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The Analytical Solution of Parabolic Volterra Integro-Differential Equations in the Infinite Domain

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Abstract: This article focuses on obtaining the analytical solutions for parabolic Volterra integro-differential equations in *d*-dimensional with different types frictional memory kernel. Based on theories of Laplace transform, Fourier transform, the properties of Fox-H function and convolution theorem, analytical solutions of the equations in the infinite domain are derived under three frictional memory kernel functions respectively. The analytical solutions are expressed by infinite series, the generalized multi-parameter Mittag-Leffler function, Fox-H function and convolution form of Fourier transform. In addition, the graphical representations of the analytical solution under different parameters are given for one-dimensional parabolic Volterra integro-differential equation with power-law memory kernel. It can be seen that the solution curves subject to Gaussian decay at any given moment.

Keywords: parabolic Volterra integro-differential equations; memory kernel; Laplace transform; Fourier transform; convolution theorem; analytical solution

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1. Introduction

In this paper, we will consider the following d-dimensional parabolic Volterra integro-differential equation with memory kernel $\mathrm{K}(t)$ [1]

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t}+\int_0^t \mathrm{K}(t-t')u(\boldsymbol{x},t')\mathrm{d}t'=\Delta u(\boldsymbol{x},t)+f(\boldsymbol{x},t), \tag{1}$$

with initial condition and boundary conditions

$$u(\boldsymbol{x},0)=g(\boldsymbol{x})\ ,\ \lim_{|\boldsymbol{x}|\to\infty}u(\boldsymbol{x},t)=0,\quad t>0\,,\ \boldsymbol{x}=(x_1,x_2,\cdots,x_d)\in R^d. \tag{2}$$

Where function $f(\boldsymbol{x},t)$ and memory kernel function $\mathrm{K}(t)$ are assumed as sufficiently smooth functions, and Δ is the d-dimensional Laplacian operator.

Parabolic Volterra integro-differential equations have many important physical applications to model dynamical systems, such as in compression of viscoelastic media[2], nuclear reactor dynamics [3], blow-up problems [4], reaction diffusion problems [5]and heat conduction materials with memory functional [6].etc. At present, analysis of numerical solution of Volterra integral-differential equations is taken into account by many authors. Dehghan et al. [6] studied numerical solution of parabolic integro-differential equations by variational iteration method. Han et al. [1] proposed the artificial boundary method to solve parabolic Volterra integro differential equations (one-dimensional) in the infinite spatial domains. Fakhar-Izadi et al. [7] considered the parabolic Volterra integro-differential equation in one dimensional finite and infinite spatial domains by spectral collocation methods. However, to the authors' knowledge, there are no studies

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on the analytical solutions of parabolic partial Volterra integro- differential equation in the infinite domain. In this article, our goal is mainly to discuss analytical solutions of Eq.(1) with three different kinds memory kernel function in the infinite domain.

This paper is organized as follows. In Sec.2, some definitions and lemmas are introduced. In Sec.3, the analytical solutions of parabolic Volterra integro-differential equation with three different kinds of memory kernel are demonstrated in the infinite domain. In Sec.4, a typical example and some graphical representations of the solution are presented. Some conclusions are given in Sec.5.

2. Preliminaries

In this section, we give some definitions and lemmas that are used throughout this paper.

Definition 1 Four-parameter Mittag-Leffler (M-L) function is defined as [8]

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n\kappa}}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!}, \tag{3}$$

where z, $\beta, \gamma, \kappa \in$ □ $, \Re(\alpha) > \max\left(\{0, \Re(\kappa)-1\}, \Re(\kappa) > 0\right)$, with Pochammer's symbol $(\gamma)_{n\kappa}$ can be expressed as

$$(\gamma)_{n\kappa} := \Gamma(\gamma+n\kappa)/\Gamma(\gamma) = \begin{cases} 1 & , if \quad n\kappa = 0 \\ \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n\kappa-1), & if \quad n\kappa \in N, \gamma \neq 0 \end{cases}.$$

It is worth noting that when $\kappa = 1$, the three parameter Mittag-Leffler function can be obtained as $E_{\alpha,\beta}^{\gamma,1}(\cdot) = E_{\alpha,\beta}^{\gamma}(\cdot)$ [8-9]

$$E_{\alpha,\beta}^{\gamma}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!}, \tag{4}$$

where $z \in$ □ $, \Re(\alpha) > 0.\ \beta, \gamma > 0$.

Note that, when $\gamma = 1$, we can obtain two parameter Mittag-Leffler function $E_{\alpha,\beta}^{1}(\cdot) = E_{\alpha,\beta}(\cdot)$ [10], there

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{5}$$

where $z, \beta \in$ □ , $\Re(\alpha) > 0$. Note that $E_{\alpha,1}(\cdot)$ reduces to Mittag-Leffler function $E_{\alpha}(\cdot)$ when $\beta = 1$, then

$$E_{\alpha,1}(z) = E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)},$$

where $z \in$ □ $, \Re(\alpha) > 0$. In particular, we can obtain regular exponential function when $\alpha = 1$.

Definition 2 An integral operator $\mathrm{E}_{a+;\alpha,\beta}^{w;\gamma,\kappa}\varphi$ is defined as [11]

$$\left(\mathrm{E}_{a+;\alpha,\beta}^{w;\gamma,\kappa}\varphi\right)(t) := \left(t^{\beta-1}E_{\alpha,\beta}^{\gamma,\kappa}(wt^{\alpha})\right) * \varphi(t) = \int_a^t (t-\tau)^{\beta-1} E_{\alpha,\beta}^{\gamma,\kappa}(w(t-\tau)^{\alpha})\varphi(\tau)\mathrm{d}\tau. \tag{6}$$

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It is worth noting that, when $w=0$ and $a=0$, integral operator $\mathrm{E}_{a+;\alpha,\beta}^{w;\gamma,\kappa}\varphi$ would correspond to the Riemann-Liouville integral operator[8].

In this subsection, we will introduce some lemmas about Laplace transform, which will be help us handling some problems in the next section.

