inequalities for $r-$preinvex functions via fractional integrals.

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This double inequality (1.1) is known in the literature as Hermite–Hadamard integral inequality for convex functions. Both inequalities hold in the reversed direction if f is concave.The inequality (1.1) has been extended and generalized for various classes of convex functions via different approaches, see [4, 7, 10, 12]. For several recent result concerning the inequality (1.1) we refer the interested reader to [1 − 11, 13, 15 − 17, 19] and references cited therein.

2. PRELIMINARES

Let K be an nonempty subset of $\mathbb{R}^n$ and let $\eta : K \times K \to \mathbb{R}^n$.

Definition 1. ([24]) *Let $u \in K$. We say K is invex at u with respect to η if, for each $v \in K$*

$$u + t\eta(v, u) \in K, \ \ t \in [0, 1]. \tag{2.1}$$

K is said to be an invex set with respect to η if K is invex at each $u \in K$.

Definition 2. ([14]) *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \ t \in [0, 1]. \tag{2.2}$$

Definition 3. ([18]) *A positive function f on the invex set K is said to be logarithmically preinvex, if*

$$f(u + t\eta(v, u)) \leq f^{1-t}(u)f^t(v) \tag{2.3}$$

for all $u, v \in K$ and $t \in [0, 1]$.

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Definition 4. ([18]) *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u+t\eta(v,u)) \leq M_r(f(u),f(v);t)$$

holds for all $u,v\in K$ and $t\in[0,1]$, where

$$M_r\left(x,y;t\right)=\begin{cases}\left[(1-t)x^r+ty^r\right]^{\frac{1}{r}} & ,\ r\neq 0\\ x^{1-t}y^t & ,\ r=0\end{cases}$$

is the weighted power mean of order r for positive numbers x,y.

Definition 5. ([19]) *Let $f\in L^1[a,b]$. The Riemann-Liouville fractional integral $J^{\alpha}_{a+}f\left(x\right)$ and $J^{\alpha}_{b-}f\left(x\right)$ of order $\alpha>0$ are defined by*

$$J^{\alpha}_{a+}\left[f\left(x\right)\right]=\frac{1}{\Gamma(\alpha)}\int_a^x\left(x-t\right)^{\alpha-1}f\left(t\right)dt \quad x>a \tag{2.4}$$

and

$$J^{\alpha}_{b-}\left[f\left(x\right)\right]=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f\left(t\right)dt \quad x<b \tag{2.5}$$

respectively, where $\Gamma\left(\alpha\right)=\int_0^\infty e^{-u}u^{\alpha-1}du$ *is Gamma function and* $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

3. Main results

Theorem 1. *Let $f:K=[a,a+\eta(b,a)]\to(0,\infty)$ be an $r-$preinvex functions on the interval of real numbers K° (the interior of K) and $a,b\in K^{\circ}$ with $a<a+\eta(b,a)$. then*

$$\left(J^{\alpha}_{(a+\eta(b,a))^-}f\right)(a)\leq\frac{[\eta(b,a)]^{\alpha}}{\Gamma\left(\alpha\right)}\left\{f^r(b)\frac{r}{\alpha r+1}+f^r(a)B\left(\alpha,\frac{1}{r}+1\right)\right\}^{\frac{1}{r}} \tag{3.1}$$

holds for $0<r\leq 1$.

Proof. Since f is an $r-$preinvex function and $r>0$, we have

$$f(a+t\eta(b,a)\leq\left[tf^r(b)+(1-t)f^r(a)\right]^{\frac{1}{r}}$$

for all $t\in[0,1]$. Then,

$$\begin{aligned}\frac{\Gamma(\alpha)}{[\eta(b,a)]^{\alpha}}\left(J^{\alpha}_{(a+\eta(b,a))^-}f\right)(a) &= \frac{1}{[\eta(b,a)]^{\alpha}}\int_a^{a+\eta(b,a)}(u-a)^{\alpha-1}f\left(u\right)du\\ &=\int_0^1 t^{\alpha-1}f(a+t\eta(b,a))dt\\ &\leq\int_0^1 t^{\alpha-1}\left[tf^r(b)+(1-t)f^r(a)\right]^{\frac{1}{r}}dt\\ &=\int_0^1\left[t^{r(\alpha-1)+1}f^r(b)+t^{r(\alpha-1)}(1-t)f^r(a)\right]^{\frac{1}{r}}dt.\end{aligned}$$

