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Revisit on Some Relations Between the Weighted Spectral Mean and the Wasserstein Mean

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Abstract: In this paper, we introduce the 2-geometric mean and explore its connections with the spectral geometric mean and the Wasserstein mean for positive definite matrices. Additionally, we revisit and establish several inequalities for these means concerning the near order relation.

Keywords: spetral geometric mean; geometric mean; Wasserstein mean; Riccati equation; central elements

MSC: [2010] 15A16, 15A45, 22E46

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1. Introduction

The study of matrix means plays a crucial role in various mathematical and applied fields, particularly in matrix analysis, optimization, and quantum information theory. Among these means, the geometric mean [22] is one of the most extensively studied. It is well known [2] that the geometric mean serves as the barycenter of two given positive definite matrices under the Riemannian metric. The spectral geometric mean of positive definite matrices was introduced by Fiedler and Pták in 1997 [11]. Later, in 2015, Kim and Lee [17] defined the weighted spectral mean and studied several of its properties. Although the spectral geometric mean has received less attention compared to the geometric mean, it remains an object of interest among researchers.

In [4], Bhatia, Jain, and Lim investigated the Bures-Wasserstein distance between positive definite matrices and solved the least squares problem to determine the barycenter of two positive definite matrices, known as the weighted Wasserstein mean. Notably, the weighted spectral geometric mean and the weighted Wasserstein mean are, in some sense, generated by the functions x^t and $1 - t + tx$, respectively. This suggests that the relationship between these means may exhibit similarities to that between the weighted geometric mean and the weighted arithmetic mean.

Recently, there has been a growing body of research on these means. Franco and Dumitru [10] introduced a near-order relation. Building on this, researchers such as J. Kim, H. Huang, and L. Gan have further explored various inequalities associated with different order relations [13,16]. More recently, Furuichi discovered connections between the spectral geometric mean and Tsallis entropy in quantum information theory [12].

In this paper, we introduce the concept of the 2-geometric mean and investigate its relationships with the spectral geometric mean and the Wasserstein mean. These formulations provide a foundation for revisiting and further exploring their properties. Additionally, we establish several inequalities satisfied by these means within the context of the near-order relation.

The paper is organized as follows: In Section 2, we recall fundamental definitions and properties of matrix means. In Section 3, we introduce new representations of the spectral geometric mean and the Wasserstein mean, along with concise proofs of their fundamental properties. In Section 4, we

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reexamine the near-order relation and derive general inequalities for these means. Furthermore, we establish norm inequalities and identify conditions under which equality holds.

2. Preliminaries

Let $\mathbb{P}_n$ be the set of all $n \times n$ positive definite matrices, $\mathbb{H}_n$ be the space of all $n \times n$ Hermitian matrices, and $\mathrm{U}(n)$ be the group of $n \times n$ unitary matrices. For $A, B \in \mathbb{P}_n$, the geometric mean $A\sharp_t B$ was firstly defined by Puzs and Woronowicz [22] in 1975. They showed that it is the unique positive definite solution to the Riccati equation

$$XA^{-1}X = B.$$

It is well-known [2] that the geometric mean $A\sharp B$ is the midpoint of the geodesic

$$A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \quad t \in [0,1],$$

joining A and B under the Riemannian metric $\delta_R(A, B) = \|\log(A^{-1/2}BA^{-1/2})\|_F$, where $\|\cdot\|_F$ denotes the Frobenius norm.

The spectral geometric mean of $A, B \in \mathbb{P}_n$ was introduced by Fiedler and Pták in 1997 [11], and one of its formulations is

$$A\natural B := (A^{-1}\sharp B)^{1/2}A(A^{-1}\sharp B)^{1/2}. \tag{2.1}$$

It is called *the spectral geometric mean* because $(A\natural B)^2$ is similar to AB and that the eigenvalues of their spectral mean are the positive square roots of the corresponding eigenvalues of AB [11, Theorem 3.2].