Lemma 1. Let $s,b,\alpha,\lambda_n\in\mathbb{R}^+$, then the following inverse Laplace transform (L^{-1}) is true [12].

$$\mathrm{L}^{-1}\left\{\frac{1}{s^2+bs^\alpha+\lambda_n}\mathrm{L}\left[g(t)\right](s)\right\}(t)=\sum_{n=0}^{\infty}(-b)^n\left(\mathrm{E}_{0+;2,(2-\alpha)(n+1)+1}^{-\lambda_n;n+1,1}g\right)(t), \tag{7}$$

where

$$0<\frac{\lambda_n}{s^2+bs^\alpha}<1 \quad , \quad 0<\frac{b}{s^{2-\alpha}}<1.$$

Lemma 2. The Laplace transform of three parameter Mittag-Leffler function is given by [13]

$$\mathrm{L}\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\pm wt^\alpha)\right](s)=\int_0^\infty e^{-st}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\pm wt^\alpha)\mathrm{d}t=\frac{s^{\alpha\gamma-\beta}}{(s^\alpha\mp w)^\gamma}, \tag{8}$$

there $\left|w/s^\alpha\right|<1$.

Lemma 3 .The Laplace transform of $e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(wt^\alpha)$ is given by the follows [13]

$$\mathrm{L}\left[e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\pm wt^\alpha)\right](s)=\frac{(s+\lambda)^{\alpha\gamma-\beta}}{\left((s+\lambda)^\alpha\mp w\right)^\gamma}, \quad (\lambda\geq 0) \tag{9}$$

where $\left|w/(s+\lambda)^\alpha\right|<1$.

In fact, by Laplace's transform definition, one easily gets

$$\mathrm{L}\left[e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\pm wt^\alpha)\right](s)=\int_0^\infty e^{-st}\left(e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\pm wt^\alpha)\right)\mathrm{d}t$$

$$=\int_0^\infty e^{-(s+\lambda)t}t^{\beta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\cdot\frac{(\pm wt^\alpha)^n}{n!}\mathrm{d}t$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n(\pm w)^n}{\Gamma(\alpha n+\beta)n!}\cdot\frac{\Gamma(\alpha n+\beta)}{(s+\lambda)^{\alpha n+\beta}}=\sum_{n=0}^{\infty}\frac{(\gamma)_n(\pm w)^n}{n!(s+\lambda)^{\alpha n+\beta}}.$$

Through simplicity, the above formula finally reduces to

$$\mathrm{L}\left[e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\pm wt^\alpha)\right](s)=\frac{(s+\lambda)^{\alpha\gamma-\beta}}{\left((s+\lambda)^\alpha\mp w\right)^\gamma}.$$

Therefore, above Lemma 3. can be proved.

It is worth noting that in case $\lambda=0$, the structure of Lemma 3 is equivalent to Lemma 2.

Lemma 4 gives one important d-dimensional integral formula about Mittag-Leffler function.

Lemma 4. For arbitrary $\alpha>0$, β is an arbitrary complex number, in addition $\mu>0$, and $a\in R$, then the following formula is established [14]

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$$\int_{R^d} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} E_{\alpha,\beta}^{(n)}(-a|\boldsymbol{k}|^{\mu})\mathrm{d}^d\boldsymbol{k} = (2\pi)^{d/2}|\boldsymbol{x}|^{1-d/2} \times \int_0^{\infty} |\boldsymbol{k}|^{d/2} E_{\alpha,\beta}^{(n)}(-a|\boldsymbol{k}|^{\mu}) J_{d/2-1}(|\boldsymbol{x}||\boldsymbol{k}|)\mathrm{d}^d|\boldsymbol{k}|, \quad (10)$$

where $J_{d/2-1}(\cdot)$ is Bessel function, and

$$(|\boldsymbol{x}||\boldsymbol{k}|)^{1-d/2} J_{d/2-1}(|\boldsymbol{x}||\boldsymbol{k}|) = 2^{1-d/2} H_{0,2}^{1,0}\left[\left(\frac{|\boldsymbol{x}||\boldsymbol{k}|}{2}\right)^2 \middle|_{(0,1)\,(1-d/2,1)}\right].\ \boldsymbol{k} = (k_1, k_2, \cdots, k_d) \in R^d$$

3. Analytical solution of parabolic Volterra integro-differential equation in the infinite domain

3.1. Analytical solution with frictional memory kernel of M-L type $\mathrm{K}(t) = \frac{1}{\tau^{\alpha\delta}} t^{\beta-1} E_{\alpha,\beta}^{\delta}(-\frac{t^{\alpha}}{\tau^{\alpha}})$.

In this case, Eq.(1) can be written as following

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t} + \frac{1}{\tau^{\alpha\delta}} \int_0^t (t-t')^{\beta-1} E_{\alpha,\beta}^{\delta}(-\frac{(t-t')^{\alpha}}{\tau^{\alpha}}) u(\boldsymbol{x},t')\mathrm{d}t = \Delta u(\boldsymbol{x},t) + f(\boldsymbol{x},t).\quad \boldsymbol{x} \in R^d,\quad t>0 \quad (11)$$

where $\alpha, \beta, \delta > 0$, τ is the memory time.

