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Article

A Procedure of Perturbation Leading to a New Class of Asymmetric Copulas

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Abstract

We propose and analyze a new class of copulas (generated by two univariate functions) that exhibit asymmetric nonlinear dependence, resulting from an antisymmetric perturbation of the independence copula. Within this framework, we construct a parametric subclass of asymmetric copulas for which we derive the analytical expression of Kendall's tau (τ). We present properties related to asymmetry and dependence measures, and provide several examples. To the best of our knowledge, this novel feature of our class represents a significant advancement in the context of the most studied perturbed copulas, which have been recently the focus of numerous scientific investigations.

Keywords: copula; perturbation; concordance measure; asymmetry measures of concordance

1. Introduction

Copulas are mathematical tools and objects that capture and model the dependence structure between random variables by constructing (bivariate for instance, or multivariate) distributions with given marginal distributions. Consider a two-dimensional random vector (X, Y) with joint distribution function F_{XY} and continuous margins F_X and F_Y . There exists a copula C such that: $F_{XY}(x, y) = C(F_X(x), F_Y(y))$

Here, C is nothing other than the distribution function of the vector (U, V) , where $U = F_X(X)$ and $V = F_Y(Y)$ are uniform random variables on $[0, 1]$. Thus, copulas allow us to describe the dependence structure between X and Y independently of their marginal distributions. They have gained significant importance and popularity in various fields, such as finance, insurance, risk management, reliability theory, and other disciplines (see [1–5]).

Over the years, a large number of copula families and various methods for their construction have been proposed and studied. For instance, see some works that have inspired our research ([6–12]).

Our study focuses on a bivariate copula construction principle based on modifying a known copula (particularly the independence copula, denoted by Π , where $\Pi(u, v) = uv$) by adding a factorial term to its expression. We are thus interested in copulas that can be expressed as :

$$C^{(P)}(u, v) = uv + P(u, v), \tag{1}$$

where P is a continuous function on the unit square with values in $\mathbb{R}$, called the "perturbation factor". In this case, the copula C is referred to as a perturbation of Π by means of P .

P can be interpreted as a disruption of independence (see Durante et al. [6]).

This type of construction underlies several copula families. For example, we may cite the well-known Farlie–Gumbel–Morgenstern **FGM** family, given by:

$$C_{\theta}^{(P)}(u, v) = uv + \theta uv(1 - u)(1 - v) \quad \forall \theta \in [-1, 1], \tag{2}$$

Several extensions of this family, along with their properties, have recently been studied by Saminger-Platz et al. [13]. When the perturbation P is not symmetric, we refer to [12] where a new family of asymmetric copulas is detailed.

The paper is organized as follows : We begin by reviewing the definitions, properties, and key results concerning copulas, which are central to our analysis. Next, we describe a particular class of asymmetric copulas based on a slight perturbation of the product copula Π , while providing necessary and sufficient conditions on the generating functions of this perturbation. Subsequently, we examine the main dependence property characterizing this copula family, relying on a specific class of weak concordance measures for bivariate copulas recently studied by Mesiar et al. [14] - which encompasses most well-known concordance measures in the literature. Finally, we investigate and quantify the asymmetry of our new copula class.

2. Preliminaries

In this contribution, for the sake of simplicity, only bivariate copulas will be considered and treated, they will simply be referred to as copulas.

Definition 1. A function $C : I^2 \rightarrow I$ is called a copula if it satisfies the following two conditions :

(C₁) 0 and 1 are, respectively, its zero element and identity argument, i.e.

$$C(x, i) = C(i, x) = \begin{cases} 0, & \text{if } i = 0; \\ x, & \text{if } i = 1. \end{cases} \quad \forall x \in I \quad (3)$$

(C₂) 2-increasing, i.e., for all $u_1, u_2, v_1, v_2 \in I$ where $u_1 \leq u_2$ and $v_1 \leq v_2$, the volume over the rectangle $R = [u_1, u_2] \times [v_1, v_2]$

$$V_C(R) = C(u_2, v_2) + C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) \geq 0, \quad (4)$$

Remark 1. 1. The condition (3) maybe compactly written

$$C(x, i) = C(i, x) = x\delta_{i1} \quad \text{for } i \in \{0, 1\},$$

2. If C has second derivatives, then the 2-increasing property (4) is equivalent to

$$\frac{\partial^2 C(u, v)}{\partial u \partial v} \geq 0 \quad \forall (u, v) \in I^2 \quad \text{See Trivedi et al. [15].}$$

In what follows, we denote by $\mathcal{C}$ the set of all (bivariate) copulas, and let $I := [0, 1]$. The set $\mathcal{C}$ possesses many interesting properties, such as being a closed convex subset of the set of functions from the unit square to the unit interval. The Fréchet-Hoeffding bounds W (the countermonotonicity copula) and M (the comonotonicity copula) are respectively its smallest and largest elements, given by : $W(u, v) = \max(u + v - 1, 0)$ and $M(u, v) = \min(u, v) \quad \forall (u, v) \in I^2$. Furthermore, $\forall C \in \mathcal{C}$, C is increasing in each variable, i.e., $C(u, v) \leq C(u', v)$ and $C(u, v) \leq C(u, v')$ for all $u \leq u'$ and $v \leq v'$ and is also 1-Lipschitz, i.e., for all $u, u', v, v' \in I$, $|C(u, v) - C(u', v')| \leq |u - u'| + |v - v'|$. It is also invariant under strictly increasing transformations.

For more details on the theory of copulas, we recommend monographs [16,17].

Each copula can be decomposed into a sum of an absolutely continuous component A_c and a singular component S_c , given respectively by:

$$A_c = \int_0^u \int_0^v \frac{\partial^2 C(x, y)}{\partial x \partial y} dx dy \quad \text{and :} \quad S_c = C - A_c,$$

when $C = A_c$ (i.e., $S_c = 0$), we say that C is absolutely continuous.

Note that any absolutely continuous copula can be represented as a perturbation of Π (see [6]).

Let us now recall the involutive operations on the set $\mathcal{F}$ of all two-dimensional random vectors (X, Y) and their connections with respective involutive operations on the set $\mathcal{C}$.

- The transformation $(X, Y) \rightarrow (X, Y)$ corresponds to the involutive transformation $i : \mathcal{C} \rightarrow \mathcal{C}$, $i(C)(u, v) = C(u, v)$,
- The transformation $(X, Y) \rightarrow (Y, X)$ corresponds to the involutive transformation $\pi : \mathcal{C} \rightarrow \mathcal{C}$, $\pi(C)(u, v) = C(v, u)$,
- The transformation $(X, Y) \rightarrow (-X, Y)$ corresponds to the involutive transformation $\phi_1 : \mathcal{C} \rightarrow \mathcal{C}$, $\phi_1(C)(u, v) = v - C(1 - u, v)$,
- The transformation $(X, Y) \rightarrow (X, -Y)$ corresponds to the involutive transformation $\phi_2 : \mathcal{C} \rightarrow \mathcal{C}$, $\phi_2(C)(u, v) = u - C(u, 1 - v)$,
- The transformation $(X, Y) \rightarrow (-X, -Y)$ corresponds to the involutive transformation $\phi : \mathcal{C} \rightarrow \mathcal{C}$, $\phi(C)(u, v) = u + v - 1 + C(1 - u, 1 - v)$,
- The transformation $(X, Y) \rightarrow (-Y, -X)$ corresponds to the involutive transformation $\sigma : \mathcal{C} \rightarrow \mathcal{C}$, $\sigma(C)(u, v) = u + v - 1 + C(1 - v, 1 - u)$.

Note that π is called the transpose, ϕ_1 the first flipping, ϕ_2 the second flipping, and ϕ is a survival operation.

The analysis of transformations applied to copulas constitutes an active research field (see [18]).

In cases where:

- $\pi(C)(u, v) = C(u, v)$, $\forall (u, v) \in I^2$, C is called symmetric,
- $\phi(C)(u, v) = C(u, v)$, $\forall (u, v) \in I^2$, C is called radially symmetric.

The latter case was intensively investigated in [19], where invariance of a copula under the transformation ϕ was deeply discussed.

- Example 1.** 1. $\forall \theta \in I$, Every member C_θ of the FGM family given by (2), is both symmetric and radially symmetric,
2. The Fréchet copulas $\lambda M + \lambda W + (1 - 2\lambda)\Pi$, $\lambda \in [0, 1/2]$, are also both symmetric and radially symmetric.

Theorem 1. A copula C_s is symmetric if and only if there exists a copula C such that

$$C_s(C)(u, v) = \frac{C(u, v) + \pi(C)(u, v)}{2}, \quad \forall (u, v) \in I^2. \quad (5)$$

It is extremely easy to see that for any symmetric copula S we have: $C_s(S) = S$. However, what appears more interesting is the inverse problem. This leads us to the fundamental **question** :

Given a symmetric copula S , do there exist other asymmetric copulas satisfying the equality $C_s(C) = S$? In this regard, the authors of [12] have provided an answer to this problem for some notable copulas: M , W and Π .

- Proposition 1.** (i) M is the unique copula C satisfying $C_s(C) = M$;
(ii) W is the unique copula C satisfying $C_s(C) = W$;
(iii) There exist numerous copulas C satisfying

$$C_s(C) = \Pi, \quad (6)$$

Proof. (i) Let C be a copula such that $C_s(C) = M$, then $C - M = M - \pi(C)$. We thus obtain two functions $C - M$ and $M - \pi(C)$ one negative and the other positive, which are equal. This allows us to conclude that $C - M = 0$, hence the uniqueness.

(ii) The same reasoning as above applies.

- (iii) For this proposition, they only proposed a parametric family of asymmetric copulas satisfying (6), given by: $\forall \theta \in I$
 $C_\theta(u, v) = uv + \theta(u - v)\min(u, v)\min(1 - u, 1 - v), \quad \forall (u, v) \in I^2. \quad \square$

Our study proposes a new class of families of copulas satisfying (6), which means the copulas whose symmetrized version C_s is identical to the product copula Π . The following result can be easily obtained.

Proposition 2. Let $P : I^2 \rightarrow \mathbb{R}$, be a continuous function. Then, C defined by $C(u, v) = \Pi(u, v) + P(u, v)$ satisfying (6) is a copula, if and only if the following conditions hold:

- (i) $P(u, 1) = P(1, v) = P(u, 0) = P(0, v) = 0, \quad \forall (u, v) \in I^2.$
(i.e., $P(u, i) = P(i, v) = \delta_{i1}\delta_{i0} \quad i \in \{0, 1\}$),
- (ii) $P(u, v) = -P(v, u) \quad \forall (u, v) \in I^2,$
- (iii) $V_\Pi(R) + V_P(R) \geq 0, \quad \text{for all rectangle } R = [u_1, u_2] \times [v_1, v_2] \subset I^2.$

In other words: The perturbation P must satisfy three requirements: First, it must vanish on the boundaries of I^2 . Secondly, it must be antisymmetric. Thirdly, the amount of negative mass that it can add to each rectangle of I^2 is bounded by the positive mass given by Π .

Remark 2. Note that any copula C naturally defines a probability measure on I^2 . If C takes the form (1), the measure V_c of a rectangle R decomposes into the sum of the measures induced by Π and P .

In what follows, we express the above characterization under specific hypotheses by selecting appropriate functions.