In 2015, Kim and Lee [17] defined the weighted spectral mean:

$$A\natural_t B := \left(A^{-1}\sharp B\right)^t A \left(A^{-1}\sharp B\right)^t, \quad t \in [0,1]. \tag{2.2}$$

It is obvious that $A\natural_t B$ is a curve joining A and B. They studied the relative operator entropy related to the spectral geometric mean and several properties similar to those of the relative entropy of Tsallis operator defined via the matrix geometric mean.

Note that in (2.2) the geometric mean $A^{-1}\sharp B$ is a main component of the weighted spectral mean $A\natural_t B$ while the middle term is A, independent of t. We define a new weighted mean below.

$$F_t(A, B) := (A^{-1}\sharp_t B)^{1/2}A^{2-2t}(A^{-1}\sharp_t B)^{1/2}, \quad t \in [0,1]. \tag{2.3}$$

It is obvious that $F_0(A, B) = A$ and $F_1(A, B) = B$, and hence $F_t(A, B)$ is a curve joining A and B. For $t = \frac{1}{2}$, $F_{\frac{1}{2}}(A, B)$ is the spectral geometric mean (2.1). We call $F_t(A, B)$ weighted F-mean and it is different from (2.2).

Although the spectral geometric mean appears to be less studied and less prominent than the geometric mean, it remains of interest to several researchers. One of the motivations is that the trace of the spectral geometric mean is the quantum fidelity:

$$\mathrm{Tr}\left((A^{-1}\sharp B)^{1/2}A(A^{-1}\sharp B)^{1/2}\right) = \mathrm{Tr}\left(A(A^{-1}\sharp B)\right) = \mathrm{Tr}\left((A^{1/2}BA^{1/2})^{1/2}\right).$$

The weighted Wasserstein mean $A \diamond_t B$ [4] is the geodesic curve joining A and B in $\mathbb{P}_n$ with respect to the Wasserstein distance:

$$A \diamond_t B = (1-t)^2 A + t^2 B + t(1-t)\left(A(A^{-1}\sharp B) + (A^{-1}\sharp B)A\right).$$

If we denote $A^{-1}\sharp B = X_0$, then by the Riccati equation, $B = X_0 A X_0$. And the weighted Wasserstein mean adopts the following form

$$A \diamond_t B = (1-t+tX_0)A(1-t+tX_0). \tag{2.4}$$

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Recently, Franco and Dumitru introduced a near-order relation defined by $A^{-1}\sharp B \leq I$. And then, T.Y. Tam, J. Kim, H. Huang, F. Furuichi, Y. Seo, and L. Gan, among others, have continued investigating these objects and exploring various inequalities within different order relations [13,16] and applications in quantum information.

3. The 2-Geometric Mean

In this section we provide a new form of the spectral geometric mean and establish new simple proofs of basic properties of it. The formula of weighted geometric mean $A\sharp_t B$ can be extended to any real number t. For $t = 2$, we notice that

$$A\sharp_2 B = A^{1/2}(A^{-1/2}BA^{-1/2})(A^{-1/2}BA^{-1/2})A^{1/2} = BA^{-1}B.$$

Also,

$$A\sharp X = B \quad \text{if and only if} \quad X = A\sharp_2 B. \tag{3.1}$$

Indeed, $X = A\sharp_2 B = BA^{-1}B$ if and only if, according to the Riccati equation, $B = A\sharp X$.

From (2.1) and (3.1), we have the following new form of the spectral geometric mean.

Defination 3.1. *For positive definite matrices A and B in $\mathbb{P}_n$, the geometric mean A is represented as*

$$A\natural B = A^{-1}\sharp_2(A^{-1}\sharp B)^{1/2} = A^{-1}\sharp_2 X_0^{1/2}, \quad \textit{where} \quad X_0 = A^{-1}\sharp B. \tag{3.2}$$

Similarly, we have the following representations:

(F_t) The new weighted spectral geometric mean $F_t(A,B)$ is represented as

$$F_t(A,B) = A^{2t-2}\sharp_2(A^{-1}\sharp_t B)^{1/2}; \tag{3.3}$$

$(\natural_t)$ The weighted spectral geometric mean $A\natural_t B$ is as

$$A\natural_t B = A^{-1}\sharp_2(A^{-1}\sharp B)^t = A^{-1}\sharp_2 X_0^t; \tag{3.4}$$

$(\diamond_t)$ The Wasserstein mean [4] is represented as

$$A \diamond_t B = (I\nabla_t X_0)A(I\nabla_t X_0) = A^{-1}\sharp_2 k(X_0), \tag{3.5}$$

where $k(x) = 1 - t + tx$.