Theorem3.1. The analytical solution of parabolic Volterra integro-differential Eq.(11) with boundary conditions and initial condition (2) can be expressed as the following form

$$u(\boldsymbol{x},t) = \frac{1}{(2\pi)^d} \int_{R^d} \sum_{i=0}^{\infty} (-\frac{1}{\tau^{\alpha\delta}})^i \sum_{n=0}^{i} \binom{i}{n} (\frac{|\boldsymbol{k}|^2}{1/\tau^{\alpha\delta}})^n \left(\mathrm{E}_{0+;\alpha,(\beta+1)(i-n)+n+1}^{-\tau^{-\alpha};\delta(i-n),1} \tilde{f}\right)(\boldsymbol{k},t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}^d\boldsymbol{k}$$

$$+\int_{R^d} G(\boldsymbol{x}-\boldsymbol{x}',t) g(\boldsymbol{x}')\mathrm{d}^d\boldsymbol{x}'. \qquad \boldsymbol{x},\, \boldsymbol{k} \in R^d \quad (12)$$

In which $G(x,t)$ is the Green function, reads as

$$G(\boldsymbol{x},t) = \frac{1}{(2\pi)^d} \int_{R^d} \left(\sum_{i=0}^{\infty} (-\frac{1}{\tau^{\alpha\delta}})^i \sum_{n=0}^{i} \binom{i}{n} (\frac{|\boldsymbol{k}|^2}{1/\tau^{\alpha\delta}})^n t^{(\beta+1)(i-n)+n} E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha}) \right) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}^d\boldsymbol{k}.$$

In general, denotes $\tilde{f}(\boldsymbol{k},t) = \int_{R^d} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x},t)\mathrm{d}^d\boldsymbol{k}$ is the Fourier transform of $f(\boldsymbol{x},t)$ with respect to the spatial variable $\boldsymbol{x}$.

Proof. Denotes $\mathrm{F}\{u(\boldsymbol{x},t)\} := \tilde{u}(\boldsymbol{k},t)$ as the Fourier transform of $u(\boldsymbol{x},t)$ with respect to variable $\boldsymbol{x}$, $\mathrm{L}\{u(\boldsymbol{x},t)\} := \hat{u}(\boldsymbol{x},s)$ as the Laplace transform of $u(\boldsymbol{x},t)$ with respect to variable t. Taking the Laplace transform with respect to the time variable t and the Fourier transform with respect to the spatial variable $\boldsymbol{x}$ to Eq.(11), it can be written as the following result

$$s\hat{\tilde{u}}(\boldsymbol{k},s) - \tilde{u}(\boldsymbol{k},0) + \frac{1}{\tau^{\alpha\delta}} \frac{s^{\alpha\delta-\beta}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}} \hat{\tilde{u}}(\boldsymbol{k},s) = -|\boldsymbol{k}|^2 \hat{\tilde{u}}(\boldsymbol{k},s) + \hat{\tilde{f}}(\boldsymbol{k},s).\quad \boldsymbol{k} \in R^d \quad (13)$$

Using the initial condition, it yields

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$$\hat{\tilde{u}}(\boldsymbol{k},s)=\frac{1}{s+\frac{1}{\tau^{\alpha\delta}}\frac{s^{\alpha\delta-\beta}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}}+|\boldsymbol{k}|^{2}}\hat{\tilde{f}}(\boldsymbol{k},s)+\frac{1}{s+\frac{1}{\tau^{\alpha\delta}}\frac{s^{\alpha\delta-\beta}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}}+|\boldsymbol{k}|^{2}}\tilde{g}(\boldsymbol{k}). \tag{14}$$

Because

$$\frac{1}{s+\frac{1}{\tau^{\alpha\delta}}\frac{s^{\alpha\delta-\beta}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}}+|\boldsymbol{k}|^{2}}=s^{-1}\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(|\boldsymbol{k}|^{2}s^{-1})^{n}(\frac{s^{\alpha\delta-\beta-1}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}})^{i-n}(\frac{1}{\tau^{\alpha\delta}})^{-n}$$

$$=\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}\frac{s^{\alpha\delta(i-n)-(\beta+1)(i-n)-n-1}}{(s^{\alpha}+\tau^{-\alpha})^{\delta(i-n)}}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}, \tag{15}$$

Employing Lemma 2, we can get

$$\frac{s^{\alpha\delta(i-n)-(\beta+1)(i-n)-n-1}}{(s^{\alpha}+\tau^{-\alpha})^{\delta(i-n)}}=\mathrm{L}\left[t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\right](s). \tag{16}$$

Substituting Eq.(16) and Eq.(15) into the first term of Eq. (14), we have

$$\frac{1}{s+\frac{1}{\tau^{\alpha\delta}}\frac{s^{\alpha\delta-\beta}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}}+|\boldsymbol{k}|^{2}}\hat{\tilde{f}}(\boldsymbol{k},s)=$$

$$\mathrm{L}\left[\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\right](s)\mathrm{L}\left[\tilde{f}(\boldsymbol{k},t)\right](s). \tag{17}$$

By applying inverse Laplace transform, the convolution definition of Laplace transform and definition 2, we can obtain the inverse Laplace transform of the first term in Eq.(14) as follows

$$\mathrm{L}^{-1}\left(\frac{1}{s+\frac{1}{\tau^{\alpha\delta}}\frac{s^{\alpha\delta-\beta}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}}+|\boldsymbol{k}|^{2}}\mathrm{L}\left(\tilde{f}(\boldsymbol{k},t)\right)(s)\right)(t)=$$

$$\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}\left(\mathrm{E}_{0+;\alpha,(\beta+1)(i-n)+n+1}^{-\tau^{-\alpha};\delta(i-n),1}\tilde{f}\right)(\boldsymbol{k},t). \tag{18}$$

It follows that the inverse Laplace transform of the second term in Eq.(14) is

$$\mathrm{L}^{-1}\left(\frac{1}{s+\frac{1}{\tau^{\alpha\delta}}\frac{s^{\alpha\delta-\beta}}{(s^{\alpha}+\tau^{-\alpha})^{\delta}}+|\boldsymbol{k}|^{2}}\tilde{g}(\boldsymbol{k})\right)(t)=$$

$$\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\tilde{g}(\boldsymbol{k}). \tag{19}$$

According to Eq.(18)-(19), we can obtain $\tilde{u}(\boldsymbol{k},t)$ from Eq.(14)

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$$\tilde{u}(\boldsymbol{k},t)=\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}\left(\mathrm{E}_{0+;\alpha,(\beta+1)(i-n)+n+1}^{-\tau^{-\alpha};\delta(i-n),1}\tilde{f}\right)(\boldsymbol{k},t)+$$

$$\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\tilde{g}(\boldsymbol{k}). \tag{20}$$