3. New Class of Copulas Based on Antisymmetric Perturbations

3.1. Characterization of the New Class of Copulas

We focus here on copulas whose representation can be formulated as follows:

$$C(u, v) = uv + f(u)g(v) - f(v)g(u), \quad (7)$$

where: $f, g : I \rightarrow \mathbb{R}$ are two non-zero continuous functions. In other words, we have chosen naturally the perturbation $P(u, v) = f(u)g(v) - f(v)g(u)$.

This form provides a flexible method for introducing new copula families by selecting appropriate functions f and g .

We now need to establish and identify the necessary and sufficient conditions to guarantee that the function defined by (7) is indeed a copula. To achieve this, we begin by stating and proving the following preliminary lemma

Lemma 1. Let $f, g : I \rightarrow \mathbb{R}$ be two non-zero continuous functions, and $P : I \rightarrow \mathbb{R}$ a non-zero function defined by $P(u, v) = f(u)g(v) - f(v)g(u)$. Then the following two propositions are equivalent:

- (i) $P(i, v) = P(u, i) = 0 \quad \forall i \in \{0, 1\}, \forall (u, v) \in I^2,$
- (ii) $f(0) = g(0) = f(1) = g(1) = 0.$

Proof. The implication (ii) $\rightarrow$ (i) is obvious, we therefore focus on the converse.

From (i) we have $P(i, v) = 0, \quad \forall i \in \{0, 1\}$, and we then obtain:

$$f(0)g(v) = f(v)g(0), \quad \forall v \in I; \quad (8)$$

$$f(1)g(v) = f(v)g(1), \quad \forall v \in I. \quad (9)$$

We assume that at least one of the two functions f and g does not vanish at 0. In this regard, we distinguish two cases.

1. If exactly one of the two functions does not vanish at 0, this means that $f(0) \neq 0$ or $g(0) \neq 0$. For $f(0) = \alpha$ and $g(0) = 0$ with $\alpha \in \mathbb{R}^*$, the hypothesis implies that $f(0)g(v) = 0$, $\forall v \in I$, therefore, $g(v) = 0$, $\forall v \in I$, which is absurd (since g is considered as a continuous non-zero function. i.e, there exists $a \in I$ such that $g(a) \neq 0$). The same reasoning applies to the other case : $f(0) = 0$ and $g(0) = \beta$ with $\beta \in \mathbb{R}^*$.
2. If both functions do not vanish at 0, then there exist two nonzero real numbers α and β such that $f(0) = \alpha$ with $g(0) = \beta$, substituting the last two equalities into (8), we find that: $g(v) = \gamma f(v)$, $\forall v \in I$, where $\gamma = \beta/\alpha$. Hence, $P(u, v) = 0$, $\forall (u, v) \in I^2$, this is impossible since P is assumed to be non-identically zero.
Thus, $f(0) = g(0) = 0$.

By following the same steps as above for equation (9), we obtain: $f(1) = g(1) = 0$. In addition, one may deduce the second point in the Lemma 1 by considering the functions $\hat{f}(x) = f(1-x)$ and $\hat{g}(x) = g(1-x)$.

It follows that, $f(0) = g(0) = f(1) = g(1) = 0$. $\square$

We are now in a position to state and prove the central theorem of this section.

Theorem 2. Let $f, g : I \rightarrow \mathbb{R}$ be two non-zero continuous functions, the function $C : I^2 \rightarrow \mathbb{R}$ given by (7) is a copula, if and only if the following conditions hold

- (1) $f(0) = g(0) = f(1) = g(1) = 0$,
- (2) f and g are absolutely continuous,
- (3) $f'(u)g'(v) - f'(v)g'(u) \geq -1$ $\forall (u, v) \in I^2$.

Proof. Note that the equivalence of the first condition of Theorem 2 with the boundary conditions (3) is a direct consequence of Lemma 1. Moreover, we have $\frac{\partial^2 C(u, v)}{\partial u \partial v} \geq 0$ (if there exists) is equivalent to $f'(u)g'(v) - f'(v)g'(u) \geq -1$. Consequently, conditions (2) and (3) lead to conclude that C is 2-increasing.

Conversely, suppose that C is a copula, for each $v \in I$ such that $g(v) \neq 0$ and $f(v) \neq 0$, the function $\mathcal{H}_v(t) = C(t, v) - \Pi(t, v) = f(t)g(v) - f(v)g(t)$ formed, respectively, by the horizontal sections of C and Π at $v \in I$ is absolutely continuous for all $t \in I$ (see [20]). There thus exists $(v_1, v_2) \in I^2$ such that $P(v_2, v_1) \neq 0$, where

$$\begin{cases} \mathcal{H}_{v_1}(t) &= f(t)g(v_1) - f(v_1)g(t); \\ \mathcal{H}_{v_2}(t) &= f(t)g(v_2) - f(v_2)g(t). \end{cases}$$

Therefore,

$$\begin{cases} f(v_2)\mathcal{H}_{v_1}(t) &= f(v_2)f(t)g(v_1) - f(v_2)f(v_1)g(t); \\ f(v_1)\mathcal{H}_{v_2}(t) &= f(v_1)f(t)g(v_2) - f(v_1)f(v_2)g(t). \end{cases}$$

Hence, $f(v_2)\mathcal{H}_{v_1}(t) - f(v_1)\mathcal{H}_{v_2}(t) = (f(v_2)g(v_1) - f(v_1)g(v_2))f(t)$.

By hypothesis ($P(v_2, v_1) \neq 0$), we obtain

$$f(t) = \frac{f(v_2)\mathcal{H}_{v_1}(t) - f(v_1)\mathcal{H}_{v_2}(t)}{f(v_2)g(v_1) - f(v_1)g(v_2)},$$

Hence the absolute continuity of f , as a linear combination of two absolutely continuous functions ($\mathcal{H}_{v_1}$ and $\mathcal{H}_{v_2}$). A similar reasoning allows us to conclude that g is also absolutely continuous.

The third property of the theorem follows immediately from (4), the proof is thus completed. $\square$

Let $\mathcal{F} = \{x \in I, f'(x) \text{ exists}\}$ and $\mathcal{G} = \{x \in I, g'(x) \text{ exists}\}$.

In the case where: $\alpha \leq f'(x) \leq \beta$ and $\gamma \leq g'(x) \leq \omega$ with $\alpha = \inf\{f'(x), x \in \mathcal{F}\} \leq 0$, $\beta =$

$\sup\{f'(x), x \in \mathcal{F}\} \geq 0$, $\gamma = \inf\{g'(x), x \in \mathcal{G}\} \leq 0$ and $\omega = \sup\{g'(x), x \in \mathcal{G}\} \geq 0$. we have, $f' \times g' \in [\min\{\alpha\omega, \beta\gamma\}, \max\{\alpha\gamma, \beta\omega\}]$, so the third condition of Theorem 2 becomes in this case

$$\mathcal{D} = \min\{\alpha\omega, \beta\gamma\} - \max\{\alpha\gamma, \beta\omega\} \geq -1, \quad (10)$$

Example 2. Let $f(x) = \frac{1}{2}x(1-x)$ and $g(x) = \frac{1}{2\pi}\sin(\pi x)$,
So, $f'(x) = \frac{1}{2}(1-2x)$ and $g'(x) = \frac{1}{2}\cos(\pi x)$,
Then, $f''(x) = -1 \leq 0$ and $g''(x) = -\frac{\pi}{2}\sin(\pi x) \leq 0 \quad \forall x \in I$,
Hence, $f' \in [f'(1), f'(0)] = [-\frac{1}{2}, \frac{1}{2}]$ and $g' \in [g'(1), g'(0)] = [-\frac{1}{2}, \frac{1}{2}]$,
From (10) we have then, $\mathcal{D} = -\frac{1}{2} - \frac{1}{2} = -1 \geq -1$,
It follows that: $C(u, v) = uv + \frac{1}{4\pi}(u(1-u)\sin(\pi v) - v(1-v)\sin(\pi u))$ defines a copula.

Example 3. Let $f(x) = x(1-x^2)$ and $g(x) = \frac{1}{2\pi}x\cos(\frac{\pi}{2}x)$,
So, $f'(x) = 1-3x^2$ and $g'(x) = \frac{1}{2\pi}\cos(\frac{\pi}{2}x) - \frac{1}{4}x\sin(\frac{\pi}{2}x)$,
Then, $f''(x) = -6x \leq 0$ and $g''(x) = -\frac{1}{2}\sin(\frac{\pi}{2}x) - \frac{\pi}{8}x\cos(\frac{\pi}{2}x) \leq 0 \quad \forall x \in I$,
Hence, $f' \in [f'(1), f'(0)] = [-2, 1]$ and $g' \in [g'(1), g'(0)] = [-\frac{1}{4}, \frac{1}{2\pi}]$,
It follows that, $\mathcal{D} = -\frac{1}{\pi} - \frac{1}{2} \geq -1$,
Thus, $C(u, v) = uv(1 + \frac{1}{2\pi}((1-u^2)\cos(\frac{\pi}{2}v) - (1-v^2)\cos(\frac{\pi}{2}u)))$ be a copula.

3.2. Subclass and Some Examples

We introduce in this section a parametric subclass of copulas derived from formulation (7), which will be the subject of a more detailed characterization later in our work.

Corollary 1. Let C be the function given by (7) with $f(x) = xg_\lambda(x)$, where $g_\lambda = \lambda^{\frac{1}{2}}g(x)$, $\forall x \in I, \lambda > 0$. Then we obtain:

$$C_\lambda(u, v) = uv + \lambda(u-v)g(u)g(v), \quad (11)$$

is a copula if and only if the following conditions hold:

- (i) $g(0) = g(1) = 0$,
- (2) f and g are absolutely continuous,
- (3) $\mathcal{V}_c = g(u)g'(v) - g(v)g'(u) + (u-v)g'(u)g'(v) \geq -1/\lambda, \quad \forall (u, v) \in I^2$.

Proof. The proof is a matter of simple computations of the derivatives for the third condition of Theorem 2. $\square$

Note that The function g plays a role similar to the generating function in Archimedian copulas see [16], and the inclusion of the coefficient λ ensures the validity of the copula construction, and its range depends only on the choice of the function g .

We now propose a few examples of functions g generating parametric families of copulas. One of these examples will be used to illustrate antisymmetry and to evaluate Kendall's tau in the following sections.

Example 4. Let m and n be two non-null real numbers such that $mn > 0$, and let g be the function defined by: $g(x) = \min(mx, \frac{1-x}{n})$, then $g'(x) = m$ if $x \leq \frac{1}{mn+1}$ and $g'(x) = \frac{-1}{n}$ if $x \geq \frac{1}{mn+1}$.