Notice that in the definition of the spectral geometric mean, Fiedler and Pták introduced a set of properties that this mean must satisfy, but they did not explain why $A^{-1}\sharp B$ is included in the definition. However, let us provide some reasons for its existence.

Proposition 3.2. *For given positive definite matrices A and B, the geometric mean $A^{-1}\sharp B$ is the unique minimizer of the loss function*

$$F(X) = \mathrm{Tr}\,(AX) + \mathrm{Tr}\,(BX^{-1})\ (X > 0).$$

Proof. Recall that the Fréchet derivative of a function $f(X)$ at X in the direction of a perturbation H is defined as:

$$Df(X)[H] = \lim_{\epsilon\to 0}\frac{f(X+\epsilon H) - f(X)}{\epsilon}. \tag{3.6}$$

Applying (3.6) to $F(X)$, for any Hermitian matrix H, we have

$$DF(X)(H) = \mathrm{Tr}\,(AH - X^{-1}BX^{-1}H).$$

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Setting $DF(X)(H) = 0$ for any Hermitian H, we obtain

$$A - X^{-1}BX^{-1} = 0,$$

or,

$$X^{-1}BX^{-1} = A.$$

From the Riccati equation, it follows that $X = B\sharp A^{-1} = A^{-1}\sharp B$ since the geometric mean is symmetric. □

In case, if A and B are two density matrices which play the role as quantum states in quantum information theory, then at the minimizer $A^{-1}\sharp B$, the function value $F(X)$ is nothing but the quantum fidelity:

$$F(A^{-1}\sharp B) = \mathrm{Tr}\,(A(A^{-1}\sharp B)) + \mathrm{Tr}\,(B(A^{-1}\sharp B)^{-1})2\mathrm{Tr}\,(A) = 2\mathrm{Tr}\,((A^{1/2}BA^{1/2})^{1/2}).$$

We collect some properties of $\sharp_2$ that we often use in the paper.

Lemma 3.3. *For positive definite matrices A, B and C,*

(A1) $A\sharp(A\sharp_2 B) = A\sharp_2(A\sharp B) = B$
(A2) $(A\sharp_2 B)^{-1} = A^{-1}\sharp_2 B^{-1}$*, and hence,* $(A^{-1}\sharp_2 B^{-1})^{-1} = A\sharp_2 B$;
(A3) $A\sharp_2(Cf(C)) = (C^{-1}AC^{-1})\sharp_2 f(C)$ *for some given function f.*
(A4) If $AB = BA$, then for any positive definite matrix X,

$$(X\sharp_2 A)\sharp(X^{-1}\sharp_2 B) = AB. \tag{3.7}$$

Proof. Properties (A1) and (A2) follow from (3.1). For (A3), we have

$$A\sharp_2(Cf(C)) = f(C)CA^{-1}Cf(C) = f(C)(C^{-1}AC^{-1})^{-1}f(C) = (C^{-1}AC^{-1})\sharp_2 f(C).$$

The identity (3.7) can be rewritten as

$$(BX^{-1}B)\sharp(CXC) = BC.$$

Multiplying both sides by $(BC)^{1/2}$, we get

$$(B^{-1/2}C^{1/2}XC^{1/2}B^{-1/2})^{-1}\sharp(B^{-1/2}C^{1/2}XC^{1/2}B^{-1/2}) = I.$$

□

Let us use the new forms to re-establish some basic properties of the spectral geometric means mentioned above.