Eq.(20) can be further manipulated by employing inverse Fourier transform

$$u(\boldsymbol{x},t)=\frac{1}{(2\pi)^{d}}\int_{R^{d}}\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}\left(\mathrm{E}_{0+;\alpha,(\beta+1)(i-n)+n+1}^{-\tau^{-\alpha};\delta(i-n),1}\tilde{f}\right)(\boldsymbol{k},t)\mathrm{d}^{d}\boldsymbol{k}+$$

$$\frac{1}{(2\pi)^{d}}\int_{R^{d}}\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\tilde{g}(\boldsymbol{k})\mathrm{d}^{d}\boldsymbol{k}. \tag{21}$$

The second term in Eq.(21) can be further manipulated as follows

$$\frac{1}{(2\pi)^{d}}\int_{R^{d}}\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\tilde{g}(\boldsymbol{k})\mathrm{d}^{d}\boldsymbol{k}$$

$$=\frac{1}{(2\pi)^{d}}\int_{R^{d}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\left(\int_{R^{d}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}'}g(\boldsymbol{x}')\mathrm{d}^{d}\boldsymbol{x}'\right)\mathrm{d}^{d}\boldsymbol{k}$$

$$=\int_{R^{d}}\left[\frac{1}{(2\pi)^{d}}\int_{R^{d}}e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\mathrm{d}^{d}\boldsymbol{k}\right]\times$$

$$g(\boldsymbol{x}')\mathrm{d}^{d}\boldsymbol{x}'. \tag{22}$$

Denote Green function $G(\boldsymbol{x},t)$ is

$$G(\boldsymbol{x}-\boldsymbol{x}',t)=\frac{1}{(2\pi)^{d}}\int_{R^{d}}e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\sum_{i=0}^{\infty}(-\frac{1}{\tau^{\alpha\delta}})^{i}\sum_{n=0}^{i}\binom{i}{n}(\frac{|\boldsymbol{k}|^{2}}{1/\tau^{\alpha\delta}})^{n}t^{(\beta+1)(i-n)+n}E_{\alpha,(\beta+1)(i-n)+n+1}^{\delta(i-n)}(-\tau^{-\alpha}t^{\alpha})\mathrm{d}^{d}\boldsymbol{k}.$$

Therefore, we complete the proof of Theorem 3.1.

3.2 Analytical solution with frictional memory kernel of power-law type $\mathrm{K}(t)=\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)},0<\alpha<1.$

In this case, Eq.(1) can be written as the following form

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t}+\int_{0}^{t}\frac{(t-t')^{-\alpha}}{\Gamma(1-\alpha)}u(\boldsymbol{x},t')\mathrm{d}t'=\Delta u(\boldsymbol{x},t)+f(\boldsymbol{x},t),\quad \boldsymbol{x}\in R^{d},\ t>0. \tag{23}$$

Theorem3.2. The analytical solution of parabolic Volterra integro-differential Eq.(23) with boundary conditions and initial condition (2) can be expressed as

$$u(\boldsymbol{x},t)=\frac{1}{(2\pi)^{d}}\int_{R^{d}}\sum_{j=0}^{\infty}(-1)^{j}\left(\mathrm{E}_{0+;1,(2-\alpha)j+1}^{-|\boldsymbol{k}|^{2};j+1,1}\tilde{f}\right)(\boldsymbol{k},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}^{d}\boldsymbol{k}$$

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$$+\int_{R^d} G(\boldsymbol{x}-\boldsymbol{x}',t)g(\boldsymbol{x}')\mathrm{d}^d\boldsymbol{x}'. \tag{24}$$

The Green function $G(\boldsymbol{x},t)$ is given as

$$G(\boldsymbol{x},t)=\frac{1}{2\pi^{d/2}|\boldsymbol{x}|^d}\sum_{j=0}^{\infty}\frac{(-t^{2-\alpha})^j}{j!}H_{1,2}^{2,0}\left[\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\middle|\begin{matrix}(1+(2-\alpha)j,1/2)\\(d/2,1/2), \quad (1+j,1/2)\end{matrix}\right]. \quad \boldsymbol{x},\boldsymbol{k}\in R^d$$

There F $\{g(\boldsymbol{x})\}:=\tilde{g}(\boldsymbol{k})$ and F $\{f(\boldsymbol{x},t)\}:=\tilde{f}(\boldsymbol{k},t)$ are the Fourier transform of $g(\boldsymbol{x})$ and $f(\boldsymbol{x},t)$, respectively.

Remark. Employing the properties of the Fox-H functions, and Green function $G(\boldsymbol{x},t)$ can be expressed as power series expansion [15]

$$G(x,t)=\frac{1}{\pi^{d/2}|\boldsymbol{x}|^d}\sum_{j=0}^{\infty}\frac{(-t^{2-\alpha})^j}{j!}\left[\sum_{k=0}^{\infty}\frac{\Gamma(1+j-k-d/2)}{\Gamma(1+(2-\alpha)j-k-d/2)}\frac{(-1)^k}{k!}\left(\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\right)^{d+2k}+\right.$$

$$\left.\sum_{k=0}^{\infty}\frac{\Gamma(d/2-j-k-1)}{\Gamma((1-\alpha)j-k)}\frac{(-1)^k}{k!}\left(\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\right)^{2(k+j+1)}\right].$$

For $\left|\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\right| \gg 1$, therefore, the power-law asymptotics behavior is given by

$$G(\boldsymbol{x},t)\approx\frac{1}{\pi^{d/2}|\boldsymbol{x}|^d}\sum_{j=0}^{\infty}\frac{(-t^{2-\alpha})^j}{j!}\left[\frac{\Gamma(1+j-d/2)}{\Gamma(1+(2-\alpha)j-d/2)}\left(\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\right)^d+\left[\frac{\Gamma(d/2-j-1)}{\Gamma((1-\alpha)j)}\left(\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\right)^{2(j+1)}\right]\right].$$