For $u \leq \frac{1}{mn+1}$ and $v \leq \frac{1}{mn+1}$
We have, $\mathcal{V}_c = m^2u - m^2v + m^2(u-v) = 2m^2(u-v) \in [-\frac{2m^2}{mn+1}, \frac{2m^2}{mn+1}]$,

For $u \geq \frac{1}{mn+1}$ and $v \leq \frac{1}{mn+1}$
 We find that, $\mathcal{V}_c = \frac{m}{n(1-u)} + \frac{m}{nv} - \frac{m}{n(u-v)} = \frac{m}{n(1-2u+2v)} \in \left[-\frac{m}{n}, \frac{m}{n}\right]$,

For $u \geq \frac{1}{mn+1}$ and $v \geq \frac{1}{mn+1}$
 $\mathcal{V}_c = -\frac{1}{n^2}(1-u) + \frac{1}{n^2}(1-v) + \frac{1}{n^2}(u-v) = \frac{2}{n^2}(u-v) \in \left[-\frac{2m}{n}(mn+1), \frac{2m}{n}(mn+1)\right]$,

For $u \leq \frac{1}{mn+1}$ and $v \geq \frac{1}{mn+1}$
 We have, $\mathcal{V}_c = -\frac{m}{n}u - \frac{m}{n}(1-v) - \frac{m}{n}(u-v) = \frac{m}{n}(2v-2u-1) \in \left[-\frac{m}{n}, \frac{m}{n}\right]$.
 Thus,

$$C_\lambda(u, v) = uv + \lambda(u-v)\min(mu, \frac{1-u}{n})\min(mv, \frac{1-v}{n}), \quad (12)$$

is a copula, where $\lambda \in]0, \frac{mn+1}{\max(2m^2, \frac{2m}{n})}]$.

Example 5. For $g(t) = \sin^n(\pi t)$, $\forall t \in I$ where $n \geq \frac{3}{4}$.
 We have, $g'(t) = n\pi \cos(\pi t) \sin^{n-1}(\pi t)$. Then,

$$\begin{aligned} \mathcal{V}_c &= n\pi(\sin^n(\pi u)\cos(\pi v)\sin^{n-1}(\pi v) - \sin^n(\pi v)\cos(\pi u)\sin^{n-1}(\pi u)) \\ &+ (n\pi)^2(u-v)\cos(\pi u)\sin^{n-1}(\pi u)\cos(\pi v)\sin^{n-1}(\pi v) \\ &= \sin^{n-1}(\pi u)\sin^{n-1}(\pi v)[n\pi\sin(\pi(u-v)) + (n\pi)^2\cos(\pi u)\cos(\pi v)], \end{aligned}$$

Hence, $\mathcal{V}_c \in [-(n\pi + (n\pi)^2), (n\pi + (n\pi)^2)]$,
 Thus, $\forall (u, v) \in I^2$, $n \geq \frac{3}{4}$

$$C_\lambda(u, v) = uv + \lambda(u-v)\sin^n(\pi u)\sin^n(\pi v), \quad (13)$$

is a copula, where $\lambda \in]0, \frac{1}{n\pi + (n\pi)^2}]$.

Remark 3. We can easily see that our subclass serves to perturb any member of the semiparametric family of symmetric copulas introduced and studied by Rodriguez-Lallena and Úbeda-Flores [8], is given by

$$C(u, v) = uv + \theta h(u)h(v), \quad \forall \theta \in [-1, 1], \quad (14)$$

which includes the Farlie–Gumbel–Morgenstern family as a particular case.

Let $g(x) = x(1-x)$, then $g'(x) = 1-2x$, hence $g(x) \in [0, g(\frac{1}{2})] = [0, \frac{1}{4}]$, $\forall x \in I$. Moreover, we have $g''(x) = -2 \leq 0$, therefore $g'(x) \in [g'(1), g'(0)] = [-1, 1]$, $\forall x \in I$. According to Corollary 1

$$\mathcal{V}_c = u(1-u)(1-2v) - v(1-v)(1-2u) + (u-v)(1-2u)(1-2v) \in \left[-\frac{3}{2}, \frac{3}{2}\right],$$

it follows that

$$C(u, v) = uv(1 + \lambda(u-v)(1-u)(1-v)), \quad \text{where } \lambda \in]0, \frac{2}{3}]. \quad (15)$$

A convex combination of the two parametric families of copulas (15) and (14) (the case where $h(x) = x(1-x)$) generates, for every $\alpha \in I$, the parametric family of copulas expressed by

$$\begin{aligned} C_{\{\alpha, \theta, \lambda\}}(u, v) &= uv + \alpha\theta uv(1-u)(1-v) + (1-\alpha)\lambda(u-v)uv(1-u)(1-v) \\ &= uv(1 + (\alpha\theta + (1-\alpha)\lambda(u-v))(1-u)(1-v)), \end{aligned}$$

where $\theta \in [-1, 1]$, $\lambda \in]0, \frac{2}{3}]$ and $\alpha \in I$.

Each copula of the **FGM** family can thus be perturbed by a convex combination with a copula that belongs to our parametric subclass, thus providing a richer modeling tool for capturing complex dependence structures observed in real data.

3.3. Illustrations

We conclude this section with a few plots to visualize the shapes of the two previous examples. More precisely, we provide the 3D surfaces and contour plots for the copulas previously defined in (12) and (13), as well as for their corresponding perturbation factors P .

As an illustration, we have chosen the parameters characterizing the copula defined in (12) as $m = -0.5$ and $n = -0.9$, whereas $n = 0.8$ for the copula defined in (13) see Figure 1.

These visualizations make it possible to analyze how the perturbations modify the dependence structure between the variables, by highlighting the deviations from the independence case.

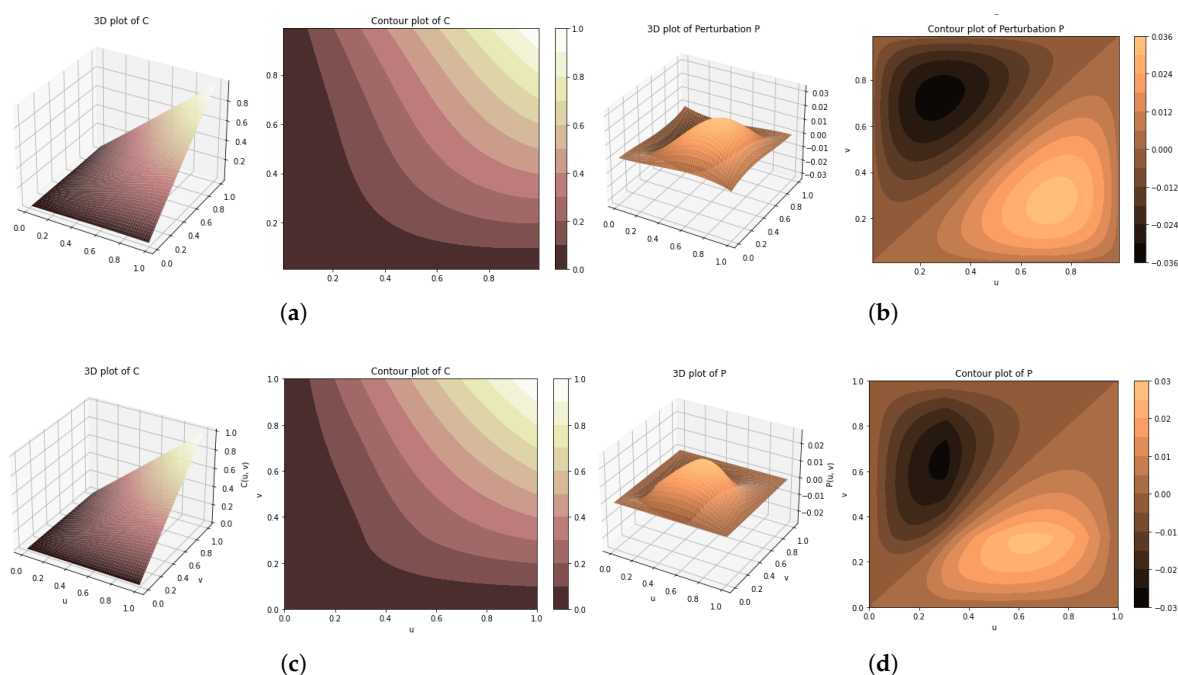


Figure 1. (a) 3D and contour plots of copula C_λ given by (12) with $m = -0.5$ and $n = -0.9$. (b) 3D and contour plots of factor perturbation P of copula C given by (13) with $m = -0.5$ and $n = -0.9$. (c) 3D and contour plots of copula C_λ given by (12) with $n = 0.8$. (d) 3D and contour plots of factor perturbation P of copula C given by (13) with $n = 0.8$.

The 3D representations in Figure 1 display slightly curved hyperbolic surfaces compared to the independence copula Π , suggesting a weak dependence introduced by the two perturbations. Regarding the contour plots, their deviations from the copula Π are subtle, they are not symmetric with respect to the diagonal $u = v$, reflecting the asymmetry of the two proposed examples.

Note that the larger the parameters m and n , the weaker the perturbation, bringing the perturbed copula closer to the independence copula.

4. Properties of the New Subclass

4.1. Preliminaries on Measures of Concordance

Let (X_1, X_2) and (Y_1, Y_2) be two continuous random vectors, if the corresponding copulas are C_1 and C_2 .

then the function Q defined by:

$$Q(C_1, C_2) = 4 \int_{I^2} C_2(u, v) dC_1(u, v) - 1, \quad (16)$$

is called the concordance function, it was introduced by [21]. Its probabilistic representation is given by:

$$P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0),$$

This function has several properties, among which we mention the following:

- **Symmetry**, $Q(C_1, C_2) = Q(C_2, C_1)$,
- **Monotonicity** in each argument, (meaning that a stronger concordance ordering between copulas implies a higher concordance value),

Formally, if $C_1 \leq C_2$ and $C_2 \leq C_3$, then

$$Q(C_1, C_3) \leq Q(C_2, C_3) \quad \text{and} \quad Q(C_1, C_2) \leq Q(C_1, C_3),$$

- The concordance function Q remains unchanged when the copulas are replaced by their survival copulas or by their transposes, while it changes sign when they are replaced by their reflections:

$$Q(C_1, C_2) = Q(\phi(C_1), \phi(C_2)) = Q(\pi(C_1), \pi(C_2)),$$

$$Q(\phi_i(C_1), \phi_i(C_2)) = -Q(C_1, C_2) \quad \forall i \in \{1, 2\}.$$

For more details, see [18]

In [22], Scarsini offers a characterization of concordance measures κ as mappings that assign a real number $\kappa(C)$ to each copula C , in accordance with axioms set forth in the following definition.

Definition 2. A function $\kappa : \mathcal{C} \rightarrow [-1, 1]$ is called a concordance measure if it satisfies the following conditions:

$$(C_1) \quad \kappa(\pi(C)) = \kappa(C),$$

$$(C_2) \quad \kappa(\phi_1(C)) = \kappa(\phi_2(C)) = -\kappa(C),$$

$$(C_3) \quad \kappa(M) = 1,$$

$$(C_4) \quad \forall C_1, C_2 \in \mathcal{C} \text{ where } C_1 \leq C_2, \text{ then } \kappa(C_1) \leq \kappa(C_2),$$

$$(C_5) \quad \text{If a sequence of copulas } (C_n)_{n \in \mathbb{N}} \text{ converges uniformly to } C \in \mathcal{C}, \text{ then } \lim_{n \rightarrow +\infty} \kappa(C_n) = \kappa(C).$$

Since the independence copula Π is invariant under both reflections, and by condition (C_2) , we deduce that

$$\kappa(\Pi) = 0, \quad (17)$$

The four most frequently used concordance measures, which also play a central role, are: Kendall's τ , Spearman's ρ , Gini's γ and Blomqvist's β .