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Using Minkowski's inequality, we have

$$\begin{aligned}
&\int_0^1 \left[t^{r(\alpha-1)+1}f^r(b)+t^{r(\alpha-1)}(1-t)f^r(a)\right]^{\frac{1}{r}}dt\\
&\leq \left\{\left[\int_0^1 t^{\alpha-1+\frac{1}{r}}f(b)dt\right]^r+\left[\int_0^1 t^{\alpha-1}(1-t)^{\frac{1}{r}}f(a)dt\right]^r\right\}^{\frac{1}{r}}\\
&=\left\{f^r(b)\frac{r}{\alpha r+1}+f^r(a)B\left(\alpha,\frac{1}{r}+1\right)\right\}^{\frac{1}{r}}.
\end{aligned}$$

Theorem 1 is proved.

Remark 1. *Under conditions of Theorem 1, $\alpha=1$, $r=1$ and $\eta(b,a)=b-a$, we have*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}.$$

□

Theorem 2. *Let $f,g:K=[a,a+\eta(b,a)]\to(0,\infty)$ be $r-$preinvex and $s-$preinvex functions respectively on the interval of real numbers K°, $a,b\in K^\circ$ with $a<a+\eta(b,a)$, then*

$$\begin{aligned}
&\left(J^{\alpha}_{(a+\eta(b,a))^-}fg\right)(a)\\
&\leq\frac{1}{2}\frac{[\eta(b,a)]^\alpha}{\Gamma(\alpha)}\left\{\left(f^r(b)\frac{r}{2\alpha+r}+f^r(a)B\left(\frac{2(\alpha-1)}{r}+1,\frac{2}{r}+1\right)\right)^{\frac{2}{r}}\right.\\
&\left.+\left(g^s(b)\frac{s}{2\alpha+s}+g^s(a)B\left(\frac{2(\alpha-1)}{s}+1,\frac{2}{s}+1\right)\right)^{\frac{2}{s}}\right\}
\end{aligned}\tag{3.2}$$

holds for $0<r,s\leq 2$.

Proof. Since f is an $r-$preinvex function and g is a $s-$preinvex function, by hypothesis of theorem, we have

$$f(a+t\eta(b,a)\leq[tf^r(b)+(1-t)f^r(a)]^{\frac{1}{r}}\tag{3.3}$$

and

$$g(a+t\eta(b,a)\leq[tg^s(b)+(1-t)g^s(a)]^{\frac{1}{s}}\tag{3.4}$$

for $t\in[0,1]$. By using the inequality (3.3) and (3.4), we get

$$\begin{aligned}
&\frac{1}{[\eta(b,a)]^\alpha}\int_a^{a+\eta(b,a)}(u-a)^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}f(u)g(u)du\\
&=\int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}f(a+t\eta(b,a))g(a+t\eta(b,a))dt\\
&\leq\int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}[tf^r(b)+(1-t)f^r(a)]^{\frac{1}{r}}[tg^s(b)+(1-t)g^s(a)]^{\frac{1}{s}}dt.
\end{aligned}\tag{3.5}$$

Using Cauchy's inequality for (3.5), we have

$$\begin{aligned}
&\int_0^1 \left[t^{\alpha} f^r(b) + t^{\alpha-1}(1-t) f^r(a)\right]^{\frac{1}{r}} \left[t^{\alpha} g^s(b) + t^{\alpha-1}(1-t) g^s(a)\right]^{\frac{1}{s}} dt \\
&\leq \frac{1}{2}\left\{\int_0^1 \left[t^{\alpha} f^r(b) + t^{\alpha-1}(1-t) f^r(a)\right]^{\frac{2}{r}} dt \right. \\
&\left. + \int_0^1 \left[t^{\alpha} g^s(b) + t^{\alpha-1}(1-t) g^s(a)\right]^{\frac{2}{s}} dt\right\} \\
&= \frac{1}{2}\left\{I_1 + I_2\right\}.
\end{aligned}$$