Proof. Employing the Laplace transform with respect to variable t and Fourier transform with respect to variable $\boldsymbol{x}$, respectively. One obtains

$$s\hat{\tilde{u}}(\boldsymbol{k},s)-\tilde{u}(\boldsymbol{k},0)+s^{\alpha-1}\hat{\tilde{u}}(\boldsymbol{k},s)=-|\boldsymbol{k}|^2\hat{\tilde{u}}(\boldsymbol{k},s)+\hat{\tilde{f}}(\boldsymbol{k},s). \quad \boldsymbol{k}\in R^d \tag{25}$$

Taking into account the initial condition, Eq.(25) can be rewritten down as

$$\hat{\tilde{u}}(\boldsymbol{k},s)=\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^2}\hat{\tilde{f}}(\boldsymbol{k},s)+\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^2}\tilde{g}(\boldsymbol{k}). \tag{26}$$

Using the technique introduced by [12], we have

$$\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^2}=\frac{1}{|\boldsymbol{k}|^2}\cdot\frac{|\boldsymbol{k}|^2 s^{1-\alpha}}{s^{2-\alpha}+1}\cdot\frac{1}{1+\frac{|\boldsymbol{k}|^2 s^{1-\alpha}}{s^{2-\alpha}+1}}. \tag{27}$$

Expanding the third section of the right of the Eq.(27) and simplifying, one easy gets

$$\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^2}=\sum_{n=0}^{\infty}(-|\boldsymbol{k}|^2)^n\frac{s^{(1-\alpha)(n+1)}}{(s^{2-\alpha}+1)^{n+1}}. \tag{28}$$

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Employing Lemma 2 on Eq.(28), then the first term of the Eq. (26) can be expressed as

$$\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^2}\hat{\tilde{f}}(\boldsymbol{k},s)=\mathrm{L}\left[\sum_{n=0}^{\infty}(-|\boldsymbol{k}|^2)^n t^n E_{2-\alpha,n+1}^{n+1}(-t^{2-\alpha})\right](s)\mathrm{L}\left[\tilde{f}(\boldsymbol{k},t)\right](s). \tag{29}$$

According to Eq. (4), we have

$$\sum_{n=0}^{\infty}(-|\boldsymbol{k}|^2)^n t^n E_{2-\alpha,n+1}^{n+1}(-t^{2-\alpha})=\sum_{n=0}^{\infty}(-|\boldsymbol{k}|^2)^n t^n\sum_{j=0}^{\infty}\frac{(n+1)_j}{\Gamma((2-\alpha)j+n+1)}\frac{(-t^{2-\alpha})^j}{j!},$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-t^{2-\alpha})^j\frac{(j+1)_n}{\Gamma((2-\alpha)j+n+1)}\frac{(-|\boldsymbol{k}|^2 t)^n}{n!},$$

$$=\sum_{j=0}^{\infty}(-t^{2-\alpha})^j E_{1,(2-\alpha)j+1}^{j+1}(-|\boldsymbol{k}|^2 t). \tag{30}$$

Applying convolution property of the Laplace transform and integral operator $\mathrm{E}_{a+;\alpha,\beta}^{w;\gamma,\kappa}\varphi$ definition, then the inverse Laplace transform of the first term in Eq.(26) can be obtained as follows

$$\mathrm{L}^{-1}\left[\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^2}\mathrm{L}\left[\tilde{f}(\boldsymbol{k},t)\right](s)\right]=\sum_{j=0}^{\infty}(-1)^j\left(\mathrm{E}_{0+;1,(2-\alpha)j+1}^{-|\boldsymbol{k}|^2;j+1,1}\tilde{f}\right)(\boldsymbol{k},t). \tag{31}$$

Noting that the relation between generalized Mittag-Leffler function and Fox-H function, the inverse Laplace transform of the second term in Eq.(26) can be expressed as[15]

$$\mathrm{L}^{-1}\left[\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^2}\tilde{g}(\boldsymbol{k})\right]=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}t^{(2-\alpha)j}H_{1,2}^{1,1}\left[|\boldsymbol{k}|^2 t\left|\begin{matrix}(-j,1)\\(0,1),\quad(-(2-\alpha)j,1)\end{matrix}\right.\right]\tilde{g}(\boldsymbol{k}). \tag{32}$$

Denote

$$\tilde{h}(\boldsymbol{k},t)=H_{1,2}^{1,1}\left[|\boldsymbol{k}|^2 t\left|\begin{matrix}(-j,1)\\(0,1),\quad(-(2-\alpha)j,1)\end{matrix}\right.\right], \tag{33}$$

and

$$\tilde{G}(\boldsymbol{k},t)=\sum_{j=0}^{\infty}\frac{(-t^{2-\alpha})^j}{j!}\tilde{h}(\boldsymbol{k},t). \tag{34}$$

Using the inverse Fourier transform to Eq.(33), we can obtain

$$h(\boldsymbol{x},t)=\frac{1}{(2\pi)^d}\int_{R^d}H_{1,2}^{1,1}\left[|\boldsymbol{k}|^2 t\left|\begin{matrix}(-j,1)\\(0,1),\quad(-(2-\alpha)j,1)\end{matrix}\right.\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}^d\boldsymbol{k}. \tag{35}$$

Using Lemma 4, we get the following result from Eq.(35)

$$h(\boldsymbol{x},t)=\frac{1}{(2\pi)^{d/2}}|\boldsymbol{x}|^{1-d/2}\int_0^{\infty}|\boldsymbol{k}|^{d/2}H_{1,2}^{1,1}\left[|\boldsymbol{k}|^2 t\left|\begin{matrix}(-j,1)\\(0,1),\quad(-(2-\alpha)j,1)\end{matrix}\right.\right]J_{d/2-1}(|\boldsymbol{x}||\boldsymbol{k}|)\mathrm{d}^d|\boldsymbol{k}|. \tag{36}$$

Employing Hankel transform and properties of the Fox-H functions [15,16,17], Then Eq.(36) can be written as

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$$h(\boldsymbol{x},t)=\frac{1}{2\pi^{d/2}|\boldsymbol{x}|^{d}}H_{1,2}^{2,0}\left[\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\left|\begin{matrix}(1+(2-\alpha)j,1/2) & \\ (d/2,1/2), & (1+j,1/2)\end{matrix}\right.\right]. \tag{37}$$