When condition (C_2) is replaced with the normalization condition (17), we obtain what Liebscher [23] calls a *weak concordance measure*. Spearman's footrule ϕ is a typical example of such a weak measure of concordance (see [24]).

Note that the set of all weak concordance measures covers all concordance measures.

A (weak) concordance measure $\kappa : \mathcal{C} \rightarrow [-1, 1]$ is said to be convex, if it satisfies the following property:

$$\kappa(\theta C_1 + (1 - \theta)C_2) = \theta \kappa(C_1) + (1 - \theta) \kappa(C_2) \quad \forall C_1, C_2 \in \mathcal{C}, \quad \forall \theta \in [0, 1],$$

which can be seen as the linearity of κ applied to convex combinations of copulas. Let $l_\kappa : I \rightarrow [-1, 1]$ defined by: $l_\kappa(\theta) = \kappa(\theta C_1 + (1 - \theta)C_2)$. κ is a convex concordance measure if and only if l_κ is a linear function, i.e., $l_\kappa(\theta) = (\kappa(C_1) - \kappa(C_2))\theta + \kappa(C_2)$. Equivalently, it means that κ is of degree 1 in the sense of Edwards and Taylor [25].

To simplify the statements, we will henceforth use the term **concordance measure** to refer to the five studied measures.

The first four concordance measures can be expressed in terms of the concordance function Q .

- **Kendall's τ** : $\tau(C) = Q(C, C) = 12 \int_{I^2} C(u, v) dC(u, v) - 3$,
- **Spearman's ρ** : $\rho(C) = 3Q(C, \Pi) = 4 \int_{I^2} C(u, v) du dv - 1$,
- **Gini's γ** : $\gamma(C) = Q(C, M) + Q(C, W) = 4 \int_I (C(u, u) + C(u, 1 - u)) du - 2$,
- **Spearman's footrule ϕ** : $\phi(C) = \frac{1}{2}(3Q(C, M) - 1) = 6 \int_{I^2} C(u, u) du - 2$,
- On the other hand, **Blomqvist's β** is given by:

$$\beta(C) = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1.$$

Example 6. For every $\theta \in [-1, 1]$, the four dependence measures ρ , τ , β and γ associated with FGM copulas given by (2) are linear functions of the parameter θ .

$$\rho(C_\theta) = \frac{1}{3}\theta \quad \tau(C_\theta) = \frac{2}{9}\theta \quad \beta(C_\theta) = \frac{1}{4}\theta \quad \gamma(C_\theta) = \frac{4}{15}\theta,$$

Furthermore, these parameters exhibit simple proportional relationships:

$$\frac{\rho(C_\theta)}{\tau(C_\theta)} = \frac{3}{2} \quad \frac{\rho(C_\theta)}{\beta(C_\theta)} = \frac{4}{3} \quad \frac{\rho(C_\theta)}{\gamma(C_\theta)} = \frac{5}{4},$$

the ranges of these measures for any $\theta \in [-1, 1]$, are given by:

$$\rho(C_\theta) \in \left[-\frac{1}{3}, \frac{1}{3}\right] \quad \tau(C_\theta) \in \left[-\frac{2}{9}, \frac{2}{9}\right] \quad \beta(C_\theta) \in \left[-\frac{1}{4}, \frac{1}{4}\right] \quad \gamma(C_\theta) \in \left[-\frac{4}{15}, \frac{4}{15}\right].$$

4.2. Single Point-Generated Convex Weak Concordance Measures

We now turn our attention to a concordance measure s whose values depend on the symmetrized copula C_s given previously by (5). This measure encompasses several well-known (weak) convex concordance measures from the literature, including Spearman's Rho, Gini's Gamma, Blomqvist's Beta, and Spearman's Footrule.

Since all copulas coincide on the boundary of the unit square, we consider a point $(u, v) \in]0, 1[^2$. The fact that C_s is a symmetric copula implies that $C_s(u, v) = C_s(v, u)$, $C_s(1 - u, v) = C_s(v, 1 - u)$, $C_s(u, 1 - v) = C_s(1 - v, u)$ and $C_s(1 - u, 1 - v) = C_s(1 - v, 1 - u)$, this allows us to focus only on points within the set $\Delta = \{(u, v) \in I \mid v \leq u\}$. Given any fixed point $(u, v) \in \Delta$, the mapping $C_s : \mathcal{C} \rightarrow \mathbb{R}$ is increasing, i.e., $\forall C_1 \leq C_2$, we have $C_s(C_1) \leq C_s(C_2)$, and consequently, $C_s(\Pi) = uv \leq v = C_s(M)$. We can therefore normalize C_s into s as follows:

$$s_{(u,v)} = \frac{C_s(u,v)(C) - C_s(u,v)(\Pi)}{C_s(u,v)(M) - C_s(u,v)(\Pi)},$$

Based on the above considerations, we introduce an alternative formulation for the family of convex concordance measures generated by a single point, recently proposed in [14].

Theorem 3. Let $(u, v) \in \Delta$. The function $s_{(u,v)} : \mathcal{C} \rightarrow \mathbb{R}$ given by:

$$s_{(u,v)}(C) = \frac{C_{s_{(u,v)}}(C) - uv}{v - uv}, \quad (18)$$

is a weak convex concordance measure.

Proof. By construction, we immediately observe that: $s_{(u,v)}(M) = 1$ and $s_{(u,v)}(\Pi) = 0$. Moreover, for all $(C_n)_{n \geq 1} \subset \mathcal{C}$, if $C_n \rightarrow C$, then $s_{(u,v)}(C_n) \rightarrow s_{(u,v)}(C)$ uniformly for each $(u, v) \in \Delta$. From our earlier discussion, the other axioms of weak concordance measures (invariance under permutation and boundedness on I^2) are easily verified.

Regarding the convexity of s : $\forall C_1, C_2 \in \mathcal{C}$ and $\forall \lambda \in I$,

$$\begin{aligned} & C_{s_{(u,v)}}(\lambda C_1 + (1 - \lambda)C_2) \\ = & \frac{C_s(\lambda C_1 + (1 - \lambda)C_2)(u, v) - uv}{v - uv} \\ = & \frac{\lambda(C_1(u, v) + \pi(C_1)(u, v)) + (1 - \lambda)(C_2(u, v) + \pi(C_2)(u, v))}{v - uv} \\ = & \lambda \frac{C_1(u, v) + \pi(C_1)(u, v) - uv}{v - uv} + (1 - \lambda) \frac{C_2(u, v) + \pi(C_2)(u, v) - uv}{v - uv} \\ = & \lambda C_{s_{(u,v)}}(C_1) + (1 - \lambda)C_{s_{(u,v)}}(C_2), \end{aligned}$$

Thus, s is convex. We omit the detailed calculations. $\square$

Remark 4. For $C = W$, $\forall (u, v) \in \Delta$

$$s_{(u,v)}(W) = \begin{cases} \frac{-u}{1-u}, & \text{if } u + v \leq 1; \\ \frac{v-1}{v}, & \text{if } u + v \geq 1. \end{cases}$$

which yields $s_{(u,v)}(W) \in]-\infty, 0[$,

which confirms that there is no lower bound for convex weak concordance measures.

Example 7. For any fixed point $(u, v) \in S = \{(a, b) \in \Delta \mid a = b\}$ we have:

$$s_{(u,u)}(C) = \frac{C_{s_{(u,u)}}(C) - u^2}{u - u^2},$$

When $u = \frac{1}{2}$, we obtain the **Blomqvist** β , i.e., $s_{(\frac{1}{2}, \frac{1}{2})}(C) = \beta$.

4.3. Representation of Basic Weak Convex Concordance Measures via s^P and s_Ψ

Any convex combination of (weak convex) concordance measures remains a (weak convex) concordance measure. Formally:

let $s_{(u_1, v_1)}, s_{(u_2, v_2)}, \dots, s_{(u_k, v_k)}$ be weak convex concordance measures. The function $s : \mathcal{C} \rightarrow \mathbb{R}$ defined by: $s(C) = \sum_{i=1}^k \lambda_i s_{(u_i, v_i)}(C)$ where $\lambda_i \in I$ with $\sum_{i=1}^k \lambda_i = 1$ is also a concordance measure. This can be interpreted as a Lebesgue integral with respect to a probability measure Ψ defined on $\mathcal{B}(\Delta)$ (the Borel subsets of I) by $\Psi(\Omega) = \sum_{(u_i, v_i) \in \Omega} \lambda_i$, i.e.,

$$s_\Psi(C) = \int_{\Delta} s_{(u,v)}(C) d\Psi(u, v), \quad (19)$$

Theorem 4. Let P be a probability measure defined on $(\Delta, \mathcal{B}(\Delta))$, then the mapping s^P given by :

$$s^P(C) = \frac{\int_{\Delta} C_{s(u,v)}(C) dP(u,v) - \int_{\Delta} uv dP(u,v)}{\int_{\Delta} v dP(u,v) - \int_{\Delta} uv dP(u,v)}, \quad (20)$$

is a convex weak concordance measure.

Proof. The proof of this statement is similar to that of Theorem 3. $\square$

Moreover, note that there exists a probability measure Ψ on $(\Delta, \mathcal{B}(\Delta))$ such that, for each copula $C \in \mathcal{C}$, we have, $s_{\Psi}(C) = s^P(C)$ where s_{Ψ} and s^P are the weak convex concordance measures given in (19) and (20) respectively.

See ([14], Theorem 4.1)

Proposition 3. Let P and Ψ be two probability measures on $(\Delta, \mathcal{B}(\Delta))$, with respective densities p and ψ .

- If $p(u,v) = 2$ and $\psi(u,v) = 24v(1-u)$, $\forall (u,v) \in \Delta$,

then, $s^P = s_{\Psi} = \rho$ (Spearman's ρ),

- If $p(u,v) = \begin{cases} \frac{1}{2}, & \text{if } (u,v) \in S_1; \\ 1, & \text{if } (u,v) \in S_2. \end{cases}$

And, $\psi(u,v) = \begin{cases} 4v(1-u), & \text{if } (u,v) \in S_1; \\ 8v(1-u), & \text{if } (u,v) \in S_2. \end{cases}$

where, $S_1 = \{v = u, u \in]0, 1[\}$ and $S_2 = \{v = 1 - u, u \in]\frac{1}{2}, 1[\}$,

then, $s^P = s_{\Psi} = \gamma$ (Gini's γ).

- If $p(u,v) = \begin{cases} 1, & \text{if } (u,v) \in S_1; \\ 0, & \text{otherwise.} \end{cases}$

and, $\psi(u,v) = \begin{cases} 6u(1-u), & \text{if } (u,v) \in S_1; \\ 0, & \text{otherwise.} \end{cases}$

then, $s^P = s_{\Psi} = \phi$ (Spearman's footrule ϕ).