Using Minkowski's inequality for I_1 and I_2, we hawe

$$\begin{aligned}
I_1 &= \int_0^1 \left[t^{\alpha} f^r(b) + t^{\alpha-1}(1-t) f^r(a)\right]^{\frac{2}{r}} dt \\
&\leq \left\{\left(\int_0^1 t^{\frac{2}{r}\alpha} f^2(b) dt\right)^{\frac{r}{2}} + \left(\int_0^1 t^{\frac{2(\alpha-1)}{r}} (1-t)^{\frac{2}{r}} f^2(a) dt\right)^{\frac{r}{2}}\right\}^{\frac{2}{r}} \\
&= \left\{f^r(b) \frac{r}{2\alpha + r} + f^r(a) B\left(\frac{2(\alpha-1)}{r} + 1, \frac{2}{r} + 1\right)\right\}^{\frac{2}{r}}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_0^1 \left[t^{\alpha+1} g^s(b) + t^{\alpha}(1-t) g^s(a)\right]^{\frac{2}{s}} dt \\
&\leq \left\{g^s(b) \frac{s}{2\alpha + s} + g^s(a) B\left(\frac{2(\alpha-1)}{s} + 1, \frac{2}{s} + 1\right)\right\}^{\frac{2}{s}}.
\end{aligned}$$

Combining I_1 and I_2 inequalities lead to (3.2). Theorem 2 is proved. □

Corollary 1. *Under conditions of Theorem 2, $r = s = 2$, we have*

$$\frac{\Gamma(\alpha)}{[\eta(b,a)]^{\alpha}} \left(J^{\alpha}_{(a+\eta(b,a))^-} fg\right)(a) \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{2(\alpha+1)}.$$

Corollary 2. *Under conditions of Theorem 2, $\eta(b,a) = b - a$ and $r = s = 2$, we have*

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J^{\alpha}_{b+} fg(a) \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{2(\alpha+1)}.$$

Corollary 3. *Under conditions of Theorem 2, $\alpha = 1$ and $r = s = 2$, we have*

$$\frac{1}{[\eta(b,a)]} \int_a^{a+\eta(b,a)} f(u) g(u) du \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{4}$$

in [18].

Corollary 4. *Under conditions of Theorem 2, $\alpha = 1$, $\eta(b,a) = b-a$ and $r = s = 2$, we have*

$$\frac{1}{[b-a]} \int_a^b f(u) g(u) du \leq \frac{f^2(a) + f^2(b) + g^2(a) + g^2(b)}{4}.$$

Corollary 5. *Under conditions of Theorem 2 with $\alpha = 1$, $r = s = 2$, and $f(x) = g(x)$, we have*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f^2(u) du \leq \frac{f^2(a) + f^2(b)}{2}$$

in [18].

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Theorem 3. *Let $f,g:K=[a,a+\eta(b,a)]\to(0,\infty)$ be $r-$preinvex and $s-$preinvex functions respectively on the interval of real numbers K°, $a,b\in K^{\circ}$ with $a<a+\eta(b,a)$. If $r>1$ and $\frac{1}{r}+\frac{1}{s}=1$, then*

$$\begin{aligned}&\left(J^{\alpha}_{(a+\eta(b,a))^{-}}fg\right)(a)\\ &\le\frac{[\eta(b,a)]^{\alpha}}{\Gamma(\alpha)}\left\{\left(f^{r}(b)\frac{1}{\alpha+1}+f^{r}(a)B(\alpha,2)\right)^{\frac{1}{r}}\right.\\ &\left.+\left(g^{s}(b)\frac{1}{\alpha+1}+g^{s}(a)B(\alpha,2)\right)^{\frac{1}{s}}\right\}.\end{aligned}$$