Proposition 3.4. *For positive definite matrices A and B in $\mathbb{P}_n$, the following are satisfied:*

(i) $A\natural_t B = B\natural_{1-t} A$;
(ii) $(A\natural_t B)^{-1} = A^{-1}\natural_t B^{-1}$;
(iii) $(A\natural_s B)\natural_t(A\natural_u B) = A\natural_{(1-t)s+tu} B$*. When $s = u$, we have* $(A\natural_s B)\natural_t(A\natural_s B) = A\natural_s B$;
(iv) $A \diamond_t B = B \diamond_{1-t} A$;
(v) $(A \diamond_t B)^{-1} = A^{-1} \diamond_t B^{-1}$ *if and only if $A = B$;*
(vi) $(A \diamond_s B) \diamond_t (A \diamond_u B) = A \diamond_{(1-t)s+tu} B$;
(vii) $A \diamond_t B \leq A\nabla_t B$

Proof. (i) Using (A3) and the fact that $B = X_0AX_0$, we have

$$B\natural_{1-t}A = B^{-1}\sharp_2 X_0^{t-1} = (X_0^{-1}A^{-1}X_0^{-1})\sharp_2 X_0^{t-1} = A^{-1}\sharp_2 X_0^{t} = A\natural_t B.$$

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(ii) Using (A2), we have

$$A^{-1}\natural_t B^{-1} = A\sharp_2 X_0^{-t} = (A^{-1}\sharp_2 X_0^t)^{-1} = (A\natural_t B)^{-1}.$$

(iii) Firstly, using (A1) we have the following

$$A\natural_s(A\natural_t B) = A^{-1}\sharp_2(A^{-1}\sharp(A^{-1}\sharp_2 X_0^t))^s = A^{-1}\sharp_2 X_0^{st} = A\natural_{st}B. \tag{3.8}$$

On account of (i), from (3.8) we have

$$(A\natural_s B)\natural_t B = B\natural_{1-t}(B\natural_{1-s}A) = B\natural_{(1-t)(1-s)}A = A\natural_{(1-t)s+t}B. \tag{3.9}$$

As a consequence of (3.8) and (3.9), we have

$$(A\natural_s B)\natural_t(A\natural_u B) = (A\natural_{s/u}(A\natural_u B))\natural_t(A\natural_u B) = A\natural_{(1-t)s/u+t}(A\natural_u B) = A\natural_{(1-t)s+tu}B.$$

(iv) Using the representation ($\diamond_t$) and (A3) we have

$$B\diamond_{1-t} A = (X_0^{-1}A^{-1}X_0^{-1})\sharp_2(X_0^{-1}k(X_0)) = A^{-1}\sharp_2 k(X_0) = A\diamond_t B.$$

(v) We have

$$(A^{-1}\diamond_t B^{-1})^{-1} = (A\sharp_2 k(X_0^{-1}))^{-1} = A^{-1}\sharp_2 k^{-1}(X_0^{-1}) = A^{-1}\sharp_2 k(X_0) \tag{3.10}$$

if and only if $k^{-1}(X_0^{-1}) = k(X_0)$. Since $k(x) = 1 - t + tx$, it is easy to check that the identity $k(x)k(x^{-1}) = 1$ if and only if $x = 1$.

(vi) The proof of this identity is similar to the proof of (iii) with the fact that

$$\begin{aligned} A\diamond_s(A\diamond_t B) = A^{-1}\sharp_2 k_s(A^{-1}\sharp(A^{-1}\sharp_2 k_t(X_0)))) = A^{-1}\sharp_2 k_s(k_t(X_0)) &= A^{-1}\sharp_2 k_{st}(X_0) \\ &= A\diamond_{st} B, \end{aligned}$$

where $k_s(X_0) = 1 - s + sX_0$, and the second identity follows from (A1).