Proof. • Considering the probability measure Ψ on $(\Delta, \mathcal{B}(\Delta))$ with density $\psi(u,v) = 24v(1-u)$, $(u,v) \in \Delta$, as $d\Psi(u,v) = 24v(1-u) du dv$. Then the corresponding convex weak concordance measure s_{Ψ} introduced in (19) can be determined as follows:

$$\begin{aligned} s_{\Psi}(C) &= \int_{\Delta} \frac{C_{s(u,v)}(C) - uv}{v - uv} d\Psi(u,v) \\ &= 24 \int_{\Delta} C_{s(u,v)}(C) du dv - 24 \int_0^1 \int_0^u uv du dv \\ &= 12 \int_{\Delta} C(u,v) + \pi(C)(u,v) du dv - 12 \int_0^1 u^3 du \\ &= 12 \int_{I^2} C(u,v) du dv - 3 \\ &= \rho(C), \end{aligned}$$

The same result can be obtained by using the representation s^P exprimed in (20).

- Let Ψ be a probability measure on $(\Delta, \mathcal{B}(\Delta))$ whose support is the set $S = \{v = u, u \in]0, 1[\} \cup \{v = 1 - u, u \in]\frac{1}{2}, 1[\}$ and density $\psi(u, v) = \begin{cases} 4v(1-u), & \text{if } (u, v) \in S_1; \\ 8v(1-u), & \text{if } (u, v) \in S_2. \end{cases}$

Then

$$\begin{aligned}
 s_{\Psi}(C) &= 4 \int_0^1 C_{s(u,u)}(C) - u^2 du + 8 \int_{1/2}^1 C_{s(u,1-u)}(C) - u(1-u) du \\
 &= 4 \int_0^1 C(u, u) du + 4 \int_{1/2}^1 C(u, 1-u) + C(1-u, u) du \\
 &\quad - 4 \int_0^1 u^2 du - 8 \int_{1/2}^1 u(1-u) du \\
 &= 4 \int_0^1 C(u, u) du + 4 \int_0^1 C(u, 1-u) du - \frac{4}{3} - \frac{2}{3} \\
 &= 4 \int_0^1 C(u, u) + C(u, 1-u) du - 2 \\
 &= \gamma(C),
 \end{aligned}$$

- The support of probability measure Ψ , is the set $S = \{v = u, u \in]0, 1[\}$. We directly obtain

$$\begin{aligned}
 s_{\Psi}(C) &= 6 \int_0^1 C_{s(u,u)}(C) - u^2 du \\
 &= 6 \int_0^1 C(u, u) du - 6 \int_0^1 u^2 du \\
 &= 6 \int_0^1 C(u, u) du - 2 \\
 &= \phi(C). \quad \square
 \end{aligned}$$

4.4. Dependence Properties

For every copula D belonging to our new subclass of copulas of the form (11), we trivially observe that $s_{(u,v)}(D) = 0$. This result follows directly from the characterizing property (6) of this subclass. It follows that the classical concordance measures — $(\rho, \gamma, \beta$ and $\phi)$ vanish identically. Regarding Kendall's tau, its expression is established in the proposition below.

Let us recall that by applying integration by parts to the expression of Kendall's τ defined by

$$\tau_C = 1 - 4 \int_{I^2} C(u, v) dC(u, v),$$

We obtain an alternative formulation that is more convenient for numerical computation:

$$\tau_C = 4 \int_{I^2} \frac{\partial C(u, v)}{\partial u} \frac{\partial C(u, v)}{\partial v} du dv - 1, \quad (21)$$

Furthermore, if the copula C is absolutely continuous, then:

$$dC(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v} du dv = c(u, v) du dv, \quad (22)$$

Where, $c(u, v)$ denotes the copula density of C .

For more detail, we refer to [16].

Proposition 4. Let (X, Y) be a random pair whose copula C_{θ} is given by (11). We have:

$$\rho(C_{\theta}) = \gamma(C_{\theta}) = \beta(C_{\theta}) = \phi(C_{\theta}) = 0,$$

Kendall's τ admits the following representation:

$$\tau_\theta = 2\theta^2 \left(\int_I g^2(u) du \right)^2,$$

Proof. The partial derivatives of the copula

$$C_\theta(u, v) = uv + \theta(u - v)g(u)g(v), \quad \theta > 0,$$

are expressed by

$$\begin{cases} \frac{\partial C_\theta(u, v)}{\partial u} = v + \theta(g(u)g(v) + (u - v)g'(u)g(v)); \\ \frac{\partial C_\theta(u, v)}{\partial v} = u + \theta(-g(u)g(v) + (u - v)g(u)g'(v)). \end{cases}$$

Using the expression (21). Kendall's τ can then be written as:

$$\begin{aligned} \tau_\theta &= 1 - 4 \int_{I^2} (v + \theta(g(u)g(v) + (u - v)g'(u)g(v)))(u + \theta(-g(u)g(v) + (u - v)g(u)g'(v))) du dv \\ &= 1 - 4 \int_{I^2} uv du dv - 4\theta \int_{I^2} vg(v)g(u) du dv + 4\theta \int_{I^2} v(u - v)g(u)g'(v) du dv \\ &\quad + 4\theta \int_{I^2} ug(u)g(v) du dv - 4\theta^2 \int_{I^2} g^2(u)g^2(v) du dv + 4\theta^2 \int_{I^2} (u - v)g(u)^2g'(v)g(v) du dv \\ &\quad + 4\theta \int_{I^2} u(u - v)g(v)g'(u) du dv - 4\theta^2 \int_{I^2} (u - v)g'(u)g(u)g^2(v) du dv \\ &\quad + 4\theta^2 \int_{I^2} (u - v)^2g'(u)g(u)g'(v)g(v) du dv, \end{aligned}$$

given that:

$$\int_{I^2} vg(v)g(u) du dv = \int_I g(u) du \int_I vg(v) dv = \int_I g(v) dv \int_I ug(u) du = \int_{I^2} ug(u)g(v) du dv,$$

hence,

$$\int_{I^2} vg(v)g(u) du dv = \int_{I^2} ug(u)g(v) du dv.$$

Analogously, we find that:

$$\int_{I^2} v(u - v)g(u)g'(v) du dv = \int_{I^2} u(u - v)g(v)g'(u) du dv.$$

First, we deduce that the Kendall's τ can be expressed as the sum of four terms:

$$\begin{aligned} \tau_\theta &= 4\theta^2 \left(\int_{I^2} g^2(u)g^2(v) du dv - \int_{I^2} (u - v)g(u)^2g'(v)g(v) du dv + \int_{I^2} (u - v) \right. \\ &\quad \left. g'(u)g(u)g^2(v) du dv - \int_{I^2} (u - v)^2g'(u)g(u)g'(v)g(v) du dv \right) \\ &= 4\theta^2(I_1 - I_2 + I_3 - I_4). \end{aligned} \tag{23}$$

On the one hand, we have:

$$\begin{aligned}
I_2 &= \int_{I^2} (u-v)g(u)^2g'(v)g(v) du dv \\
&= \int_I ug(u)^2 du \int_I g'(v)g(v) dv - \int_I g(u)^2 du \int_I vg'(v)g(v) dv \\
&= \int_I vg(v)^2 dv \int_I g'(u)g(u) du - \int_I g(v)^2 dv \int_I ug'(u)g(u) du \\
&= \int_{I^2} (v-u)g(v)^2g'(u)g(u) du dv,
\end{aligned}$$

consequently,

$$\begin{aligned}
I_3 - I_2 &= \int_{I^2} (u-v)g'(u)g(u)g^2(v) du dv - \int_{I^2} (u-v)g(u)^2g'(v)g(v) du dv \\
&= 2 \int_{I^2} ug'(u)g(u)g^2(v) du dv - 2 \int_{I^2} vg(v)^2g'(u)g(u) du dv.
\end{aligned} \quad (24)$$

On the other hand,

$$\int_{I^2} g^2(u)g^2(v) du dv = \int_I g^2(v) \left(\int_I g^2(u) du \right) dv,$$

where, by integration by parts, we have:

$$\int_I g^2(u) du = \left[ug^2(u) \right]_0^1 - 2 \int_I ug'(u)g(u) du = -2 \int_I ug'(u)g(u) du,$$

thus,

$$\int_{I^2} g^2(u)g^2(v) du dv = -2 \int_{I^2} ug'(u)g(u)g^2(v) du dv. \quad (25)$$

Moreover, the last integral in (23) can be decomposed as follows:

$$\begin{aligned}
I_4 &= \int_{I^2} (u-v)^2g'(u)g(u)g'(v)g(v) du dv \\
&= \int_{I^2} u^2g'(u)g(u)g'(v)g(v) du dv - 2 \int_{I^2} uv g'(u)g(u)g'(v)g(v) du dv \\
&\quad + \int_{I^2} v^2g'(v)g(v)g'(u)g(u) du dv \\
&= 2 \int_{I^2} u^2g'(u)g(u)g'(v)g(v) du dv - 2 \left(\int_I ug'(u)g(u) du \right)^2,
\end{aligned}$$

using integration by parts,

$$\begin{aligned}
I_4 &= \int_I g'(v)g(v) \left(\left[u^2g(u)^2 \right]_0^1 - 2 \int_I ug(u)^2 du \right) dv - 2 \left(\left[\frac{1}{2}ug(u)^2 \right]_0^1 - \frac{1}{2} \int_I g^2(u) du \right)^2 \\
&= -2 \int_{I^2} ug(u)^2g'(v)g(v) du dv - \frac{1}{2} \left(\int_I g^2(u) du \right)^2.
\end{aligned} \quad (26)$$

Combining (24), (25) and (26) into (23), we obtain:

$$\tau_\theta = 2\theta^2 \left(\int_I g^2(u) du \right)^2. \quad \square$$

It is not surprising that Kendall's tau behaves differently from the other concordance measures — ρ, γ, β , and ϕ . This dissimilarity is explained by the linearity of the latter under convex combinations, whereas Kendall's τ exhibits a quadratic behavior.

Example 8. We revisit the copula generated by the function $g_\lambda(x) = \lambda^{1/2} \sin(\pi x)^n$, where $\lambda \in]0, \frac{1}{n\pi + (n\pi)^2}]$, as given by (13), for different values of the shape parameter n , which controls the concentration of mass around $x = 0.5$, as shown in Figure 2.

Figure 2 also illustrates the evolution of τ_{C_θ} , whose graph displays three curves, each corresponding to a specific value of n and its associated $\theta_{\max}$. The vertical dashed lines indicate the value of $\theta_{\max}$ for each n .

- For $n = 0.8$, $\theta_{\max}$ is the largest (approximately 0.1133).
- For the other two cases, $n = 1$ and $n = 2$, $\theta_{\max}$ decreases (approximately 0.0769 and 0.0219, respectively).