Substituting Eq.(37) to the inverse Fourier transform of Eq.(34), we can obtain

$$G(\boldsymbol{x},t)=\frac{1}{2\pi^{d/2}|\boldsymbol{x}|^{d}}\sum_{j=0}^{\infty}\frac{(-t^{2-\alpha})^{j}}{j!}H_{1,2}^{2,0}\left[\frac{|\boldsymbol{x}|}{2t^{\frac{1}{2}}}\left|\begin{matrix}(1+(2-\alpha)j,1/2) & \\ (d/2,1/2), & (1+j,1/2)\end{matrix}\right.\right].$$

Then Eq.(32) can also be written formally as

$$\mathrm{L}^{-1}\left[\frac{1}{s+s^{\alpha-1}+|\boldsymbol{k}|^{2}}\tilde{g}(\boldsymbol{k})\right]=\mathrm{L}^{-1}\left[\mathrm{F}\left[G(\boldsymbol{x},s)\right](\boldsymbol{k})\tilde{g}(\boldsymbol{k})\right]. \tag{38}$$

Applying an inverse Laplace transform to the Eq.(26), we can finally find

$$\tilde{u}(\boldsymbol{k},t)=\sum_{j=0}^{\infty}(-1)^{j}\left(\mathrm{E}_{0+;1,(2-\alpha)j+1}^{-|\boldsymbol{k}|^{2};j+1,1}\tilde{f}\right)(\boldsymbol{k},t)+\mathrm{F}\left[G(\boldsymbol{x},t)\right](\boldsymbol{k})\tilde{g}(\boldsymbol{k}). \tag{39}$$

The Eq.(39)can be further manipulated by employing inverse Fourier transform and Fourier convolution theorem, respectively. Accordingly, the Theorem 3.2 is clearly demonstrated.

3.3.Analytical solution with frictional memory kernel of exponential factor type $\mathrm{K}(t)=t^{\beta}e^{-\lambda t},\ \beta>-1.$

In this case, Eq.(1) can be written in the following form

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t}+\int_{0}^{t}(t-t')^{\beta}e^{-\lambda(t-t')}u(\boldsymbol{x},t')\mathrm{d}t'=\Delta u(\boldsymbol{x},t)+f(\boldsymbol{x},t),\quad \lambda>0\,,\boldsymbol{x}\in R^{d}. \tag{40}$$

Theorem3.3. The analytical solution of parabolic Volterra integro-differential Eq.(40) with boundary conditions and initial condition (2) can be expressed as the following analysis formula

$$u(\boldsymbol{x},t)=\frac{1}{(2\pi)^{d}}\int_{R^{d}}\left\{\left(\sum_{n=0}^{\infty}(-1)^{n}e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^{r}(|\boldsymbol{k}|^{2})^{n-r}t^{n}E_{\beta+2,n+1}^{n+1}(-k_{\beta}t^{\beta+2})\right)*\tilde{f}(\boldsymbol{k},t)\right\}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}^{d}\boldsymbol{k}$$
$$+\int_{R^{d}}G(\boldsymbol{x}-\boldsymbol{x}',t)g(\boldsymbol{x}')\mathrm{d}^{d}\boldsymbol{x}',\qquad \boldsymbol{x},\boldsymbol{k}\in R^{d}. \tag{41}$$

Denoting $k_{\beta}=\Gamma(\beta+1)$, using the asterisk($*$)denotes a Laplace convolution, the Green function $G(\boldsymbol{x},t)$ is given by

$$G(\boldsymbol{x},t)=\frac{1}{(2\pi)^{d}}\int_{R^{d}}\left(\sum_{n=0}^{\infty}(-1)^{n}e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^{r}(|\boldsymbol{k}|^{2})^{n-r}t^{n}E_{\beta+2,n+1}^{n+1}(-k_{\beta}t^{\beta+2})\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}^{d}\boldsymbol{k}.$$

Proof. Applying the Laplace and Fourier transform with respect to the time variable t and spatial variable x to Eq.(40), respectively, and using the initial conditions, then Eq.(40) can be written as

$$s\hat{\tilde{u}}(\boldsymbol{k},s)-\tilde{u}(\boldsymbol{k},0)+k_{\beta}(s+\lambda)^{-(1+\beta)}\hat{\tilde{u}}(\boldsymbol{k},s)=-|\boldsymbol{k}|^{2}\hat{\tilde{u}}(\boldsymbol{k},s)+\hat{\tilde{f}}(\boldsymbol{k},s),\quad \boldsymbol{k}\in R^{d}. \tag{42}$$

From Eq.(42),we have

$$\hat{\tilde{u}}(\boldsymbol{k},s)=\frac{1}{s+k_{\beta}(s+\lambda)^{-(1+\beta)}+|\boldsymbol{k}|^{2}}\hat{\tilde{f}}(\boldsymbol{k},s)+\frac{1}{s+k_{\beta}(s+\lambda)^{-(1+\beta)}+|\boldsymbol{k}|^{2}}\tilde{g}(\boldsymbol{k}). \tag{43}$$

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Applying power series expanding, one obtains

$$\frac{1}{s+k_\beta(s+\lambda)^{-(1+\beta)}+|\boldsymbol{k}|^2}=\sum_{n=0}^{\infty}(-1)^n\frac{(|\boldsymbol{k}|^2-\lambda)^n(s+\lambda)^{(\beta+1)n+\beta+1}}{((s+\lambda)^{\beta+2}+k_\beta)^{n+1}}. \tag{44}$$

Combining Lemma 3, Eq.(44) can be expressed as

$$\frac{1}{s+k_\beta(s+\lambda)^{-(1+\beta)}+|\boldsymbol{k}|^2}=\mathrm{L}\left[\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}(|\boldsymbol{k}|^2-\lambda)^n t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right](s). \tag{45}$$

From Eq.(45), the first term in Eq.(43) can be written as formally

$$\frac{1}{s+k_\beta(s+\lambda)^{-(1+\beta)}+|\boldsymbol{k}|^2}\hat{\tilde{f}}(\boldsymbol{k},s)=$$

$$\mathrm{L}\left[\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right](s)\mathrm{L}\left[\tilde{f}(\boldsymbol{k},t)\right](s). \tag{46}$$