(vii) For any $t \in (0,1)$, the inequality is equivalent to the following

$$(1-t)^2 A + t^2 B + t(1-t)(A(A^{-1}\sharp B) + (A^{-1}\sharp B)A) \le (1-t)A + tB,$$

which is equivalent to

$$A + X_0 A X_0 \ge A X_0 + X_0 A,$$

where B is replaced with $X_0 A X_0$. The last inequality is obvious since

$$(I - X_0)A(I - X_0) \ge 0.$$

□

Recall that the arithmetic mean and the harmonic mean are dual, that means, $A!B = (A^{-1}\nabla B^{-1})^{-1}$, and the representing functions satisfy the relation $f_!(x) = x f_\nabla^{-1}(x)$. Between the arithmetic mean, the geometric mean and the harmonic mean there are two main relations:

(R1) $A!_t B \le A\sharp_t B \le A\nabla_t B$ for any $t \in [0,1]$;

(R2) $A!B = (A\sharp B)(A\nabla B)(A\sharp B)$.

Dinh, Le and Vo [6] proved that for an arbitrary Kubo-Ando matrix mean σ such that $\sigma \ge \sharp$, for positive definite matrices X and Y,

$$X\sigma^\perp Y = (X\sharp Y)(X\sigma Y)^{-1}(X\sharp Y),$$

or, equivalently,

$$(X\sigma Y)\sharp(X\sigma^{\perp}Y) = X\sharp Y. \tag{3.11}$$

This identity (3.11) has an interesting geometrical interpretation: the barycenter of $X\sigma Y$ and $X\sigma^{\perp}Y$ is coincide with the barycenter of X and Y with respect to the Riemannian distance.

Recently, Franco [5] obtained a similar identity for the spectral geometric mean and the Wasserstein mean as

$$(A^{-1} \diamond B^{-1})^{-1}\natural(A \diamond B) = A\natural B. \tag{3.12}$$

In the following theorem, we establish a more general result of (3.12).

Theorem 3.5. *Let $A, B \in \mathbb{P}_n$. Then*

$$(A^{-1} \diamond_t B^{-1})^{-1}\natural(A \diamond_t B) = (k^{1/2}(X_0)k^{-1/2}(X_0^{-1})X_0^{-1/2})(A\natural B)(X_0^{-1/2}k^{-1/2}(X_0^{-1})k^{1/2}(X_0)).$$

Proof. Let $X = (A^{-1} \diamond_t B^{-1})^{-1} = A^{-1}\sharp_2 k^{-1}(X_0^{-1})$ and $Y = A \diamond_t B = A^{-1}\sharp_2 k(X_0)$. Then on account of (A4) we have

$$X^{-1}\sharp Y = (A\sharp_2 k(X_0^{-1}))\sharp(A^{-1}\sharp_2 k(X_0)) = k(X_0^{-1})k(X_0).$$

Consequently,

$$\begin{aligned}
X\natural Y &= X^{-1}\sharp_2(X^{-1}\sharp Y)^{1/2}\\
&= (A\sharp_2 k(X_0^{-1}))\sharp_2(k^{1/2}(X_0^{-1})k^{1/2}(X_0))\\
&= (k^{1/2}(X_0^{-1})k^{1/2}(X_0))(k^{-1}(X_0^{-1})Ak^{-1}(X_0^{-1}))(k^{1/2}(X_0^{-1})k^{1/2}(X_0))\\
&= (k^{-1/2}(X_0^{-1})k^{1/2}(X_0)X_0^{-1/2})(X_0^{1/2}AX_0^{1/2})(X_0^{-1/2}k^{-1/2}(X_0^{-1})k^{1/2}(X_0))\\
&= (k^{1/2}(X_0)k^{-1/2}(X_0^{-1})X_0^{-1/2})(A\natural B)(X_0^{-1/2}k^{-1/2}(X_0^{-1})k^{1/2}(X_0)),
\end{aligned}$$

where the last identity follows from the fact that $X_0^{1/2}AX_0^{1/2} = A^{-1}\sharp_2 X_0^{1/2} = A^{-1}\sharp_2(A^{-1}\sharp B)^{1/2} = A\natural B$. □

4. The Near Order Relation and Means

Recently, Franco and Dumitru introduced the near order relation $\preceq$ on $\mathbb{P}_n$ as follows:

$$A \preceq B \quad \text{if and only if} \quad A^{-1}\sharp B \geq I.$$

This order is weaker than the traditional Loewner order, which states that for Hermitian matrices A and B, $A \leq B$ if and only if $B - A \geq 0$. From Proposition 3.2, $A \preceq B$ if the inverse of the minimizer of $F(X)$ is a contraction.