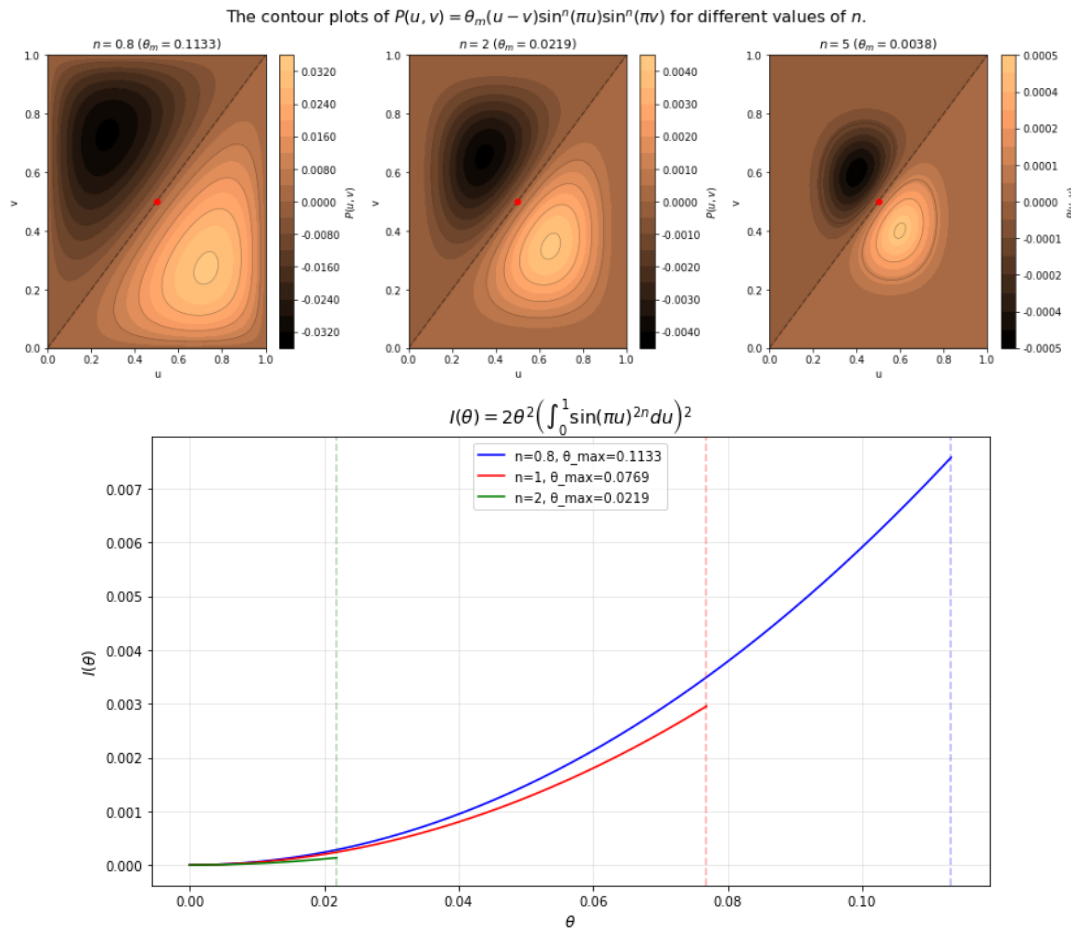


Figure 2. The evolution of Kendall's tau for the copula defined by (13), over the set of values $n \in \mathcal{N} = \{0.8, 1, 2\}$.

Consequently, the larger the value of n , the smaller the value of θ , and the lower the maximum value of τ_{C_θ} . These observations are consistent with the behavior of the function g , as n increases, g becomes more concentrated around $u = 0.5$ and vanishes more rapidly elsewhere, which leads to a reduction in $I(\theta)$.

We now study a partial order on $C \in \mathcal{C}$. Let C_1 and C_2 be two copulas. C_2 is said to be more concordant than C_1 , denoted $C_1 \preceq C_2$, if $C_1(u, v) \leq C_2(u, v)$, $\forall (u, v) \in I^2$ (see [16]). Let C_1 and C_2 be two copulas such that $C_1(u, v) = uv + \theta_1(u - v)g_1(u)g_2(v)$ and $C_2(u, v) = uv + \theta_2(u - v)f_1(u)f_2(v)$, $\forall (u, v) \in I^2$, where $\theta_1 > 0$ and $\theta_2 > 0$, then $C_1 \preceq C_2$, if and only if,

$$\begin{cases} g_1(u)g_2(v) \leq f_1(u)f_2(v), & \text{si } u \geq v; \\ g_1(u)g_2(v) \geq f_1(u)f_2(v), & \text{si } u \leq v. \end{cases}$$

The following two theorems consider continuous random pairs (X, Y) associated with the copula $C \in \mathcal{C}$ and characterize the pairs that satisfy certain well-known positive dependence properties. In what follows, we will use some concepts of positive dependence (for more details, see [16]). A copula C is said to be positively quadrant dependent (**PQD**) if $C \geq \Pi$, and negatively quadrant dependent (**NQD**) if $C \leq \Pi$. It should be noted that copulas exhibiting this property concentrate more mass near the main diagonal than near the opposite diagonal.

For our own parametric family of copulas given by (11), we have the following result, the proof of which is straightforward.

Theorem 5. Let $\theta > 0$ and (X, Y) be a continuous random pair associated with the copula C_θ given by (11). Then X and Y are **PQD**, if and only if either $g(u)$ and $g(v)$ have the same sign $\forall (u, v) \in \Delta_1 = \{(x, y) \in I^2 \mid x \geq y\}$, or $g(u)$ and $g(v)$ have opposite signs $\forall (u, v) \in \Delta_2 = \{(x, y) \in I^2 \mid x \leq y\}$.

The interconnections between the dependence properties described in the following and a probabilistic interpretation of these relationships are discussed in [26]. For this, each concept of positive dependence studied in the following theorem implies **PQD**. We may therefore assume, without loss of generality, that the generating function g is concave with $g(x) = g(1 - x)$, $\forall x \in I$, which allows us to deduce that

$$g(x) \geq 0, \quad \forall x \in I, \quad (27)$$

And that g is symmetric with respect to $v = \frac{1}{2}$. Consequently, $g'(x) \geq 0$ for all $x \in [0, \frac{1}{2}]$, and $g'(x) \leq 0$, $\forall x \in [\frac{1}{2}, 1]$. Under these considerations, we find that C_θ is **PQD** $\forall (u, v) \in \Delta_1$.

This result is perfectly illustrated by (Figure 1, (b) and (d)), which provide a particularly insightful graphical representation of it.

Theorem 6. Let $\theta > 0$ and (X, Y) be a continuous random pair associated with the copula C_θ given by (11), where g is a concave function with $g(x) = g(1 - x)$, $\forall x \in I$. Then:

1. Y is left tail decreasing in X (**LTD**($Y|X$)), if and only if $ug'(u) \leq g(u) \leq u(1 - u)g'(u)$, for almost all u , and **LTD**($X|Y$) holds, if and only if either $g'(v) \geq 0$ and $g(v) \geq vg'(v)$ or $v(1 - v)g'(v) \leq g(v) \leq vg'(v)$, for almost all v .
2. Y is right tail increasing in X (**RTI**($Y|X$)), if and only if either $g'(u) \leq 0$ and $g(u) \geq (u - 1)g'(u)$ or $u(1 - u)g'(u) \leq g(u) \leq (u - 1)g'(u)$, for almost all u , and (**RTI**($X|Y$)) holds, if and only if, $(v - 1)g'(v) \leq g(v) \leq v(v - 1)g'(v)$, for almost all v .
3. Y is stochastically increasing in X (**SI**($Y|X$)), if and only if $\forall u \in I$, $g'(u) \leq \frac{1-u}{2}g''(u)$ and (**SI**($X|Y$)) holds, if and only if $\forall v \in I$, $g'(v) \geq \frac{v}{2}g''(v)$.

Proof. 1. According to (corollary 5.2.6, nelsen2006introduction), **LTD**($Y|X$) holds if and only if, for any $v \in I$

$$\frac{\partial C_\theta(u, v)}{\partial u} \leq \frac{C_\theta(u, v)}{u}, \quad \text{For almost all } u. \quad (28)$$

Thus, we have: $\frac{\partial C_\theta(u, v)}{\partial u} = v + \theta(g(u)g(v) + (u - v)g'(u)g(v))$.

And $\frac{C_\theta(u, v)}{u} = v + \theta \frac{u-v}{u} g(u)g(v)$.

The inequality (28) therefore becomes $\theta g(v) \left[\frac{v}{u} g(u) + (u - v)g'(u) \right] \leq 0$, by rearranging the terms, and from (27), we obtain $\left[\frac{1}{u} g(u) - g'(u) \right] v + ug'(u) \leq 0$.

Let $\mathcal{A}(v) = \left[\frac{1}{u} g(u) - g'(u) \right] v + ug'(u)$ (an affine function with respect to v).

If $\frac{1}{u} g(u) - g'(u) \geq 0$, this implies that $\mathcal{A}$ is increasing, consequently, $\mathcal{A}$ reaches its maximum at $v = 1$.

Hence, $\mathcal{A}(1) \leq 0$ is equivalent to $g(u) \leq u(1 - u)g'(u)$, for almost all u .

If $\frac{1}{u} g(u) - g'(u) \leq 0$, this implies that $\mathcal{A}$ is decreasing, hence $\mathcal{A}$ reaches its maximum at $v = 0$.

Therefore, $\mathcal{A}(0) \leq 0$ is equivalent to $g'(u) \leq 0$ which contradicts $g'(u) \geq \frac{1-u}{2}g''(u) \geq 0$, for almost all u .

Similarly, for **LTD**($X|Y$). Following the same steps as before, we find that

$$\frac{\partial C_\theta(u, v)}{\partial v} \leq \frac{C_\theta(u, v)}{v} \quad \text{For almost all } v,$$

this leads to $[\frac{1}{v}g(v) - g'(v)]u + vg'(v) \geq 0$.

Let $\mathcal{B}(u) = [\frac{1}{v}g(v) - g'(v)]u + vg'(v)$.

If $\frac{1}{v}g(v) - g'(v) \geq 0$, which implies that $\mathcal{B}$ is increasing, hence $\mathcal{B}$ reaches its minimum at $u = 0$.

Then $\mathcal{B}(0) \geq 0$ is equivalent to $g(v) \leq v(1-v)g'(v)$, for almost all v .

If $\frac{1}{v}g(v) - g'(v) \leq 0$, then $\mathcal{B}$ is decreasing, consequently $\mathcal{B}$ reaches its minimum at $u = 1$.

Thus, $\mathcal{B}(1) \geq 0$ is equivalent to $g(v) \geq v(1-v)g'(v)$, for almost all v .

2. **RTI(Y|X)** holds if and only if, for any $v \in I$,

$$\frac{\partial C_{\theta}(u, v)}{\partial u} \geq \frac{v - C_{\theta}(u, v)}{1 - u} \text{ for almost all } u,$$

this leads to $[\frac{1}{1-u}g(u) + g'(u)]v - \frac{1}{1-u}g(u) - ug'(u) \leq 0$.

Let $\mathcal{T}(v) = [\frac{1}{1-u}g(u) + g'(u)]v - \frac{1}{1-u}g(u) - ug'(u)$.

If $\frac{1}{1-u}g(u) + g'(u) \leq 0$, then $\mathcal{T}$ reaches its maximum at $v = 0$. $\mathcal{T}(0) \leq 0$ is equivalent to $g(u) \geq u(1-u)g'(u)$, for almost all u .

If $\frac{1}{1-u}g(u) + g'(u) \geq 0$, then $\mathcal{T}$ reaches its maximum at $v = 1$.

Then, $\mathcal{T}(1) \leq 0$ is equivalent to $g'(u) \leq 0$, for almost all u . (where $\mathcal{T}(1) = (1-u)g'(u)$).

RTI(X|Y) can be proved using similar arguments.

3. **SI(Y|X)** holds if and only if, for any $v \in I$, $C(u, v)$ is a concave function of u .