Finally, the inverse Laplace transform of the first term in Eq.(43) can be rewritten down as

$$\mathrm{L}^{-1}\left[\frac{1}{s+k_\beta(s+\lambda)^{-(1+\beta)}+|\boldsymbol{k}|^2}\mathrm{L}\left[\tilde{f}(\boldsymbol{k},t)\right](s)\right]=$$

$$\left(\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right)*\tilde{f}(\boldsymbol{k},t). \tag{47}$$

After analogous manipulation, the inverse Laplace transform of the second term of Eq.(43) can be expressed as

$$\mathrm{L}^{-1}\left[\frac{1}{s+k_\beta(s+\lambda)^{-(1+\beta)}+|\boldsymbol{k}|^2}\tilde{g}(\boldsymbol{k})\right]=$$

$$\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\tilde{g}(\boldsymbol{k}). \tag{48}$$

Performing an inverse Laplace transform in Eq.(43), we finally get

$$\tilde{u}(\boldsymbol{k},t)=\left(\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right)*\tilde{f}(\boldsymbol{k},t)+$$

$$\left(\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right)\tilde{g}(\boldsymbol{k}). \tag{49}$$

The solution is now obtained by performing inverse Fourier transform in Eq.(49), easy gets

$$u(\boldsymbol{x},t)=\frac{1}{(2\pi)^d}\int_{R^d}\left(\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right)*\tilde{f}(\boldsymbol{k},t)\mathrm{d}^d\boldsymbol{k}+$$

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$$\int_{R^d}\left[\frac{1}{(2\pi)^d}\int_{R^d} e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\left(\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right)\mathrm{d}^d\boldsymbol{k}\right]\times$$

$$g(\boldsymbol{x}')\mathrm{d}^d\boldsymbol{x}', \tag{50}$$

where the Green function is denoted as

$$G(\boldsymbol{x}-\boldsymbol{x}',t)=\frac{1}{(2\pi)^d}\int_{R^d} e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\left(\sum_{n=0}^{\infty}(-1)^n e^{-\lambda t}\sum_{r=0}^{n}\binom{n}{r}(-\lambda)^r(|\boldsymbol{k}|^2)^{n-r}t^n E_{\beta+2,n+1}^{n+1}(-k_\beta t^{\beta+2})\right)\mathrm{d}^d\boldsymbol{k}.$$

Therefore, we complete the proof of Theorem 3.3.

4. Example

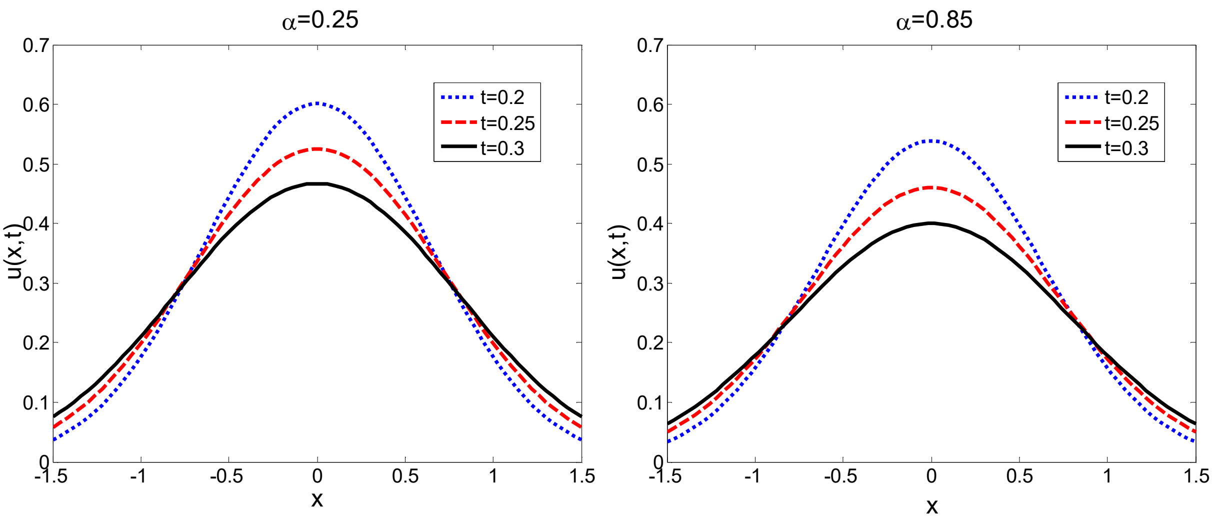
We select one-dimensional parabolic Volterra integro-differential equation with power-law memory kernel under special initial conditions (Dirac delta function) in the infinite domain. Let $f(x,t)=0$, the initial boundary value problem in section 3.2 are

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}+\displaystyle\int_0^t\frac{(t-t')^{-\alpha}}{\Gamma(1-\alpha)}u(x,t')\mathrm{d}t'=\Delta u(x,t)\quad,\quad 0<\alpha<1, x\in R, t>0\\ u(x,0)=\delta(x),\quad \displaystyle\lim_{x\to\infty}u(x,t)=0,\quad t>0\ , x\in R.\end{cases} \tag{51}$$

According to theorem3.2, the analytical solution can be expressed as following

$$u(x,t)=\frac{1}{2\pi^{1/2}|x|}\sum_{j=0}^{\infty}\frac{(-t^{2-\alpha})^j}{j!}H_{1,2}^{2,0}\left[\frac{|x|}{2t^{\frac{1}{2}}}\middle|\begin{matrix}(1+(2-\alpha)j,1/2)\\(1/2,1/2), \quad (1+j,1/2)\end{matrix}\right]. \tag{52}$$

The graphical representation of solution (52) for the different parameters of α and t are plotted in Figure.1, Figure.2 and Figure.3, Figure.4, respectively.