In this section we review some inequalities in this the near order relation. Fortunately, many results can be obtained easily with the 2-geometric mean form for the spectral geometric mean and the Wasserstein mean.

The following lemma is crucial when we consider inequalities with respect to the near order.

Lemma 4.1. *Let A be a positive definite matrix, and let $0 < f(x) \leq g(x)$ for any positive x. Then, for any positive definite matrix X,*

(i) $A^{-1}\sharp_2 f(X) \preceq A^{-1}\sharp_2 g(X)$;

(ii) $\|A^{-1}\sharp_2 f(X)\| \leq \|A^{-1}\sharp_2 g(X)\|$, *where $\|\cdot\|$ is the operator norm.*

Proof. On account of Lemma 3.3, inequality (i) is equivalent to the following

$$\begin{aligned}
(A^{-1}\sharp_2 f(X_0))^{-1}\sharp(A^{-1}\sharp_2 g(X_0)) &= (A\sharp_2 f^{-1}(X_0))\sharp(A^{-1}\sharp_2 g(X_0))\\
&= f^{-1}(X_0)g(X_0) \geq I
\end{aligned}$$

which is true because of assumption.

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(ii) Suppose that $A^{-1}\sharp_2 g(X) \leq I$, we have $A \leq g^{-2}(X) \leq f^{-2}(X)$. That means, $A^{-1}\sharp_2 f(X) \leq I$. □

The following result was obtained by Huang and Gan [15] which is an obvious consequence of Lemma 4.1.

Theorem 4.2. *The following are equivalent for $A, B \in \mathbb{P}_n$, and for $s, t \in \mathbb{R}$:*

(i) $A \preceq B$;
(ii) $A \diamond_s B \leq A \diamond_t B$ *whenever* $s \leq t$;
(iii) $A \natural_s B \leq A \natural_t B$ *whenever* $s \leq t$.

Proof. Observe that the condition $A \preceq B$ is equivalent to that $X_0 = A^{-1}\sharp B \geq I$. Also, the representing function of $\diamond_t$ is $k(x) = 1 - t + tx$. This function is increasing in t if and only if $x \geq 1$. Similarly, the representing function of $\natural_t$ is x^t. This function is increasing for any x. □

In the following theorem, we prove a similar relation to (R2) which was obtained in [5].

Theorem 4.3. *Let $A, B \in \mathbb{P}_n$, and $t \in [0, 1]$. Then,*

$$(A^{-1} \diamond_t B^{-1})^{-1} \preceq A \natural_t B \preceq A \diamond_t B. \tag{4.1}$$

Proof. The inequalities in (4.1) are equivalent to

$$A^{-1}\sharp_2 k^{-1}(X_0^{-1}) \preceq A^{-1}\sharp_2 f(X_0) \preceq A^{-1}\sharp_2 k(X_0).$$

According to Lemma 4.1, it is enough to show that $k^{-1}(x^{-1}) \leq x^t \leq k(x)$ for any $x > 0$ and for any $t \in (0, 1)$. We show, for example, the second inequality. It is not difficult to see that the function $k(x) - x^t = 1 - t + tx - x^t$ attains minimum at $x = 1$ for any $t \in (0, 1)$. That means, $k(x) \geq x^t$. □

Remark 4.4. *Notice that the second inequality in (4.1) was proved by Huang and Gan in [15]. Our proof is essential shorter. Also, Huang and Gan showed that if $A \preceq B$, then $A \diamond_t B \preceq A \natural_t B$, and if $B \preceq A$ then for any $t \in (-\infty, 0)$, $A \diamond_t B \preceq A \natural_t B$. Again, these statements can be proved based on analyzing the sign of the function $k(x) - x^t$, in term of x or t on $\mathbb{R}$.*

In the following we establish some norm inequalities for the spectral geometric mean and the Wasserstein mean.