We have

$$\frac{\partial^2 C(u, v)}{\partial u^2} = 2g'(u)g(v) + (u-v)g''(u)g(v),$$

Then $\frac{\partial^2 C(u, v)}{\partial u^2} \leq 0$, implies that $-g''(u)v + 2g'(u) + ug''(u) \leq 0$, $\forall u \in I$.

Let $\mathcal{F}(v) = -g''(u)v + 2g'(u) + ug''(u)$, since $g''(u) \leq 0$, $\forall u \in I$.

This leads to $\mathcal{F}$ is increasing, i.e, $\mathcal{F}$ reaches its maximum at $v = 1$.

Hence $\mathcal{F}(1) \leq 0$ is equivalent to $g'(u) \leq \frac{1-u}{2}g''(u)$, $\forall u \in I$.

The same reasoning applies to **SI(Y|X)**, hence the result. $\square$

4.5. Tail Dependence

Tail dependence quantifies the joint likelihood of extreme events in a bivariate distribution's tails (see [3]).

Let X and Y be continuous random variables with respective distribution functions F and G . The upper λ_U and lower λ_L tail dependence coefficients are respectively

$$\lambda_U = \lim_{u \rightarrow 1^-} P[Y > G^{(-1)}(t) | X > F^{(-1)}(t)],$$

and,

$$\lambda_L = \lim_{u \rightarrow 0^+} P[Y \leq G^{(-1)}(t) | X \leq F^{(-1)}(t)].$$

According to ([16], Theorem 5.4.2). If the limits exist, the upper and lower tail dependence coefficients λ_U and λ_L can also be characterized in terms of the copula associated with the joint distribution of (X, Y) . Let $C(u, v)$ denote the copula of the pair (X, Y) . Then:

$$\lambda_U = \lim_{u \rightarrow 1^-} 2 - \frac{1 - C_{\theta}(u, u)}{1 - u},$$

and

$$\lambda_L = \lim_{u \rightarrow 0^+} \frac{C_{\theta}(u, u)}{u}.$$

Proposition 5. The copula C_{θ} of the form (11) has a total tail independence, i.e,

$$\lambda_U(C_{\theta}) = \lambda_L(C_{\theta}) = 0.$$

Proof. It is immediate to verify that $C_\theta(u, u) = u^2$, where C_θ is of the form (11). Hence,

$$\lambda_U = \lim_{u \rightarrow 1^-} 2 - \frac{1 - u^2}{1 - u} = \lim_{u \rightarrow 1^-} 2 - 1 - u = 0,$$

and

$$\lambda_L = \lim_{u \rightarrow 0^+} \frac{u^2}{u} = \lim_{u \rightarrow 0^+} u = 0. \quad \square \quad (29)$$

5. Asymmetry Analysis of Our New Class of Family of Copulas

In this section, we analyze the asymmetry of copulas belonging to our new family through the relationships between an asymmetry measure (*non-exchangeability*), denoted by μ_∞ , and the four convex concordance measures ρ, ϕ, γ and β , recently introduced by Bukovšek et al. [27].

The topic of asymmetry or non-exchangeability of copulas has recently received considerable attention and plays a prominent role in current research [28–31].

A function $\nu : \mathcal{C} \rightarrow [0, \infty[$ is called a measure of asymmetry (or non-exchangeability) for a copula C if it satisfies the following properties:

- (A₁) There exists $M \in \mathbb{R}^+$ such that for all $C \in \mathcal{C}$, we have $\nu(C) \leq M$,
- (A₂) $\nu(C) = 0$ if and only if, $C = \pi(C)$ (C is symmetric),
- (A₃) $\nu(C) = \nu(\pi(C)) = \nu(\phi(C))$, $\forall C \in \mathcal{C}$,
- (A₄) If $(C_n)_{n \in \mathbb{N}}$ and C are in $\mathcal{C}$, if $(C_n)_{n \in \mathbb{N}}$ converges uniformly to C , then $\nu(C_n)_{n \in \mathbb{N}}$ converges to $\nu(C)$.

As an example, a broad class of asymmetry measures was proposed by Nelsen [32], for a complementary reference, see [33].

Let d_p denote the classical L_p distance on $\mathcal{C}$ for $p \in [1, +\infty]$. For all $C_1, C_2 \in \mathcal{C}$.

- For $p \in [1, +\infty[$:

$$d_p(C_1, C_2) = \left(\int_{I^2} |C_1(u, v) - C_2(u, v)|^p du dv \right)^{\frac{1}{p}}.$$

- For $p = +\infty$:

$$d_{+\infty}(C_1, C_2) = \max_{(u, v) \in I^2} |C_1(u, v) - C_2(u, v)|.$$

The measure $\nu_p : \mathcal{C} \rightarrow \mathbb{R}^+$ is then defined as $\nu_p(C) = d_p(C, \pi(C))$, and can be normalized using ν_∞ , see [34].

Property (A₄) implies the continuity of ν with respect to the topology induced by the L_∞ norm on $\mathcal{C}$ and the euclidean norm on $\mathbb{R}$. The compactness of $\mathcal{C}$ therefore guarantees the existence of a maximum for ν , which justifies the existence of the following result.

Proposition 6. Let $\nu_p : \mathcal{C} \rightarrow \mathbb{R}^+$ for $p \in [1, +\infty]$, be a measure of asymmetry. Then, there exists a constant $M_{\nu_p} \in \mathbb{R}^+$ and a copula $C_{\nu_p} \in \mathcal{C}$ such that, $\nu_p(C_{\nu_p}) = M_{\nu_p}$, and $\nu_p(C_{\nu_p}) \geq \nu_p(C)$ for all copula $C \neq C_{\nu_p}$.

Proof. According to result ([32], Lemme 2.1), for any copula C and any $(u, v) \in I^2$, we have:

$$|C(u, v) - \pi(C)(u, v)| \leq d^*(u, v), \text{ where } d^*(u, v) = \min\{u, v, 1 - u, 1 - v, |v - u|\},$$

by integrating both sides of this inequality over the unit square, we obtain, $\forall p \in [1, +\infty[$:

$$\nu_p(C) \leq \left(\frac{2 \times 3^{-p}}{(p+1)(p+2)} \right)^{\frac{1}{p}}. \quad \square$$

Note that $\nu_p \leq \frac{1}{3}$ and $\lim_{p \rightarrow +\infty} \left(\frac{2 \times 3^{-p}}{(p+1)(p+2)} \right)^{\frac{1}{p}} = \frac{1}{3}$.

Now choose $(a, b) \in I^2, \alpha \in I$ such that $0 \leq \alpha \leq d^*(a, b)$. From the convexity of $\mathcal{C}$, it readily follows that there exists a copula $C \in \mathcal{C}$ satisfying

$$C(a, b) - C(b, a) = \alpha. \quad (30)$$

We adopt the notation $x^+ := \max(0, x)$ to denote the positive part of x .

Theorem 7. For each $\alpha \in [0, d^*(a, b)]$, the set of copulas satisfying (30) admits extremal elements, $C_{L\alpha}^{(a,b)}$ and $C_{U\alpha}^{(a,b)}$. i.e, for every copula C satisfying (30), we have:

$$C_{L\alpha}^{(a,b)} \leq C \leq C_{U\alpha}^{(a,b)}, \quad (31)$$

where,

$$C_{L\alpha}^{(a,b)} = \max\{W(u, v), \min\{c_1 + (u - a)^+ + (v - b)^+\}\}, \quad (32)$$

and

$$C_{U\alpha}^{(a,b)} = \min\{M(u, v), \max\{c_2 + (u - b)^+ + (v - a)^+\}\}, \quad (33)$$

$$\text{with : } c_1 := c_{1,\alpha}^{(a,b)} = W(a, b) + \alpha \text{ and } c_2 := c_{2,\alpha}^{(a,b)} = M(a, b) - \alpha.$$

Proof. If $u \geq a$, by the 1-increasing property, we have $C(u, v) - C(a, v) \geq 0$.
If $u < a$, by the 1-Lipschitz property, we obtain

$$C(u, v) - C(a, v) \geq u - a,$$

hence, it follows that

$$C(u, v) - C(a, v) \geq (u - a)^+. \quad (34)$$

Similarly, by the same properties. It follows that

If $v \geq b$, then $C(a, v) - C(a, b) \geq 0$.

If $v < b$, then $C(a, v) - C(a, b) \geq v - b$,

consequently,

$$C(a, v) - C(a, b) \geq (v - b)^+. \quad (35)$$

From (34) and (35), we obtain

$$(u - a)^+ + (v - b)^+ \leq C(u, v) - C(a, b),$$

by (30), and $W(a, b) = W(b, a) \leq C(b, a)$, thus,

$$c_1 + (u - a)^+ + (v - b)^+ \leq C(u, v) \quad \text{With } c_1 = W(a, b) + \alpha.$$

In a manner analogous to the previous one, we find that

$$\begin{cases} C(u, v) - C(b, v) \leq u - b, & \text{if } u > b; \\ C(u, v) - C(b, v) \leq 0, & \text{if } u \leq b. \end{cases} \quad \text{And } \begin{cases} C(b, v) - C(b, a) \leq v - a, & \text{if } v > a; \\ C(b, v) - C(b, a) \leq 0, & \text{if } v \leq a. \end{cases}$$

This leads us to

$$C(u, v) - C(b, a) \leq (u - b)^+ + (v - a)^+,$$

hence, $C(u, v) \leq c_2 + (u - b)^+ + (v - a)^+$, where $c_2 = M(a, b) - \alpha$.

Incorporating the Fréchet-Hoeffding bounds completes the proof. $\square$

We call (32) and (33) *local Fréchet-Hoeffding bounds*.