Proof. Since f is an $r-$preinvex function and g is a $s-$preinvex function for $t\in[0,1]$, we have

$$f(a+t\eta(b,a)\le[tf^{r}(b)+(1-t)f^{r}(a)]^{\frac{1}{r}}\tag{3.6}$$

and

$$g(a+t\eta(b,a)\le[tg^{s}(b)+(1-t)g^{s}(a)]^{\frac{1}{s}}.\tag{3.7}$$

From (3.6) and (3.7), we get

$$\begin{aligned}&\frac{1}{[\eta(b,a)]^{\alpha}}\int_{a}^{a+\eta(b,a)}(u-a)^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}f(u)\,g(u)du\\ &=\int_{0}^{1}t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}f(a+t\eta(b,a))g\left(a+t\eta(b,a)\right)dt\\ &\le\int_{0}^{1}t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{s}\right)}\left[tf^{r}(b)+(1-t)f^{r}(a)\right]^{\frac{1}{r}}\left[tg^{s}(b)+(1-t)g^{s}(a)\right]^{\frac{1}{s}}dt.\end{aligned}$$

By virtue of Hölder's inequality, we have

$$\begin{aligned}&\int_{0}^{1}\left[t^{\alpha}f^{r}(b)+t^{\alpha-1}(1-t)f^{r}(a)\right]^{\frac{1}{r}}\left[t^{\alpha}g^{s}(b)+t^{\alpha-1}(1-t)g^{s}(a)\right]^{\frac{1}{s}}dt\\ &\le\left\{\int_{0}^{1}\left[t^{\alpha}f^{r}(b)+t^{\alpha-1}(1-t)f^{r}(a)\right]dt\right\}^{\frac{1}{r}}\\ &+\left\{\int_{0}^{1}\left[t^{\alpha}g^{s}(b)+t^{\alpha-1}(1-t)g^{s}(a)\right]dt\right\}^{\frac{1}{s}}\\ &=\frac{[\eta(b,a)]^{\alpha}}{\Gamma(\alpha)}\left\{\left(f^{r}(b)\frac{1}{\alpha+1}+f^{r}(a)B(\alpha,2)\right)^{\frac{1}{r}}\right.\\ &\left.+\left(g^{s}(b)\frac{1}{\alpha+1}+g^{s}(a)B(\alpha,2)\right)^{\frac{1}{s}}\right\}.\end{aligned}$$

The proof is done. □

Corollary 6. *Under conditions of Theorem 2, if $r=s=2$, we have*

$$\begin{aligned}&\left(J^{\alpha}_{(a+\eta(b,a))^{-}}fg\right)(a)\\ &\le\frac{[\eta(b,a)]^{\alpha}}{\Gamma(\alpha)}\left\{\sqrt{\left(f^{2}(b)\frac{1}{\alpha+1}+f^{2}(a)B(\alpha,2)\right)}\right.\\ &\qquad\left.+\sqrt{g^{2}(b)\frac{1}{\alpha+1}+g^{2}(a)B(\alpha,2)}\right\}.\end{aligned}$$

Corollary 7. *Under conditions of Theorem 2, if $r = s = 2$, $\eta(b,a) = b - a$, we have*

$$\frac{(b-a)^{\alpha}}{\Gamma(\alpha)} J_{b^+}^{\alpha} fg(a) \leq \left\{ \sqrt{\left(f^2(b) \frac{1}{\alpha+1} + f^2(a) B(\alpha,2) \right)} + \sqrt{g^2(b) \frac{1}{\alpha+1} + g^2(a) B(\alpha,2)} \right\}.$$

Corollary 8. *Under conditions of Theorem 2, if $r = s = 2$, $\eta(b,a) = b - a$ and $\alpha = 1$, we have*

$$\frac{1}{b-a} \int_a^b f(u) g(u) du \leq \sqrt{\frac{f^2(a) + f^2(b)}{2}} \sqrt{\frac{g^2(a) + g^2(b)}{2}}.$$

Corollary 9. *Under conditions of Theorem 2, if $r = s = 2$ and $\alpha = 1$, we have*

$$\frac{1}{[\eta(b,a)]} \int_a^{a+\eta(b,a)} f(u) g(u) du \leq \sqrt{\frac{f^2(a) + f^2(b)}{2}} \sqrt{\frac{g^2(a) + g^2(b)}{2}}.$$

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