Theorem 4.5. *Let $A, B \in \mathbb{P}_n$, and $t \in (0, 1)$. Then*

(i) $\|A\sharp B\| \leq \|A\natural B\|$;
(ii) $\|(A^{-1} \diamond_t B^{-1})^{-1}\| \leq \|A\natural_t B\| \leq \|A \diamond_t B\|$,

where $\|\cdot\|$ is the operator norm.

Proof. (i) can be found in [20]. Here we provide a short direct proof of it. Suppose that $A\natural B \leq I$, or, $X_0^{1/2} A X_0^{1/2} \leq I$, we have $B = X_0 A X_0 \leq X_0$ and $A \leq X_0^{-1}$. By monotonicity of the geometric mean, we have $A\sigma B \leq X_0^{-1}\sharp X_0 = I$. From here, $\|A\sharp B\| \leq \|A\natural B\|$.

The second inequality in (ii) was proved in [14]. However, all inequalities in (ii) are direct consequences of Lemma 4.1. □

To finish this note, we establish new characterizations of central elements in the algebra $\mathbb{M}_n$. We need the following fact which is a special case of the main result in [23]. Notice that some global characterizations of trace and commutativity were studied by the fourth authors, and his co-authors in [1,8,9,21].

Lemma 4.6. *Let $A \in \mathbb{P}_n$. Suppose that for any Hermitian matrix X:*

$$A \leq X \quad \Longrightarrow \quad A^2 \leq X^2.$$

Then A is a scalar multiple of the identity.

Theorem 4.7. *Let A be a positive definite matrix. Then*

(i) $\|A\sharp X\| = \|A\natural X\|$ *for any positive definite matrix X if and only if A is scalar multiple of the identity matrix;*

(ii) $\|A\natural_t B\| = \|A \diamond_t B\|$ *if and only if $A = B$;*

(iii) $\|(A^{-1} \diamond_t B^{-1})^{-1}\| = \|A\natural_t B\|$ *if and only if $A = B$.*

Proof. (i) was obtained in [20]. Here we provide a direct proof. Let $\|A\sharp B\| = \|A\natural B\|$. It follows that $A\sharp B \le I$ if and only if $A\natural B \le I$. That means, $Y^{1/2} \le A^{-1}$ if and only if $X_0^{1/2} A X_0^{1/2} \le I$, where $Y = A^{-1/2} B A^{-1/2}$. Since $B = X_0 A X_0 \le X_0^{1/2} X_0^{1/2} A X_0^{1/2} X_0^{1/2} \le X_0$, we have that $Y \le A^{-1/2} X_0 A^{-1/2} \le A^{-1/2} A^{-1} A^{-1/2} = A^{-2}$. That means, $Y^{1/2} \le A^{-1}$ if and only if $Y \le A^{-2}$. From here, it follows that A and Y^{-1} are commuting, and hence, $A^{1/2}$ and Y are commuting. As a consequence, we can see that B and $A^{-1/2}$ are commuting. By Lemma 4.6, A and B commuting for any B. Thus, A is scalar multiple of the identity matrix.

(ii) From the assumption it implies that $k(X_0)Ak(X_0) \le I$ if and only if $X_0^t A X_0^t \le I$. The first inequality is equivalent to that $X_0^t A X_0^t \le X_0^{2t} k^{-2}(X_0)$. This occurs if and only if $X_0^{2t} k^{-2}(X_0) = I$ which implies that $X_0 = I$. Consequently, $A^{-1}\sharp B = I$. From here, one can see that $A = B$.

(iii) can be proved by using similar arguments. □

Remark 4.8. *We would like to obtain a similar result for the equality $\|A\sharp_t B\| = \|A\natural_t B\|$. Unfortunately, our proof could not be transferred to the general case.*

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