An alternative proof, structured around five distinct cases according to the regions defined by the scatterplots, is developed in ([27], Theorem 2)

The calculation of the concordance measures for the local Fréchet-Hoeffding bounds, defined respectively by (32) and (33), requires the evaluation of the concordance function Q (given by (16)) applied to these two copulas, as well as to W, Π and M . The synthesis of the graphical observations (Figure 3) and the preliminary calculations then reveals that

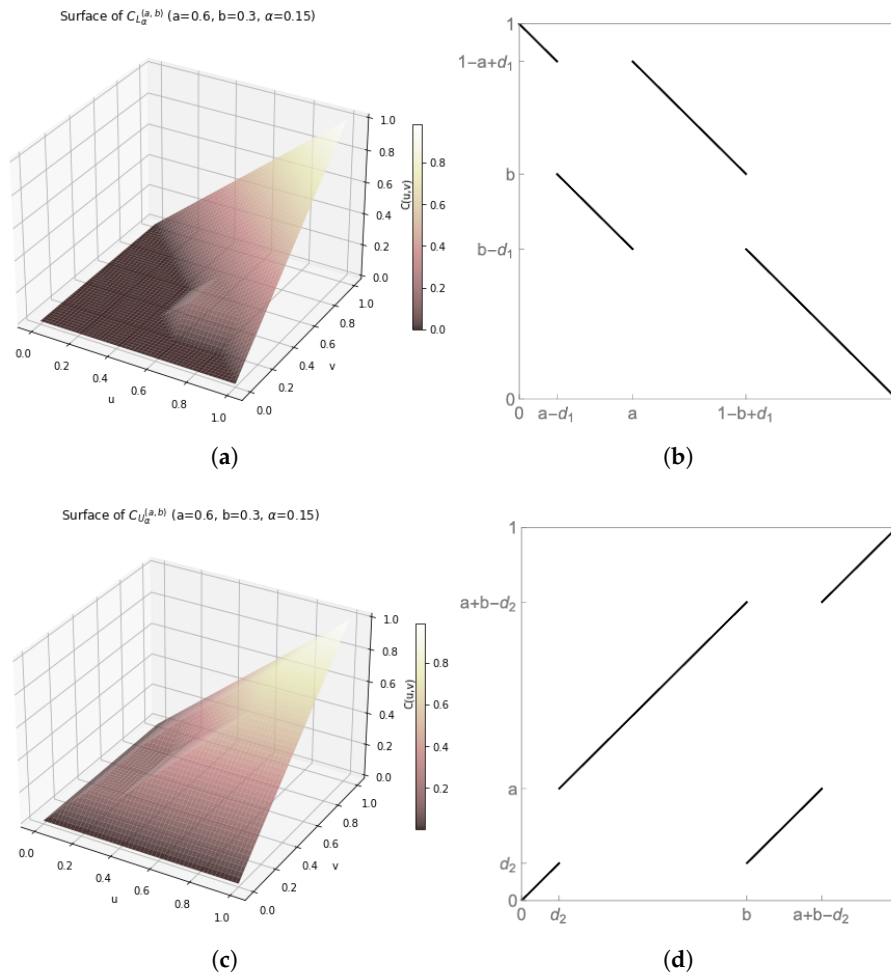


Figure 3. (a) and (c) The respective 3D representations of $C_{L\alpha}^{(a,b)}$ and $C_{UL\alpha}^{(a,b)}$. (b) and (d) their scatter plots.

$$\begin{aligned}
 Q(A, C_{L\alpha}^{(a,b)}) &= 4 \int_0^1 A(u, v) dC_{L\alpha}^{(a,b)} - 1 \\
 &= 4 \int_0^{a-c_1} A(u, 1-u) du + 4 \int_{a-c_1}^a A(u, a+b-c_1-u) du \\
 &+ 4 \int_a^{1-b+c_1} A(u, 1+c_1-u) du + 4 \int_{1-b+c_1}^1 A(u, 1-u) du,
 \end{aligned}
 \tag{36}$$

and

$$\begin{aligned} Q(A, C_{U\alpha}^{(a,b)}) &= 4 \int_0^1 A(u, v) dC_{U\alpha}^{(a,b)} - 1 \\ &= 4 \int_0^{c_2} A(u, u) du + 4 \int_{c_2}^b A(u, u + a - c_2) du \\ &\quad + 4 \int_b^{a+b-c_2} A(u, u - b + c_2) du + 4 \int_{a+b-c_2}^1 A(u, u) du. \end{aligned} \quad (37)$$

In accordance with ([27], Properties 4 and 5), through which the authors derive formulas (36) and (37) for every $A \in \{W, \Pi, M\}$, the results obtained are symmetric with respect to both the main diagonal and the counter-diagonal. i.e,

Let $(a, b) \in I, \alpha \in [0, \min\{a, b, 1 - a, 1 - b\}]$ and $\forall A \in \{W, \Pi, M\}$.

$$Q(A, C_{L\alpha}^{(a,b)}) = Q(A, C_{L\alpha}^{(b,a)}) = Q(A, C_{L\alpha}^{(1-a, 1-b)}), \quad (38)$$

and

$$Q(A, C_{U\alpha}^{(a,b)}) = Q(A, C_{U\alpha}^{(b,a)}) = Q(A, C_{U\alpha}^{(1-a, 1-b)}). \quad (39)$$

Let $C \in \mathcal{C}$ be any copula with $v_\infty(C) = m$, there exists a pair $(a_0, b_0) \in I$ such that

$$|C(a_0, b_0) - C(b_0, a_0)| = m,$$

then, $C(a_0, b_0) - C(b_0, a_0) = m \geq 0$ (If $C(a_0, b_0) - C(b_0, a_0) \leq 0$ we replace C with $\pi(C)$).

By virtue of Theorem 7, it follows that

$$C_{Lm}^{(a_0, b_0)} \leq C \leq C_{Um}^{(a_0, b_0)},$$

as an immediate consequence of monotonicity, we have $\forall \kappa \in \{\rho, \phi, \gamma\}$

$$\kappa(C_{Lm}^{(a_0, b_0)}) \leq \kappa(C) \leq \kappa(C_{Um}^{(a_0, b_0)}).$$

Given the symmetries discussed in (38) and (39), we fix m . The study can therefore be restricted to the set $\mathcal{S} = \{(a, b) \in [0, 1] \mid a \leq b, b \leq 1 - a, d^*(a, b) \geq m\}$ without loss of generality.

The analysis is therefore limited to the triangle $\Delta EFG := \mathcal{S} = \{(a, b) \in [0, 1] \mid a \geq m, b - a \geq m, b + a \leq 1 - 2m\}$ with the vertices $E(m, 2m), F(\frac{1-m}{2}, \frac{1+m}{2})$ and $G(m, 1 - m)$.

It results that

$$\min_{(a_0, b_0) \in \mathcal{S}} \kappa(C_{Lm}^{(a_0, b_0)}) \leq \kappa(C) \leq \max_{(a_0, b_0) \in \mathcal{S}} \kappa(C_{Um}^{(a_0, b_0)}),$$

Following ([27], Corollary 22), the minimum value of $\kappa(C_{Lm}^{(a_0, b_0)})$ and the maximum value of $\kappa(C_{Um}^{(a_0, b_0)})$ on the set $\mathcal{S}$ are attained at the same point, denoted $G(m, 1 - m)$ and $E(m, 2m)$, respectively. The preceding developments make it possible to establish the link between the measure v_∞ and each measure $\kappa \in \{\rho, \phi, \gamma, \beta\}$.

Theorem 8. Let $C \in \mathcal{C}$ be any copula with $v_\infty(C) = m$. Then

$$\kappa(C) \in [\kappa(C_{Lm}^{(m, 1-m)}), \kappa(C_{Um}^{(m, 2m)})], \quad \forall \kappa \in \{\rho, \phi, \gamma, \beta\},$$

In particular,

- For $\kappa = \rho$, we have $\rho(C) \in [f(m), g(m)]$, where

$$f(m) = 12m^3 - 1 \text{ and } g(m) = 1 - 36m^3,$$

- For $\kappa = \phi$, we have $\phi(C) \in [f(m), g(m)]$, where

$$f(m) = \begin{cases} -\frac{1}{2}, & \text{if } 0 \leq m \leq \frac{1}{4}; \\ 24m^2 - 12m + 1, & \text{if } \frac{1}{4} \leq m \leq \frac{1}{3}. \end{cases} \text{ and } g(m) = 1 - 12m^2$$

- For $\kappa = \gamma$, we have $\gamma(C) \in [f(m), g(m)]$, where

$$f(m) = \begin{cases} 4m^2 - 1, & \text{if } 0 \leq m \leq \frac{1}{4}; \\ 20m^2 - 8m, & \text{if } \frac{1}{4} \leq m \leq \frac{1}{3}. \end{cases}$$

And

$$g(m) = \begin{cases} 1 - 8m^2, & \text{if } 0 \leq m \leq \frac{1}{6}; \\ -44m^2 + 12m, & \text{if } \frac{1}{6} \leq m \leq \frac{1}{5}; \\ -19m^2 + 2m + 1, & \text{if } \frac{1}{5} \leq m \leq \frac{1}{4}; \\ -3m^2 - 6m + 2, & \text{if } \frac{1}{4} \leq m \leq \frac{1}{3}. \end{cases}$$

- For $\kappa = \beta$, we have $\beta(C) \in [f(m), g(m)]$, where

$$f(m) = \begin{cases} -1, & \text{if } 0 \leq m \leq \frac{1}{4}; \\ 8m - 3, & \text{if } \frac{1}{4} \leq m \leq \frac{1}{3}. \end{cases}$$

And

$$g(m) = \begin{cases} 1, & \text{if } 0 \leq m \leq \frac{1}{6}; \\ -12m + 3, & \text{if } \frac{1}{6} \leq m \leq \frac{1}{4}; \\ -4m + 1, & \text{if } \frac{1}{4} \leq m \leq \frac{1}{3}. \end{cases}$$

Note that, If copula C takes the maximal possible asymmetry $\nu_{\infty}(C) = \frac{1}{3}$. Then

$$\rho(C) \in [-\frac{5}{9}, -\frac{1}{3}], \gamma(C) \in [-\frac{4}{9}, -\frac{1}{3}] \text{ and } \phi(C) = \beta(C) = -\frac{1}{3}.$$

Regarding our new class of family of copulas, the use of inverses immediately yields the following result.

Corollary 2. Let C be the copula satisfying the property (6),

$$\text{If } \kappa = \rho, \text{ then } 0 \leq \nu_{\infty}(C) \leq \sqrt[3]{\frac{1}{36}}.$$

$$\text{If } \kappa = \phi, \text{ then } 0 \leq \nu_{\infty}(C) \leq \sqrt{\frac{1}{12}}.$$

$$\text{If } \kappa = \gamma, \text{ then } 0 \leq \nu_{\infty}(C) \leq \frac{\sqrt{15}}{3} - 1.$$

$$\text{If } \kappa = \beta, \text{ then } 0 \leq \nu_{\infty}(C) \leq \frac{1}{4}.$$

Remark 5. for all copula $C \in \mathcal{C}$. If $\tau(C) \in [-\frac{5}{9}, \frac{1}{9}]$, then C takes the maximal possible asymmetry $\nu_{\infty}(C) = \frac{1}{3}$. See [32]. The inductive inference, based on the data from Figure 2, reveals that for any copula C belonging to the parametric family given by (13), we have $0 \leq \nu_{\infty}(C) \leq \frac{1}{3}$.

Let C be a copula of the form $C(u, v) = uv + P(u, v)$, where $P(u, v) = -P(v, u)$, $\forall (u, v) \in I^2$, we have $|C(u, v) - C(v, u)| = 2|P|$, $\forall (u, v) \in I^2$. The corollary 2 directly implies that: $0 \leq |P| \leq \frac{1}{8}$. Which appears to be a good indicator of the mildness of the perturbation.

6. Conclusion and Directions for Further Research

This article defines and studies a class of parametric antisymmetric copula families and establishes their dependence and asymmetry properties. The methodology developed for the analysis of concordance and dependence measures relies on a family of Convex weak concordance measures, allowing the inference of a nonlinear dependence structure characterizing the considered copula class. This, in turn, provides an optimal modeling framework for various phenomena in which weak dependencies reveal critical nonlinear structures. In line with our previous approach, our future work will consist in characterizing and analyzing a family of copulas arising from an anti-radially symmetric perturbation of the independence copula. It is worth noting that this perturbative approach is extensible to the multivariate case, which will be addressed separately in future work. Another research direction is to identify and study another class of copulas whose symmetrized part is Archimedean.

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Abbreviations

The following abbreviations are used in this manuscript:

FGM	Farlie-Gumbel-Morgenstern
PQD	Positively Quadrant Dependent
LTD	Left Tail Decreasing
RTI	Right Tail Increasing
SI	Stochastically Increasing

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