(a) (b)

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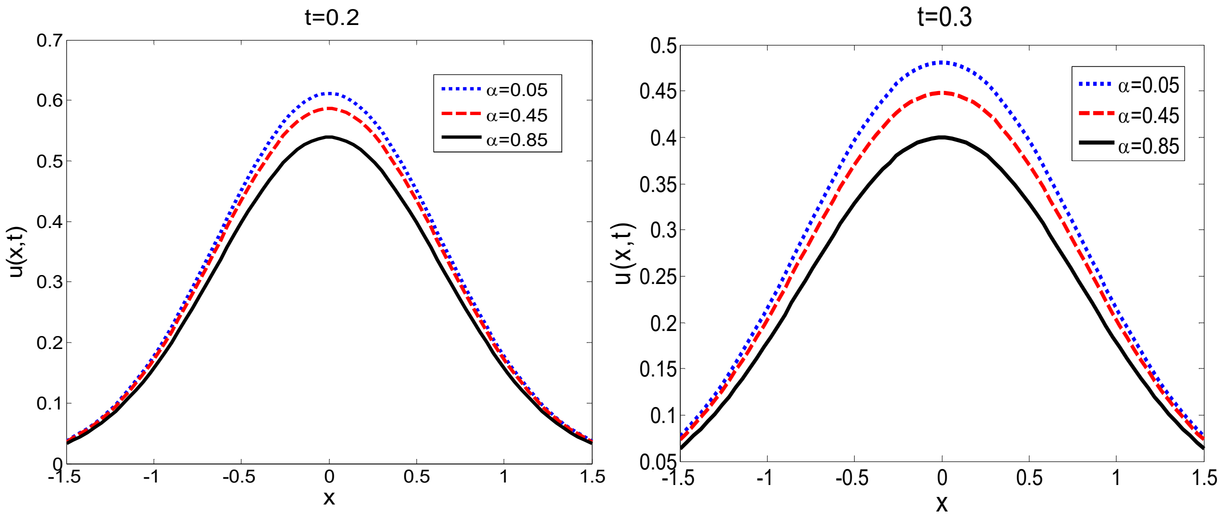
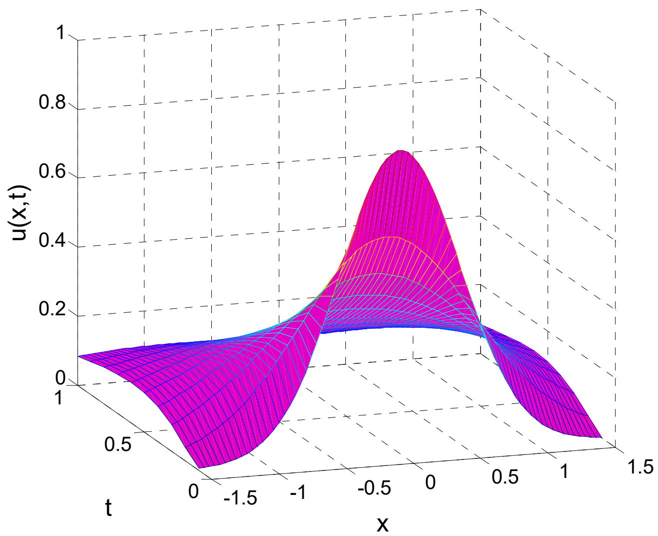
Figure 1. Graphical representation of the solution at different times with $\alpha = 0.25, 0.85$.



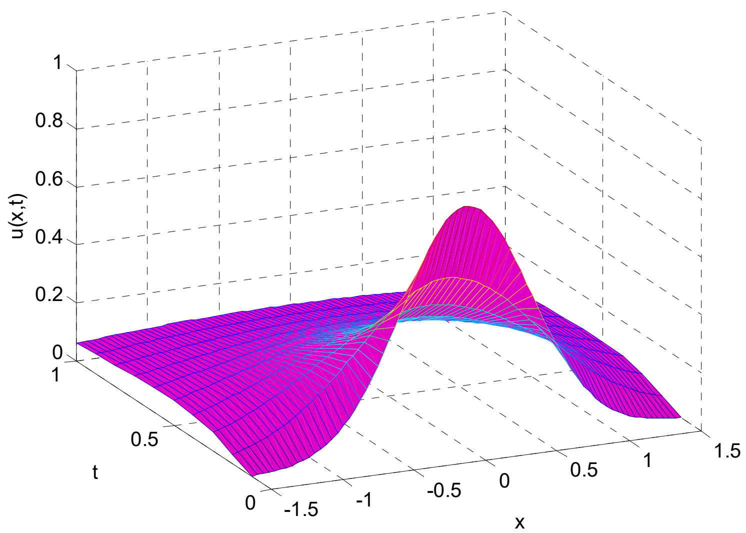
(c) (d)

Figure2. Graphical representation of the solution with different parameters α at times $t = 0.2, 0.3$.



(e)

Figure 3. Graphical representation of the solution with α=0.25.



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(f)

Figure 4. Graphical representation of the solution with α=0.85.

5. Conclusions

In a practical application, different types of the frictional memory kernel $\mathrm{K}(t)$ have been used to describe a wide variety of complex dynamics and physical phenomena with memory effects. In this paper, by applying the method of the Laplace transform with respect to the time variable and Fourier transform with respect to the spatial variable, we obtained the analytical solutions of parabolic Volterra integro-differential equation with three different kinds of memory kernel in the infinite domain. The analytical solutions of the parabolic Volterra integro-differential equation are consist of some special functions, such as multi-parameter Mittag-Leffler function and Fox-H function. It is worth mentioning that the analytical solution provided in Eq.(24) can also be obtained by taking the method of references [18-20] to Eq.(23). In the end, some curves of analytical solution are given. Meanwhile, the analytical solutions we obtained from parabolic Volterra integro-differential equation with different types frictional memory kernel provide much convenience for practical applications.

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Author Contributions: Yun Zhao conceived and designed derivation of the formula of the problem, and wrote the paper. Feng-Qun Zhao contributed the paper of correction and analyzed experiment results. All authors agreed with the final manuscript.

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